London Academy of Mathematics

Edexcel - AS-level Maths
Pure
Integration
Worked solutions

Compiled by Chris Nunes, created by Pearson Edexcel

Answer ALL questions. Write your answers in the spaces provided.

1. Find

$$\int\left(\frac{2}{3}x^3 - 6\sqrt{x} + 1\right)\mathrm{d}x$$

giving your answer in its simplest form.

(4)

$= \int \frac{2}{3}x^3 - 6x^{1/2} + 1 \; dx$

$= \frac{2}{3} \times \frac{x^3}{3} - 6x^{3/2} \times \frac{2}{3} + x + C$

$= \frac{2}{9}x^3 - 4x^{3/2} + x + C$

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Question 1 continued

(Total for Question 1 is 4 marks)

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15.

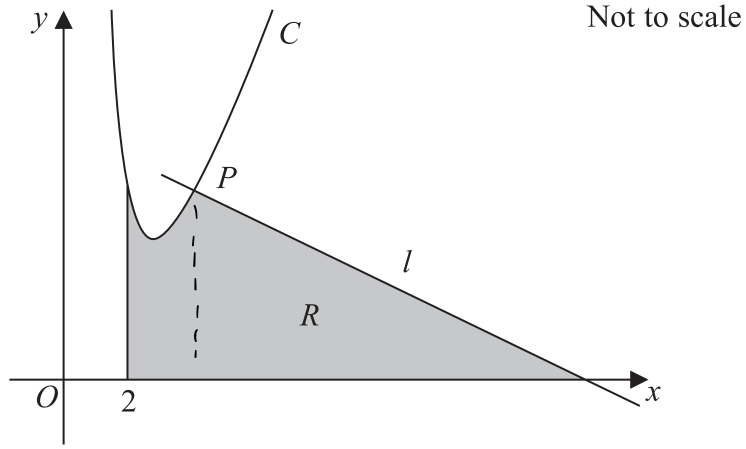

Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{32}{x^2} + 3x - 8, \qquad x > 0$$

The point P (4, 6) lies on C.
The line l is the normal to C at the point P.

The region R, shown shaded in Figure 4, is bounded by the line l, the curve C, the line with equation $x = 2$ and the x-axis.

Show that the area of R is 46
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(10)

First find equation of normal:

$y = 32x^{-2} + 3x - 8$

$\frac{dy}{dx} = -64x^{-3} + 3$

$m_T = \frac{dy}{dx}\Big|_{x=4} = -64(4)^{-3} + 3$

$= 2$

$m_N = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}x + c$

$6 = -\frac{1}{2}(4) + c \Rightarrow 6 = -2 + c \quad c = 8$

Question 15 continued

$y = -\frac{1}{2}x + 8$

Total Area $= \int_2^4 y \, dx$ + area of triangle

$$\int_2^4 y\,dx = \int_2^4 32x^{-2} + 3x - 8 \, dx$$

$$= \left[-32x^{-1} + \frac{3}{2}x^2 - 8x\right]_2^4$$

$$= \left(-32(4)^{-1} + \frac{3}{2}(4)^2 - 8(4)\right) - \left(-32(2)^{-1} + \frac{3}{2}(2)^2 - 8(2)\right)$$

$$= (-16) - (-26)$$

$$= 10$$

Area of triangle.

First find x intercept:

$y = 0 \Rightarrow -\frac{1}{2}x + 8 = 0 \Rightarrow \frac{1}{2}x = 8$

$\Rightarrow x = 16$

Area of triangle $= \frac{1}{2}(16-4)(6) = 36$

$\therefore$ Area of $R = 10 + 36 = 46$

3. (a) Given that k is a constant, find

$$\int \left(\frac{4}{x^3} + kx \right) dx$$

simplifying your answer.

(3)

(b) Hence find the value of k such that

$$\int_{0.5}^{2} \left(\frac{4}{x^3} + kx \right) dx = 8$$

(3)

a) $\int 4x^{-3} + kx \; dx$

$= \frac{4x^{-2}}{-2} + \frac{kx^2}{2} + c$

$= -2x^{-2} + \frac{k}{2}x^2 + c$

b) $\left[-2x^{-2} + \frac{k}{2}x^2 \right]_{0.5}^{2} = 8$

$\left(-2(2)^{-2} + \frac{k}{2}(2)^2 \right) - \left(-2(0.5)^{-2} + \frac{k}{2}(0.5)^2 \right) = 8$

$\left(-\frac{1}{2} + 2k \right) - \left(-8 + \frac{k}{8} \right) = 8$

$-\frac{1}{2} + 8 + 2k - \frac{k}{8} = 8$

$\frac{15}{2} + \frac{15k}{8} = 8$

$60 + 15k = 64 \qquad 15k = 4 \qquad \Rightarrow k = 4/15$

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Question 3 continued

(Total for Question 3 is 6 marks)

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13.

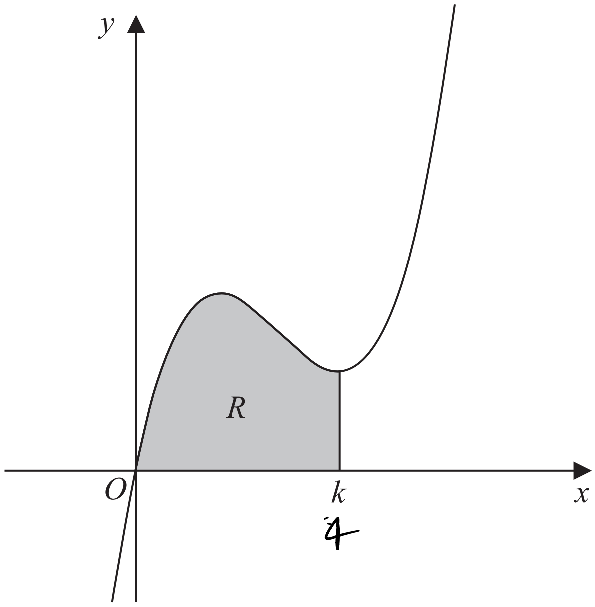

Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2x^3 - 17x^2 + 40x$$

The curve has a minimum turning point at $x = k$.

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and the line with equation $x = k$.

Show that the area of R is $\dfrac{256}{3}$

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(7)

First find k.

$\frac{dy}{dx} = 6x^2 - 34x + 40$

$6x^2 - 34x + 40 = 0$

$3x^2 - 17x + 20 = 0$

$3x^2 - 12x - 5x + 20 = 0$

$3x(x-4) - 5(x-4) = 0$

$(3x-5)(x-4) = 0$

$x = 4, x = \frac{5}{3}$ $\Rightarrow k = 4$ Since minimum has larger x-coordinate by inspection.

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Question 13 continued

$$\int_0^4 2x^3 - 17x^2 + 40x \, dx$$

$$= \left[\frac{2x^4}{4} - \frac{17x^3}{3} + \frac{40}{2}x^2 \right]_0^4$$

$$= \left[\frac{1}{2}x^4 - \frac{17}{3}x^3 + 20x^2 \right]_0^4$$

$$= \left(\frac{1}{2}(4)^4 - \frac{17}{3}(4)^3 + 20(4)^2 \right) - (0)$$

$$= \frac{256}{3}$$

7. Given that k is a positive constant and $\int_1^k \left(\frac{5}{2\sqrt{x}} + 3\right) dx = 4$

(a) show that $3k + 5\sqrt{k} - 12 = 0$

(4)

(b) Hence, using algebra, find any values of k such that

$$\int_1^k \left(\frac{5}{2\sqrt{x}} + 3\right) dx = 4$$

(4)

a) $\int_1^k \left(\frac{5}{2}x^{-1/2} + 3\right) dx = 4$

$\left[5x^{1/2} + 3x\right]_1^k = 4$

$\left(5k^{1/2} + 3k\right) - \left(5 + 3\right) = 4$

$5k^{1/2} + 3k = 12$

$3k + 5\sqrt{k} - 12 = 0$

b) $3k + 5\sqrt{k} - 12 = 0$

Sub $x = \sqrt{k}$ in: $3x^2 + 5x - 12 = 0$

$3x^2 + 9x - 4x - 12 = 0$

$3x(x+3) - 4(x+3) = 0$

$(3x-4)(x+3) = 0$

$x = -3$, $x = 4/3$

$\sqrt{k} = -3$ $\quad \sqrt{k} = 4/3 \Rightarrow k = \frac{16}{9}$

ignore, since $\sqrt{k} \geq 0$ for all $k \geq 0$.

Note for $k < 0$, $\sqrt{k} \notin \mathbb{R}$

i.e. $\sqrt{k}$ is not a real number.

Question 7 continued

(Total for Question 7 is 8 marks)

3. Find

$$\int \frac{3x^4 - 4}{2x^3}\,dx$$

writing your answer in simplest form.

(4)

$$\int \frac{3x^4}{2x^3} - \frac{4}{2x^3}\,dx$$ Split up

$$= \int \frac{3}{2}x - 2x^{-3}\,dx$$ Simplify.

$$= \frac{3x^2}{4} - \frac{2x^{-2}}{-2} + c$$

$$= \frac{3}{4}x^2 + x^{-2} + c$$

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Question 3 continued

(Total for Question 3 is 4 marks)

9. Find the value of the constant k, $0 < k < 9$, such that

$$\int_k^9 \frac{6}{\sqrt{x}}\,dx = 20$$

(4)

$$\int_k^9 6x^{-\frac{1}{2}}\,dx = 20$$

$$\left[12x^{\frac{1}{2}}\right]_k^9 = 20$$

$$12(9)^{\frac{1}{2}} - 12(k)^{\frac{1}{2}} = 20$$

$$36 - 12\sqrt{k} = 20$$

$$12\sqrt{k} = 16$$

$$\sqrt{k} = \frac{4}{3}$$

$$k = \frac{16}{9}$$

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Question 9 continued

(Total for Question 9 is 4 marks)

14. A curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = -3x^2 + 12x + 8$$

(a) Write $\mathrm{f}(x)$ in the form

$$a(x + b)^2 + c$$

where a, b and c are constants to be found.

(3)

The curve C has a maximum turning point at M.

(b) Find the coordinates of M.

(2)

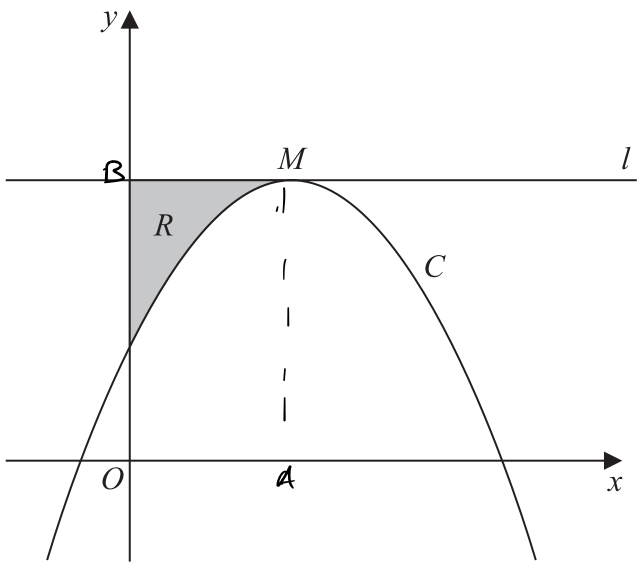

Figure 3

Figure 3 shows a sketch of the curve C.

The line l passes through M and is parallel to the x-axis.

The region R, shown shaded in Figure 3, is bounded by C, l and the y-axis.

(c) Using algebraic integration, find the area of R.

(5)

a) $= -3(x^2 - 4x) + 8$

$= -3((x-2)^2 - 4) + 8$

$= -3(x-2)^2 + 12 + 8$

$= -3(x-2)^2 + 20$

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Question 14 continued

b) by part (a) $(2, 20)$

c) $\int_0^2 -3x^2+12x+8\,dx = \left[-\frac{3x^3}{3} + \frac{12x^2}{2} + 8x\right]_0^2$

$= \left[-x^3+6x^2+8x\right]_0^2$ Simplify

$= \left(-(2)^3+6(2)^2+8(2)\right) - (0)$

$= 32$

Area of rectangle OAMB $= 2\times 20 = 40$

Area of R $= 40 - 32 = \underline{\underline{8}}$

P 6 6 5 8 5 A 0 3 5 4 4

Turn over

1. Find

$$\int\left(8x^3 - \frac{3}{2\sqrt{x}} + 5\right)\mathrm{d}x$$

giving your answer in simplest form.

(4)

$= \int 8x^3 - \frac{3}{2}x^{-1/2} + 5 \; dx$

$= \frac{8x^4}{4} - \frac{3}{2} \times x^{1/2} \times 2 + 5x + c$

$= 2x^4 - 3x^{1/2} + 5x + c$

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Question 1 continued

(Total for Question 1 is 4 marks)

10.

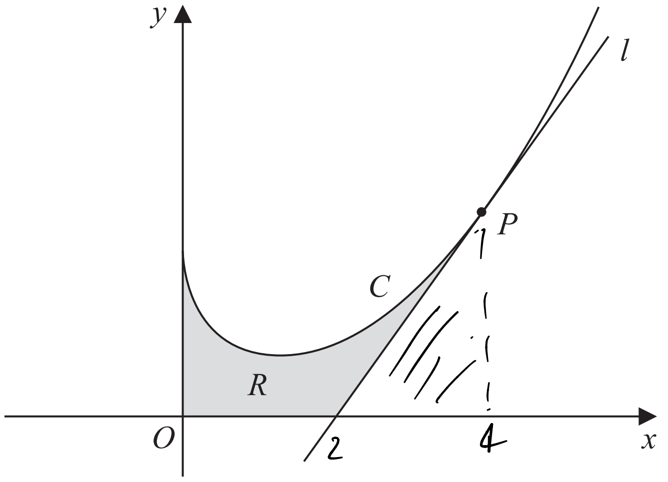

Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 2 shows a sketch of part of the curve C with equation

$$y = \frac{1}{3}x^2 - 2\sqrt{x} + 3 \qquad x \geqslant 0$$

The point P lies on C and has x coordinate 4

The line l is the tangent to C at P.

(a) Show that l has equation

$$13x - 6y - 26 = 0$$

(5)

The region R, shown shaded in Figure 2, is bounded by the y-axis, the curve C, the line l and the x-axis.

(b) Find the exact area of R.

(5)

a) $\frac{dy}{dx} = \frac{2}{3}x - x^{-1/2}$

$m_T = \left.\frac{dy}{dx}\right|_{x=4} = \frac{2}{3}(4) - (4)^{-1/2}$

$= \frac{13}{6}$

P.T.O

Question 10 continued

$\therefore \quad y = \frac{13}{6}x + c \qquad (\dagger)$

we want to sub (x,y) coordinate of P in to find c.
First find y coordinate by subbing $x=4$ into

$y = \frac{1}{3}x^2 - 2\sqrt{x} + 3$

$x=4, \quad y = \frac{1}{3}(4)^2 - 2\sqrt{4} + 3 = \frac{13}{3}$

So, subbing back into $(\dagger)$:

$\frac{13}{3} = \frac{13}{6}(4) + c \qquad \Rightarrow c = \frac{13}{3} - \frac{26}{3} = -\frac{13}{3}$

$\therefore \quad y = \frac{13}{6}x - \frac{13}{3}$

$6y = 13x - 26$

$13x - 6y - 26 = 0$

b) $\int_0^4 \frac{1}{3}x^2 - 2x^{1/2} + 3 \, dx$

$= \left[\frac{1}{3}\frac{x^3}{3} - 2x^{3/2} \times \frac{2}{3} + 3x \right]_0^4$

$= \left[\frac{1}{9}x^3 - \frac{4}{3}x^{3/2} + 3x \right]_0^4$

$= \left(\frac{1}{9}(4)^3 - \frac{4}{3}(4)^{3/2} + 3(4) \right) - (0)$

$= \frac{76}{9}$

Area of triangle $= \frac{1}{2} \times (4-2) \times \frac{13}{3} = \frac{13}{3}$

Area of $R = \frac{76}{9} - \frac{13}{3} = \frac{37}{9}$

5. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

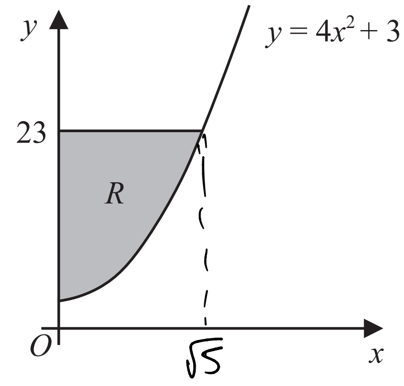

Figure 2

The finite region R, shown shaded in Figure 2, is bounded by the curve with equation $y = 4x^2 + 3$, the y-axis and the line with equation $y = 23$

Show that the exact area of R is $k\sqrt{5}$ where k is a rational constant to be found.

(5)

$y = 23$, $23 = 4x^2 + 3$

$20 = 4x^2$

$5 = x^2$

$x = \pm\sqrt{5}$

$x = \sqrt{5}$ since $x > 0$

$$\int_0^{\sqrt{5}} (4x^2 + 3)\,dx = \left[\frac{4x^3}{3} + 3x\right]_0^{\sqrt{5}}$$

$$= \frac{4(\sqrt{5})^3}{3} + 3(\sqrt{5}) - 0$$

$$= \frac{29\sqrt{5}}{3}$$

Question 5 continued

Area of R = Area of rectangle − Area under curve

$= 23\sqrt{5} - \frac{29\sqrt{5}}{3}$

$= \frac{40}{3}\sqrt{5}$

$\therefore k = \frac{40}{3}$

(Total for Question 5 is 5 marks)

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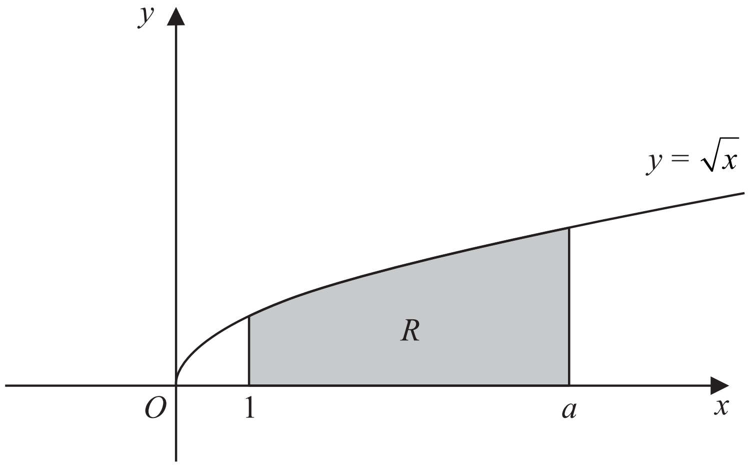

Figure 2

Figure 2 shows a sketch of the curve with equation $y = \sqrt{x}, \quad x \geqslant 0$

The region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 1$, the x-axis and the line with equation $x = a$, where a is a constant.

Given that the area of R is 10

(a) find, in simplest form, the value of

(i) $\int_1^a \sqrt{8x}\, dx$

(ii) $\int_0^a \sqrt{x}\, dx$

(4)

(b) show that $a = 2^k$, where k is a rational constant to be found.

(4)

ai) $\int_1^a \sqrt{8x}\, dx = \int_1^a \sqrt{8}\sqrt{x}\, dx$

$= 2\sqrt{2}\int_1^a \sqrt{x}\, dx$ by taking constant out

$= 2\sqrt{2} \times 10 = 20\sqrt{2}$

(ii) $\int_0^a \sqrt{x}\, dx = \int_0^1 \sqrt{x}\, dx + \int_1^a \sqrt{x}\, dx$

$= \int_0^1 x^{1/2}\, dx + 10$

$= \left[x^{3/2} \times \frac{2}{3}\right]_0^1 + 10$

Question 8 continued

$$= \left[\frac{2}{3}x^{3/2}\right]_0^1 + 10$$

$$= \frac{2}{3}(10)^{3/2} + 10$$

$$= \frac{20}{3}\sqrt{10} + 10$$

$$\int_1^a \sqrt{x}\,dx = 10$$

$$\left[\frac{2}{3}x^{3/2}\right]_1^a = 10$$

$$\left(\frac{2}{3}(a)^{3/2}\right) - \left(\frac{2}{3}(1)^{3/2}\right) = 10$$

$$\frac{2}{3}a^{3/2} - \frac{2}{3} = 10$$

$$2a^{3/2} - 2 = 30$$

$$2a^{3/2} = 32$$

$$a^{3/2} = 16$$

$$(a^{3/2})^{2/3} = 16^{2/3}$$

$$a = (2^4)^{2/3}$$

$$= 2^{8/3} \qquad k = 8/3$$

(Total for Question 8 is 8 marks)

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14.

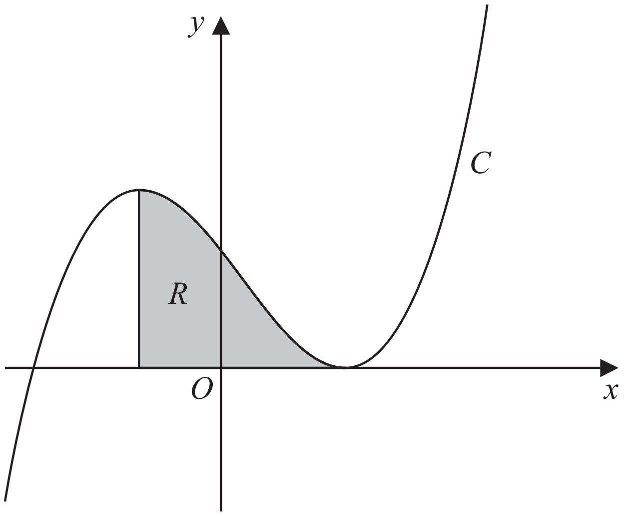

Figure 5

Figure 5 shows a sketch of the curve C with equation $y = (x-2)^2(x+3)$

The region R, shown shaded in Figure 5, is bounded by C, the vertical line passing through the maximum turning point of C and the x-axis.

Find the exact area of R.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(9)

First find bounds:

$y = (x^2 - 4x + 4)(x+3)$

$= x^3 - 4x^2 + 4x + 3x^2 - 12x + 12$

$= x^3 - x^2 - 8x + 12$

$\frac{dy}{dx} = 3x^2 - 2x - 8$

$0 = 3x^2 - 2x - 8$

$= 3x^2 - 6x + 4x - 8$

$= 3x(x-2) + 4(x-2)$

$= (3x+4)(x-2)$

$x = 2$, $x = -\frac{4}{3}$

Question 14 continued

Area of R = $\int_{-4/3}^{2} x^3 - x^2 - 8x + 12 \; dx$

$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 4x^2 + 12x \right]_{-4/3}^{2}$

$= \left(\frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - 4(2)^2 + 12(2) \right) - \left(\frac{1}{4}(-4/3)^4 - \frac{1}{3}(-4/3)^3 - 4(-4/3)^2 + 12(-4/3) \right)$

$= \frac{28}{3} - \left(-\frac{1744}{81} \right)$

$= \frac{2500}{81}$