London Academy of Mathematics

Edexcel - A-level Further Maths
Further Pure
The t-formulae
Worked solutions

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1. Let F(x) be the function $F(x) = 12 - 10\sin(12x) - 5\cos(24x)$

a) By using a suitable trigonometric substitution show that

$$\frac{dF}{dx} = \frac{120(t^2 - 1)(t^2 - 4t + 1)}{(1 + t^2)^2}$$

b) Hence find the exact coordinates of the points A, B and C on the graph of F(x) in Figure 1.

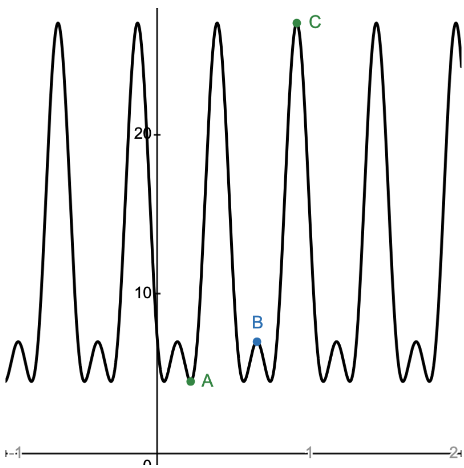


Figure 1

a) $\frac{dF}{dx} = -120\cos(12x) + 120\sin(24x)$

$= -120\cos(12x) + 240\sin(12x)\cos(12x)$

$= -120\cos(12x)\left(1 - 2\sin(12x)\right)$ $\qquad t = \tan(6x)$

$= -120\left(\frac{1-t^2}{1+t^2}\right)\left(1 - 2\left(\frac{2t}{1+t^2}\right)\right)$

$$= -120\left(\frac{1-t^2}{1+t^2}\right)\left(\frac{1+t^2}{1+t^2} - \frac{4t}{1+t^2}\right)$$

$$= \frac{-120(1-t^2)(t^2-4t+1)}{(1+t^2)^2}$$

$$= \frac{120(t^2-1)(t^2-4t+1)}{(1+t^2)^2}$$

b) $\frac{dF}{dx} = 0$ at these points.

$\therefore \cancel{120}(t^2-1)(t^2-4t+1) = 0$

$t = 1, -1, 2+\sqrt{3}, 2-\sqrt{3}$ $\qquad t = \tan(6x)$

$t=1$: $6x = \frac{1}{4}\pi, \frac{5}{4}\pi, \frac{9}{4}\pi, \ldots$

$x = \frac{1}{24}\pi, \frac{5}{24}\pi, \frac{9}{24}\pi, \ldots$

$t=-1$: $6x = -\frac{1}{4}\pi, \frac{3}{4}\pi, \frac{7}{4}\pi, \ldots$

$x = \frac{-1}{24}\pi, \frac{3}{24}\pi, \frac{7}{24}\pi$

$t = 2+\sqrt{3}$: $6x = \frac{5}{12}\pi, \frac{17}{12}\pi, \frac{29}{12}\pi, \ldots$

$x = \frac{5}{72}\pi, \frac{17}{72}\pi, \frac{29}{72}\pi, \ldots$

$t = 2-\sqrt{3}$: $6x = \frac{1}{12}\pi, \frac{13}{12}\pi, \ldots$

$x = \frac{1}{72}\pi, \frac{13}{72}\pi, \ldots$

in increasing order,

$$x = \frac{1}{72}\pi, \frac{1}{24}\pi, \frac{5}{72}\pi, \frac{3}{24}\pi, \frac{13\pi}{72}, \frac{5\pi}{24}, \frac{17\pi}{72}, \frac{7\pi}{24}, \ldots$$

A is third stationary point.

$\therefore A = \left(\frac{5}{72}\pi, 4-5\right)$

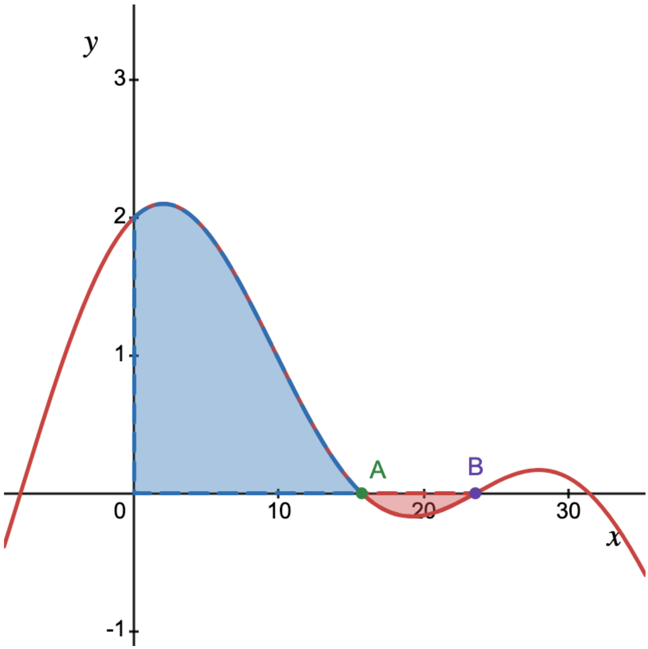
B is sixth stationary point

$\therefore B = \left(\frac{5}{24}\pi, 7\right)$

C is eighth stationary point

$C = \left(\frac{7}{24}\pi, 27\right)$

2. The curve $y = f(x) = \sin(\frac{x}{10}) + \cos(\frac{x}{10}) + \cos(\frac{x}{5})$ is drawn below



a) By using the trigonometric substitution $t = \tan(\frac{x}{20})$ show that points A and B have coordinates $A(a\pi, 0), B(b\pi, 0)$, respectively, where a and b are rational coefficients to be found.

b) Hence, find the area of the shaded region.

a) $f(x) = \sin\left(\frac{1}{10}x\right) + \cos\left(\frac{1}{10}x\right) + \cos\left(\frac{1}{5}x\right)$

$t = \tan\left(\frac{x}{20}\right)$

$0 = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 2\left(\frac{1-t^2}{1+t^2}\right)^2 - 1$

$0 = 2t(1+t^2) + (1-t^2)(1+t^2) + 2(1-t^2)^2 - (1+t^2)^2$

$$0 = 2t + 2t^3 + 1 - t^4 + 2(1 - 2t^2 + t^4) - (1 + 2t^2 + t^4)$$

$$= \cancel{2t} + \cancel{2t^3} + 1 \cancel{- t^4} + 2 \cancel{- 4t^2} + \cancel{2t^4} - 1 \cancel{- 2t^2} \cancel{- t^4}$$

$$= 2t^3 - 6t^2 + 2t + 2$$

$$0 = t^3 - 3t^2 + t + 1$$

$$t = 1 - \sqrt{2},\ 1 + \sqrt{2},\ 1$$

$$t = \tan\left(\frac{x}{20}\right)$$

$t = 1 - \sqrt{2}$:

$$\frac{x}{20} = -\frac{1}{8}\pi,\ \frac{7}{8}\pi, \ldots$$

$$x = -\frac{5}{2}\pi,\ \frac{35}{2}\pi, \ldots$$

$t = 1 + \sqrt{2}$

$$\frac{x}{20} = \frac{3}{8}\pi,\ \frac{11}{8}\pi, \ldots$$

$$x = \frac{15}{2}\pi,\ \frac{55}{2}\pi, \ldots$$

$t = 1$

$$\frac{x}{20} = \frac{1}{4}\pi,\ \frac{5}{4}\pi, \ldots$$

$$x = 5\pi,\ 25\pi$$

$$A(5\pi, 0) \qquad B\left(\frac{15\pi}{2}, 0\right)$$

$$\text{Area} = \int_0^{5\pi} f(x)\,dx - \int_{5\pi}^{\frac{15\pi}{2}} f(x)\,dx$$

$$= \int_0^{5\pi} \sin\left(\tfrac{1}{10}x\right) + \cos\left(\tfrac{1}{10}x\right) + \cos\left(\tfrac{1}{5}x\right)dx + \int_{\frac{15\pi}{2}}^{5\pi} \sin\left(\tfrac{1}{10}x\right) + \cos\left(\tfrac{1}{10}x\right) + \cos\left(\tfrac{1}{5}x\right)dx$$

$$= \left[-10\cos\left(\tfrac{1}{10}x\right) + 10\sin\left(\tfrac{1}{10}x\right) + 5\sin\left(\tfrac{1}{5}x\right)\right]_0^{5\pi} + \left[\;\text{''} \quad \cdots \quad \text{''}\;\right]_{\frac{15\pi}{2}}^{5\pi}$$

$$= 2(10) - (-10) - \left(-10\cos\left(\tfrac{3\pi}{4}\right) + 10\sin\left(\tfrac{3\pi}{4}\right) + 5\sin\left(\tfrac{3\pi}{2}\right)\right)$$

$$= 20 + 10 - \left(-5 + 10\sqrt{2}\right)$$

$$= 35 - 10\sqrt{2}$$

3. The amplitude of a music note can be modelled by the function $A(x) = \sin(2\pi f x)$

where f is the frequency of the note in Hz and x is the time elapsed in seconds.
When you play two or more notes simultaneously the amplitude is found by **adding the sine waves together.**
The frequency of an Octave 1 A is 55Hz and the frequency of an Octave 2 A is 110Hz.

a) Find the exact time in seconds that the amplitude is zero for the third time (not including x=0).

b) using a trigonometric substitution, show that when the notes 1A and 2A are played together $\frac{dA}{dx}$ is given by

$$\frac{dA}{dx} = \frac{110\pi(t^4 - 12t^2 + 3)}{(1+t^2)^2}$$

c) Hence, find the first time that the maximum amplitude is reached in milliseconds to three significant figures.

a) $\sin(110\pi x) + \sin(220\pi x) = 0$

$\sin(110\pi x) + 2\sin(110\pi x)\cos(110\pi x) = 0$

$\sin(110\pi x)(1 + 2\cos(110\pi x)) = 0$

$\sin(110\pi x) = 0$ $\qquad$ $\cos(110\pi x) = -\frac{1}{2}$

$110\pi x = 0, \pi, 2\pi, \ldots$ $\qquad$ $110\pi x = \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{8}{3}\pi, \ldots$

$x = 0, \frac{1}{110}, \frac{2}{110}$ $\qquad$ $x = \frac{1}{165}, \frac{2}{165}$

2 $\quad$ 4 $\qquad$ 1 $\quad$ 3

$x = \frac{2}{165}$ s

b) $A(x) = \sin(110\pi x) + \sin(220\pi x)$

$$\frac{dA}{dx} = 110\pi\cos(110\pi x) + 220\pi\cos(220\pi x)$$

$t = \tan(55\pi x)$

$$\frac{dA}{dx} = 110\pi\left(\frac{1-t^2}{1+t^2}\right) + 220\pi\left(1 - 2\left(\frac{2t}{1+t^2}\right)^2\right)$$

$$= 110\pi\left(\frac{1-t^2}{1+t^2} + 2 - 4\left(\frac{2t}{1+t^2}\right)^2\right)$$

$$= \frac{110\pi}{(1+t^2)^2}\left((1-t^2)(1+t^2) + 2(1+t^2)^2 - 4(2t)^2\right)$$

$$= \frac{110\pi}{(1+t^2)^2}\left(1-t^4 + 2(1+2t^2+t^4) - 16t^2\right)$$

$$= \frac{110\pi}{(1+t^2)^2}\left(1-t^4 + 2 + 4t^2 + 2t^4 - 16t^2\right)$$

$$= \frac{110\pi(t^4 - 12t^2 + 3)}{(1+t^2)^2}$$

c) $\frac{dA}{dx} = 0$

$\cancel{110\pi}(t^4 - 12t^2 + 3) = 0$

$t = \pm 3.427034089$, $t = \pm 0.505408105 9$

$t = 3.427...$ $\quad 55\pi x = 1.286....$

$x = 0.0074477....$ Seconds

$= 7.45$ milliseconds

$t = -3.427...$ $\quad 55\pi x = \cancel{-1.2868}....,\ -1.2868...+\pi$

$x = 0.0107...$ seconds

ignore.

$t = 0.505....$ $\quad 55\pi x = 0.4679...$

$x = 0.002708....$

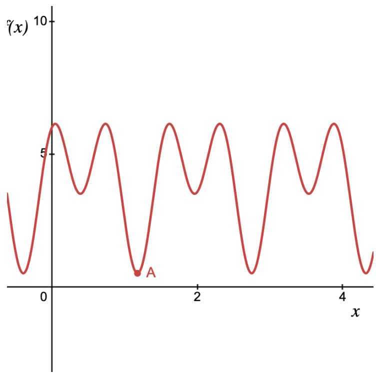
$= 2.71$ seconds

$t = -0.505$ too lage like $t = -3.427....$

$\therefore x = 2.71$ seconds

4. Mario decides to use the function h(x), where x is the width in cm, to trace out what is known to him and his customers as a tile: a series of consecutive letters 'M' on a blank piece of paper. The printing tool has plotted the function h onto this paper starting at the point (0,6), heading in the positive x direction.

$h(x) = 1.5\sin(4x) + 2\cos(8x) + 4$



a) show, using the substitution $t = \tan(kx)$ where k is a constant to be deduced, that the derivative of h with respect to x is given by

$$\frac{dh}{dx} = \frac{2(1-t^2)(3t^2-32t+3)}{(1+t^2)^2}$$

After the printing job is finished Mario cuts the piece of paper vertically at the point A. This is so that he doesn't give an incomplete M to the customer such as the one from $0 \le x \le x_A$ where x_A is the x coordinate of the point A

b) show that the exact minimum width of paper that Mario used (including the piece he cut off) to finish with exactly 15 complete Ms, is $b\pi$, where b is a constant to be found.

c) Given that there must be exactly 0.5cm of clearance above and below the Ms, find the cost of the paper he used to print 1000 of tiles to 3 significant figures. You should use the two following facts in this question:

- his supplier sells him the paper at 0.23p per centimetre squared
- like x, h(x) is also measured in centimetres

a) $t = \tan(2x)$

$$\frac{dh}{dx} = 6\cos(4x) - 16\sin(8x)$$

$$= 6\cos(4x) - 32\sin(4x)\cos(4x)$$

$$= 2\cos(4x)\left(3 - 16\sin(4x)\right)$$

$$= 2\left(\frac{1-t^2}{1+t^2}\right)\left(3 - 16\left(\frac{2t}{1+t^2}\right)\right)$$

$$= \frac{2(1-t^2)}{(1+t^2)}\left(\frac{3(1+t^2)}{1+t^2} - \frac{32t}{1+t^2}\right)$$

$$= \frac{2(1-t^2)(3t^2-32t+3)}{(1+t^2)^2}$$

b) $\frac{dh}{dx} = 0 \quad \Rightarrow \quad (1-t^2)(3t^2-32t+3) = 0$

$t = \frac{16 \pm \sqrt{247}}{3}, \pm 1$

$t = 1 \quad : \quad 2x = \pi/4, 5\pi/4, \ldots$

$x = \pi/8, \frac{5\pi}{8}, \ldots$

$t = -1 \quad : \quad 2x = -\pi/4, 3\pi/4, 7\pi/4, \ldots$

$x = -\pi/8, 3\pi/8, 7\pi/8$

$t = \frac{16+\sqrt{247}}{3} \quad : \quad 2x = 1.4764\ldots, 1.4764+\pi, \ldots$

$x = 0.7382\ldots, 2.309\ldots, \ldots$

$t = \frac{16-\sqrt{247}}{3} \quad 2x = 0.0943\ldots, 0.0943+\pi, \ldots$

$x = 0.04715, 1.618, \ldots$

$x_A = \frac{3\pi}{8}$ as it is the fourth stationary point

one wavelength $= \frac{7\pi}{8} - \frac{3\pi}{8} = \frac{\pi}{2}$

$\therefore 15 \times \frac{\pi}{2} + \frac{3\pi}{8} = \frac{63\pi}{8}$

c) $f(0.047154\ldots) = 1.5\sin(4\times 0.047\ldots) + 2\cos(8\times 0.047\ldots)+4$

$= 6.14\ldots$

$f(\frac{317}{8}) = \ldots = 0.5$

$\therefore$ Vertical length paper $= 6.14\ldots + 0.5$

$= 6.641$ to 3dp.

Area of 1 tile $= 6.641\times \frac{63}{8}\pi = 164.284\ldots cm^2$

Area of 1000 tiles $= 164289.46\ldots \ cm^2$

Cost of 1000 tiles $= £378$ to 3sf.