London Academy of Maths

# Method of Differences

Question Paper

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**1.** (a) Show that, for $r > 0$

$$\frac{r+2}{r(r+1)} - \frac{r+3}{(r+1)(r+2)} = \frac{r+4}{r(r+1)(r+2)}$$

**(2)**

(b) Hence show that

$$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \frac{n(an+b)}{c(n+1)(n+2)}$$

where $a$, $b$ and $c$ are integers to be determined.

**(4)**



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**Question 1 continued**

**2.** Given that

$$\frac{2n+1}{n^2(n+1)^2} \equiv \frac{A}{n^2} + \frac{B}{(n+1)^2}$$

(a) determine the value of $A$ and the value of $B$

**(1)**

(b) Hence show that, for $n \geqslant 5$

$$\sum_{r=5}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n^2+an+b}{c(n+1)^2}$$

where $a$, $b$ and $c$ are integers to be determined.

**(4)**

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**Question 2 continued**

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**3.** (a) Express

$$\frac{1}{(n+3)(n+5)}$$

in partial fractions.

**(2)**

(b) Hence, using the method of differences, show that for all positive integer values of $n$,

$$\sum_{r=1}^{n} \frac{1}{(r+3)(r+5)} = \frac{n(pn+q)}{40(n+4)(n+5)}$$

where $p$ and $q$ are integers to be determined.

**(4)**

(c) Use the answer to part (b) to determine, as a simplified fraction, the value of

$$\frac{1}{9\times11} + \frac{1}{10\times12} + \ldots + \frac{1}{24\times26}$$

**(2)**

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**Question 3 continued**

**4.** **In this question you must show all stages of your working.**

**Solutions relying on calculator technology are not acceptable.**

(a) Show that, for $r \geqslant 2$

$$\frac{2}{\sqrt{r} + \sqrt{r-2}} = \sqrt{r} - \sqrt{r-2}$$

**(2)**

(b) Hence use the method of differences to determine

$$\sum_{r=2}^{n} \frac{2}{\sqrt{r} + \sqrt{r-2}}$$

giving your answer in simplest form.

**(3)**

(c) Hence show that

$$\sum_{r=4}^{50} \frac{2}{\sqrt{r} + \sqrt{r-2}} = A + B\sqrt{2} + C\sqrt{3}$$

where $A$, $B$ and $C$ are integers to be determined.

**(2)**

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**Question 4 continued**

**5.** Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{2}{(r+4)(r+6)} = \frac{n(an+b)}{30(n+5)(n+6)}$$

where $a$ and $b$ are integers to be determined.

**(6)**

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**Question 5 continued**

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**6.** (a) Express $\dfrac{2}{r(r^2-1)}$ in partial fractions.

**(3)**

(b) Hence find, in terms of $n$,

$$\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$$

Give your answer in the form

$$\frac{n^2 + An + B}{Cn(n+1)}$$

where $A$, $B$ and $C$ are constants to be found.

**(5)**

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**Question 6 continued**

**7.** (a) Express

$$\frac{1}{(2n-1)(2n+1)(2n+3)}$$

in partial fractions.

**(2)**

(b) Hence, using the method of differences, show that for all integer values of $n$,

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)(2r+3)} = \frac{n(n+2)}{a(2n+b)(2n+c)}$$

where $a$, $b$ and $c$ are integers to be determined.

**(4)**

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**Question 7 continued**

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**8.** (a) Write $\dfrac{3r+1}{r(r-1)(r+1)}$ in partial fractions.

**(2)**

(b) Hence find

$$\sum_{r=2}^{n} \frac{3r+1}{r(r-1)(r+1)} \qquad n \geqslant 2$$

giving your answer in the form

$$\frac{an^2+bn+c}{2n(n+1)}$$

where $a$, $b$ and $c$ are integers to be determined.

**(5)**

(c) Hence determine the exact value of

$$\sum_{r=15}^{20} \frac{3r+1}{r(r-1)(r+1)}$$

**(2)**

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**Question 8 continued**

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**9.** Prove that, for $n \in \mathbb{Z}$, $n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where $a$, $b$ and $c$ are integers to be found.

**(5)**

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**Question 9 continued**

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**10.** (a) Use the method of differences to prove that for $n > 2$

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right)$$

**(4)**

(b) Hence find the exact value of

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$$

Give your answer in the form $a\ln\left(\frac{b}{c}\right)$ where $a$, $b$ and $c$ are integers to be determined.

**(3)**

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**Question 10 continued**

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**11.** (a) Show that

$$n^5 - (n-1)^5 \equiv 5n^4 - 10n^3 + 10n^2 - 5n + 1$$

**(2)**

(b) Hence, using the method of differences, show that for all integer values of $n$,

$$\sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1)(2n+1)(an^2 + bn + c)$$

where $a$, $b$ and $c$ are integers to be determined.

**(7)**

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**Question 11 continued**

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**12.** Given that $A > B > 0$, by letting $x = \arctan A$ and $y = \arctan B$

(a) prove that

$$\arctan A - \arctan B = \arctan\left(\frac{A - B}{1 + AB}\right)$$

**(3)**

(b) Show that when $A = r + 2$ and $B = r$

$$\frac{A - B}{1 + AB} = \frac{2}{(1 + r)^2}$$

**(2)**

(c) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \arctan\left(\frac{2}{(1 + r)^2}\right) = \arctan(n + p) + \arctan(n + q) - \arctan 2 - \frac{\pi}{4}$$

where $p$ and $q$ are integers to be determined.

**(4)**

(d) Hence, making your reasoning clear, determine

$$\sum_{r=1}^{\infty} \arctan\left(\frac{2}{(1 + r)^2}\right)$$

giving the answer in the form $k\pi - \arctan 2$, where $k$ is a constant.

**(2)**

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**Question 12 continued**