London Academy of Maths

Complex Numbers: Roots of Unity and De Moivre's Theorem

Mark Scheme

WFM02 Further Pure Mathematics F2 Mark Scheme

Question Number	Scheme	Notes	Marks
1	$z^5 - 32\text{i} = 0 \Rightarrow r^5 = 32 \Rightarrow r = 2$	Correct value for r. May be shown explicitly or used correctly.	B1
	$5\theta = \frac{\pi}{2} + 2n\pi \Rightarrow \theta = \frac{\pi}{10} + \frac{2n\pi}{5}$	Applies a correct strategy for establishing at least 2 values of θ. This can be awarded if if the initial angle $\left(\frac{\pi}{2} \text{ or } \frac{\pi}{10}\right)$ is incorrect but otherwise their strategy is correct.	M1
	$z = 2\text{e}^{\text{i}\frac{\pi}{10}}, 2\text{e}^{\text{i}\frac{\pi}{2}}, 2\text{e}^{\text{i}\frac{9\pi}{10}}, 2\text{e}^{\text{i}\frac{13\pi}{10}}, 2\text{e}^{\text{i}\frac{17\pi}{10}}$ or	At least 2 correct, follow through their r	A1ft
	$z = 2\text{e}^{\left(\frac{\pi}{10}+\frac{2n\pi}{5}\right)\text{i}}, \quad n = 0,1,2,3,4$	All correct. Must have $r = 2$	A1
			(4)
			Total 4

Question Number	Scheme	Marks
2		
(a)	$\lvert 18\sqrt{3}-18\mathrm{i}\rvert = 18\sqrt{(3+1)} = 36$	B1
	$\tan\theta = \dfrac{-18}{18\sqrt{3}}$ $\quad \theta = -\dfrac{\pi}{6}$, $\quad 18\sqrt{3}-18\mathrm{i} = 36\left(\cos\left(-\dfrac{\pi}{6}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{6}\right)\right)$	M1,A1cao (3)
(b)	$z^4 = 36\left(\cos-\dfrac{\pi}{6}+\mathrm{i}\sin-\dfrac{\pi}{6}\right) = 36\left(\cos\left(2k\pi-\dfrac{\pi}{6}\right)+\mathrm{i}\sin\left(2k\pi-\dfrac{\pi}{6}\right)\right)$	M1
	$z = \sqrt{6}\left(\cos\left(\dfrac{12k\pi-\pi}{24}\right)+\mathrm{i}\sin\left(\dfrac{12k\pi-\pi}{24}\right)\right)$	M1
	$k=0 \quad z_0 = \sqrt{6}\left(\cos\left(\dfrac{-\pi}{24}\right)+\mathrm{i}\sin\left(\dfrac{-\pi}{24}\right)\right) = \sqrt{6}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{24}\right)}$	B1
	$k=1 \quad z_1 = \sqrt{6}\left(\cos\left(\dfrac{11\pi}{24}\right)+\mathrm{i}\sin\left(\dfrac{11\pi}{24}\right)\right) = \sqrt{6}\mathrm{e}^{\mathrm{i}\frac{11\pi}{24}}$	A1ft
	$k=2 \quad z_2 = \sqrt{6}\left(\cos\left(\dfrac{23\pi}{24}\right)+\mathrm{i}\sin\left(\dfrac{23\pi}{24}\right)\right) = \sqrt{6}\mathrm{e}^{\mathrm{i}\frac{23\pi}{24}}$	
	$k=-1 \quad z_3 = \sqrt{6}\left(\cos\left(-\dfrac{13\pi}{24}\right)+\mathrm{i}\sin\left(-\dfrac{13\pi}{24}\right)\right) = \sqrt{6}\mathrm{e}^{\mathrm{i}\left(-\frac{13\pi}{24}\right)}$	A1ft (5)
		[8]
(a)		
B1	Correct modulus	
M1	Attempt argument using $\tan\theta = \dfrac{\pm 18}{18\sqrt{3}}$ or other valid method. Can be implied by $\theta = \pm\dfrac{\pi}{6}$	
A1cao	Correct answer in the required form.	
(b)		
M1	Valid method for generating at least 2 roots, rotation through $\dfrac{\pi}{2}$ accepted	
M1	Apply de Moivre or use the rotation method	
B1	Any one correct root	
A1ft	Second root in required form	
A1ft	All 4 roots in the required form Follow through their $\sqrt[4]{36}$ but 36 not acceptable.	
NB	Argument in degrees – M1M1B1A0A0 (ie treat as mis-read) Incorrect argument: B0A1ftA1ft available Answers in $r(\cos\theta+\mathrm{i}\sin\theta)$ form – deduct final A marks	

Question	Scheme	Marks
3(a)	$r=\sqrt{(-4)^2+\left(-4\sqrt{3}\right)^2}=...$	M1
	$\tan\theta=\frac{-4\sqrt{3}}{-4}\Rightarrow\theta=\tan^{-1}\left(\sqrt{3}\right)\pm\pi$	M1
	$8\left(\cos\left(-\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right)$	A1
		(3)
(b)	$z=r\mathrm{e}^{\mathrm{i}\theta}\Rightarrow\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^3=-4-4\sqrt{3}\Rightarrow r^3\left(\mathrm{e}^{3\mathrm{i}\theta}\right)=8\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$	
	$\Rightarrow r=\sqrt[3]{8}=2$	M1
	$3\theta=-\frac{2\pi}{3}(+2k\pi)\Rightarrow\theta=-\frac{2\pi}{9}+\left(\frac{2k\pi}{3}\right)$	M1
	So $z=2\mathrm{e}^{-\frac{8\pi}{9}\mathrm{i}},2\mathrm{e}^{-\frac{2\pi}{9}\mathrm{i}},2\mathrm{e}^{\frac{4\pi}{9}\mathrm{i}}$	A1ft A1
		(4)
		(7 marks)

Notes:

(a)
M1: For a correct attempt at the modulus, implied by a correct modulus if no method seen and allow recovery if correct answer follows a minor slip in notation.

M1: For an attempt to find a value of θ in the correct quadrant. Accept $\tan^{-1}\left(\sqrt{3}\right)\pm\pi$ or $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\pm\pi$

May be implied by sight of an of $-\frac{2}{3}\pi,\frac{4}{3}\pi,-\frac{5}{6}\pi,\frac{7}{6}\pi$.

A1: cao as in scheme, no other solution.

(b)

M1: Applies De Moivre's Theorem and proceeds to find a value for r ie $\left(\text{their } 8\right)^{\frac{1}{3}}$

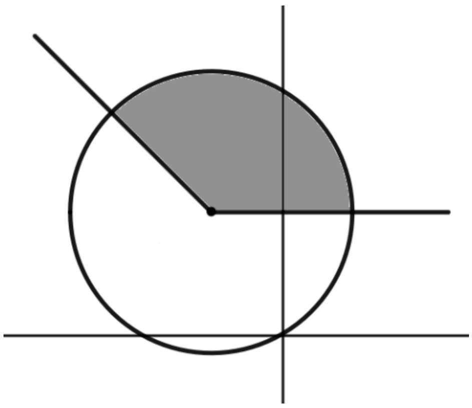
M1: Proceeds to find at least one value for θ – ie their argument/3.

A1ft: At least two roots correct for their r and θ. (Must come from correct method, watch for correct roots coming from an incorrect angle due to errors.)

A1: All three correct roots and no others. Accept e.g $2\mathrm{e}^{\mathrm{i}-\frac{8\pi}{9}}$ as a slip in notation, so allow marks.

Question Number	Scheme	Notes	Marks
4(a)	**(i)** $z = 6 - 6\sqrt{3}\mathrm{i} \Rightarrow \lvert z \rvert = \sqrt{6^2 + \left(6\sqrt{3}\right)^2} = 12$	+12 only. Accept if just stated	**B1**
	(ii) e.g., $\arg z = -\arctan\frac{6\sqrt{3}}{6}$ Attempts an expression for a relevant angle. Look for $\pm\arctan\left(\pm\frac{6\sqrt{3}}{6}\right)$ or e.g., $\pm\tan^{-1}\left(\pm\frac{1}{\sqrt{3}}\right)$ If arctan is not seen allow e.g., $\tan\alpha = \frac{6\sqrt{3}}{6} \Rightarrow \alpha = \frac{\pi}{3}$ with α correct for their $\tan\alpha$ If using sin or cos the hypotenuse must be their 12		**M1**
	$\arg z$ or arg or argument (of z) $= -\frac{\pi}{3}$ * A correct proof with no incorrect work/statements. LHS required. **Allow** "$\theta =$" **if consistent**, e.g., $\theta = -\frac{\pi}{3}$ cannot follow "$\tan\theta = +\sqrt{3}$"		**A1***
(ii) Way 2	$z = 12\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) = 12\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ or $12\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}$ or $\cos\theta = \frac{1}{2}$ or $\sin\theta = -\frac{\sqrt{3}}{2}$ [M1] $\Rightarrow \arg z = -\frac{\pi}{3}$ [A1*] M1: Factorises out 12 and writes in trig or exp form or identifies $\cos\theta = \frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$ A1: Acceptable statement with all work correct		
(ii) Way 3	$z = 12\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ or $12\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}$ or $12\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) = 6 - 6\sqrt{3}\mathrm{i}$ [M1] $\Rightarrow \arg z = -\frac{\pi}{3}$ [A1*] M1: Assumes result, writes correctly for their 12 and attempts $a + \mathrm{i}b$ form A1: Obtains $6 - 6\sqrt{3}\mathrm{i}$ and makes acceptable statement with all work correct		
			(3)
(b)	$z = \text{"}12\text{"}\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ or $\text{"}12\text{"}e^{-\frac{\pi}{3}\mathrm{i}}$ [no missing "i" unless recovered] Correct trig or exp. form with their 12. Could be implied by their z^4 in trig or exp. form e.g., $(\text{"}12\text{"}e^{-\frac{\pi}{3}\mathrm{i}})^4$ Allow equivalent values of θ e.g. $\frac{5\pi}{3}$ and use of e.g., $\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$. Condone poor bracketing. Allow this mark if $+2k\pi, -2k\pi, \pm 2k\pi$ appears with argument		**M1**
	$z^4 = 20736\left(\cos\left(-\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{3}\right)\right)$ or $20736\left(\cos -\frac{4\pi}{3} + \mathrm{i}\sin -\frac{4\pi}{3}\right)$ or $20736e^{-\frac{4\pi}{3}\mathrm{i}}$ Correct z^4 in any form. 12^4 evaluated and arg. of $-\frac{4\pi}{3}$ (not just $4\times -\frac{\pi}{3}$) or $\frac{2\pi}{3}$ only although may use e.g., $\sin\left(-\frac{4\pi}{3}\right) = -\sin\left(\frac{4\pi}{3}\right)$. No "$k$"s. Condone an "unclosed" bracket. Only accept $-10368 + 10368\sqrt{3}\mathrm{i}$ or $20736\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)$ **provided evidence of de Moivre.**		**A1**
			(2)

Question Number	Scheme	Notes	Marks
4(c)	$w = z^{\frac{1}{2}} = (\pm)\sqrt{\text{"}12\text{"}}\left(\cos\left(\frac{-\frac{\pi}{3}}{2}\right) + \text{i}\sin\left(\frac{-\frac{\pi}{3}}{2}\right)\right)$ or e.g., $(\pm)\text{"}2\sqrt{3}\text{"}e^{-\frac{\pi}{6}\text{i}}$ [no missing "i" unless recovered] Correct use of de Moivre's theorem with $-\frac{\pi}{3}$ and their 12 to attempt one square root. Allow work with argument of $\frac{5\pi}{3}$ for $-\frac{\pi}{3}$ and use of e.g., $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$. Condone poor bracketing. M0 if z^4 used for z. Allow this mark if $+2k\pi, -2k\pi, \pm 2k\pi$ appears with argument		**M1**
	$w = 3-\sqrt{3}\text{i}, -3+\sqrt{3}\text{i}$ oe A1ft: One correct exact root in $a + \text{i}b$ or $c(a + \text{i}b)$ form (a, b, c may be unsimplified but not numerical trig expressions) **ft their 12 only** i.e. $(\pm)\sqrt{\text{"}12\text{"}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}\right)$ A1: Both exact roots (no others) correct in $a + \text{i}b$ form – a and b may be unsimplified (but not numerical trig expressions) e.g. accept $a = (\pm)\sqrt{12}\frac{\sqrt{3}}{2}, (\pm)\frac{\sqrt{36}}{2} \quad b = (\mp)\frac{\sqrt{12}}{2}, (\mp)\frac{2\sqrt{3}}{2}$ Accept $\pm\left(3-\sqrt{3}\text{i}\right)$ but just $\pm 3-\sqrt{3}\text{i}$ is A1 A0. Just $\pm\sqrt{3}\left(\sqrt{3}-\text{i}\right)$ is A1 A0		**A1ft** **A1**
	Note: $w^2 = r^2(\cos 2\theta + i\sin 2\theta) = z \Rightarrow r, \theta, w = ...$ is an acceptable approach		**(3)**
Alt	$w^2 = z \Rightarrow (a+\text{i}b)^2 = a^2 - b^2 + 2ab\text{i} = 6 - 6\sqrt{3}\text{i} \Rightarrow a^2 - b^2 = 6,\ 2ab = -6\sqrt{3}$ $b = -\frac{3\sqrt{3}}{a} \Rightarrow a^2 - \frac{27}{a^2} = 6 \Rightarrow a^4 - 6a^2 - 27 = (a^2-9)(a^2+3) = 0 \Rightarrow a^2 = 9,\ a = \pm 3,\ b = \mp\sqrt{3}$ **M1**: From a correct starting point, expands and equates real and imaginary parts to form two equations in a and b and obtains at least one value for both a and b $w = 3-\sqrt{3}\text{i}, -3+\sqrt{3}\text{i}$ **A1**: One correct exact root in $a + \text{i}b$ or $c(a + \text{i}b)$ form (a, b, c may be unsimplified) **A1**: Both exact roots (no others) correct in $a + \text{i}b$ form – a and b may be unsimplified		
			Total 8

Question Number	Scheme	Marks
5(a)	$\left(\cos\frac{5\pi}{12}+\mathrm{i}\sin\frac{5\pi}{12}\right)^4=\cos\frac{20\pi}{12}+\mathrm{i}\sin\frac{20\pi}{12}$ or/and $\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)^3=\cos(-\pi)+\mathrm{i}\sin(-\pi)$	M1 / A1
	$(z_1=)\dfrac{\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}}{\cos(-\pi)+\mathrm{i}\sin(-\pi)}=\cos\left(\frac{5\pi}{3}-(-\pi)\right)+\mathrm{i}\sin\left(\frac{5\pi}{3}-(-\pi)\right)$ Alt : $(z_1=)\dfrac{\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}}{-1}=-\cos\frac{5\pi}{3}-\mathrm{i}\sin\frac{5\pi}{3}$	M1
	$=\cos\frac{8\pi}{3}+\mathrm{i}\sin\frac{8\pi}{3}\quad=\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$ * Alt: if denominator −1 used via e.g. $\cos\left(\frac{5\pi}{3}-\pi\right)+\mathrm{i}\sin\left(-\frac{5\pi}{3}\right)$	A1*
		(4)
(b)	$\lvert z-z_1\rvert\leqslant 1\qquad 0\leqslant\arg(z-z_1)\leqslant\frac{3\pi}{4}$	
	 A circle in any position (may just see the minor arc)	M1
	A pair of **half**–lines in correct directions from their centre, one with negative gradient and one parallel to (but not) the x–axis. *If full lines are used the M marks can be implied by their shading*	M1
	Area shaded inside their circle between the two half lines from from the parallel one anticlockwise to the negative gradient line.	M1
	Fully correct shaded sector. See notes.	A1
		(4)
(c)	$\arctan\left(\dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)=\ldots\qquad\frac{\pi}{3}\quad(\text{or }60^\circ)$	M1 A1
		(2)
		Total 10

Notes

(a)

M1: One correct use of de Moivre in polar (or exponential) form. Allow use of $\mathrm{e}^{\mathrm{i}\theta}$ for $\cos\theta+\mathrm{i}\sin\theta$ throughout until the final A.

A1: Both correct (unsimplified) in polar form. Accept for both marks use of $\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)^3\to\cos\pi-\mathrm{i}\sin\pi$ but denominator directly to $\cos\pi+\mathrm{i}\sin\pi$ with no evidence of dealing with the negative between terms is A0.

M1: A correct method shown for the division of the two complex numbers. No need to simplify for this mark, a difference of argument in the trig terms is fine. Look for the subtraction of the arguments. Sight of the $\frac{8\pi}{3}$ can imply the mark if no incorrect work is seen. Note it is M0 if the arguments are added (so

$\cos\left(\frac{5\pi}{3}-\pi\right)+\mathrm{i}\sin\left(\frac{5\pi}{3}-\pi\right)$ is M0 unless the denominator has clearly been written as $\cos\pi+\mathrm{i}\sin\pi$ first.)

May write the denominator as −1 first, which is correct, score for $-\cos\frac{5\pi}{3}-\mathrm{i}\sin\frac{5\pi}{3}$.

Accept methods that convert both numbers into exact Cartesian form, apply a correct process to realise the denominator and convert back to polar form.

Do not allow going straight to $\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$ for this mark as it is a given answer. Justification is required and an incorrect method is M0.

A1cso: Obtains $\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$ with suitable intermediate step shown and no errors in the work. Must have scored the preceding 3 marks. This will usually be via $\cos\frac{8\pi}{3}+\mathrm{i}\sin\frac{8\pi}{3}$ unless equivalent suitable working has been shown to justify the correct modulus (e.g. proceeding via −1 in the denominator).

(b)

Do not be concerned about lines being dashed or dotted in this part. Accept either.

M1: A circle in any position. You may see just the minor arc of a circle, which is acceptable as long as it is clearly an arc of a circle (e.g. implied by their shading) and not just a angle demarcation.

M1: Draws or indicates a pair of half–lines from their centre (which need not be in quadrant 2) with one with negative gradient proceeding up and left and one parallel to the *x*–axis proceeding right but not the *x*-axis itself. If full lines are used this can be implied by the shading of the correct region between lines.

M1: Shades the area between their pair of rays (the second ray may have positive gradient for this mark) and inside their circle, anticlockwise from the horizontal line. The half line need not stem from the centre of the circle for this mark, and accept the *x*-axis as the horizontal line for this mark.

A1: Fully correct shaded sector. Must

- be in quadrants 1 and 2
- have approximately correct gradients for the half-lines (−1 and 0)
- have circle with centre in quadrant 2 and (if whole circle shown) passing roughly through the origin. If only an arc is shown apply bod as long as the position is reasonable.

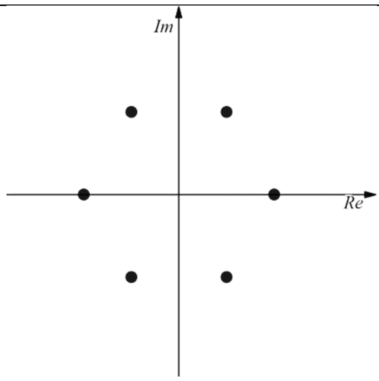
Ignore any centre coordinates and axes intersections of any major sector. If extra regions are shaded they must make clear which their region *R* is in order to access the mark.

(c)

M1: Identifies the correct point for their sector, which must be from a circle with centre in quadrant 1 or 2 (above the real axis) and ray parallel to the real axis, and attempts the relevant angle. Look for selecting the point (*c*, *d*) at the “3 o’clock” position having identified a suitable sector and proceeding to find a relevant positive angle (e.g., allow if $\pi-\theta$ is found) (accept $\arctan\pm\frac{c}{d}$ or $\arctan\pm\frac{d}{c}$ as an attempt at the angle).

Their point must be in quadrants 1 or 2. If no shading was shown in (b) allow for attempts at the relevant point of horizontal ray and circle intersection. Could use other trig.

A1 : Either correct value. Mark final answer.

Question	Scheme	Marks	AOs
6(a)	$z = \mathrm{e}^{\frac{k\pi}{3}\mathrm{i}}, \quad k = 0, 1, 2, 3, 4, 5$	M1 A1	1.1b 1.1b
		(2)	
(b)		B1 dB1	2.2a 1.1b
		(2)	
(c)	e.g. $\left(\sqrt{3}+\mathrm{i}\right)^6 = \left(2\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}\right)^6 = 64\mathrm{e}^{\mathrm{i}\pi} = -64*$ **or** $\left[2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right]^6 = 2^6(\cos\pi+\mathrm{i}\sin\pi) = 64(-1) = -64*$ **or** $\left(\sqrt{3}+\mathrm{i}\right)^6 = \left(\sqrt{3}\right)^6 + 6\left(\sqrt{3}\right)^5\mathrm{i} - 15\left(\sqrt{3}\right)^4 - 20\left(\sqrt{3}\right)^3\mathrm{i} + 15\left(\sqrt{3}\right)^2 + 6\sqrt{3}\mathrm{i} + \mathrm{i}^6$ $= 27 - 135 + 45 - 1 = -64*$ **or** $\left(\sqrt{3}+\mathrm{i}\right)^6 = = 27 + 54\sqrt{3}\mathrm{i} + 135\mathrm{i}^2 + 60\sqrt{3}\mathrm{i}^3 + 45\mathrm{i}^4 + 6\sqrt{3}\mathrm{i}^5 + \mathrm{i}^6$ $= 27 + 54\sqrt{3}\mathrm{i} - 135 - 60\sqrt{3}\mathrm{i} + 45 + 6\sqrt{3}\mathrm{i} - 1 = -64*$	M1 A1*	1.1b 2.1
		(2)	
(d)	$r = 2$	B1	2.2a
	$z = 2\mathrm{e}^{\frac{\pi}{6}\mathrm{i}} \times \mathrm{e}^{\frac{k\pi}{3}\mathrm{i}}, \quad k = 0, 1, 2, 3, 4, 5$	M1	3.1a
	$z = 2\mathrm{e}^{\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)\mathrm{i}}, \quad k = 0, 1, 2, 3, 4, 5$	A1	1.1b
		(3)	
			(9 marks)

Notes

(a)

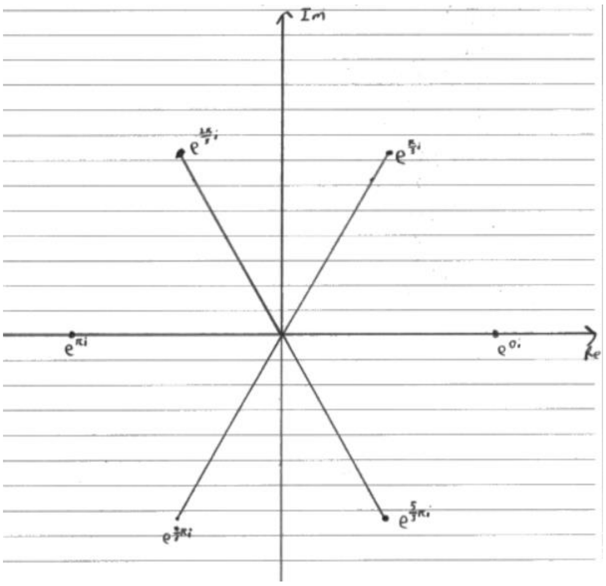
M1: For sight of $\mathrm{e}^{\frac{k\pi}{3}\mathrm{i}}$
Accept any value for k

A1: All six roots fully defined as shown or listed separately with their values of θ within the given range with no incorrect or extra values. Ensure i and π are present in each term.

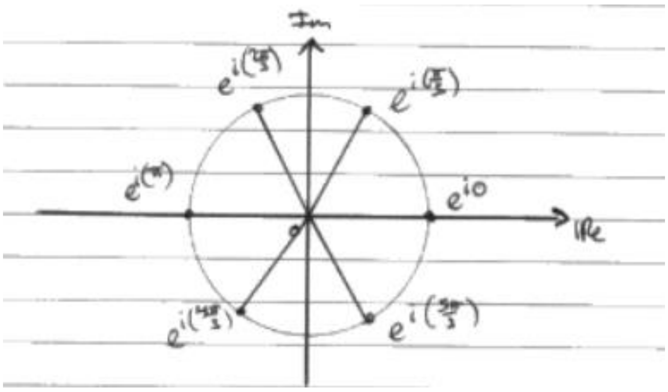
Note: Roots if listed are $\mathrm{e}^0, \mathrm{e}^{\frac{\pi}{3}\mathrm{i}}, \mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}, \mathrm{e}^{\pi\mathrm{i}}, \mathrm{e}^{\frac{4\pi}{3}\mathrm{i}}, \mathrm{e}^{\frac{5\pi}{3}\mathrm{i}}$, condone 1 for e^0 and/or -1 for $\mathrm{e}^{\pi\mathrm{i}}$

(b)
B1: Plots 6 points that form a hexagon, with a point on the positive real axis and a point on the negative real axis, and one point in each quadrant.
Do not be concerned about the position of each point from the centre, however the sketch must convey a hexagon.
dB1: The points form a hexagon, centre the origin (see diagram), axes need not be labelled.
Look for the axes acting as lines of symmetry.
(Drawing line/vectors to each point is acceptable but not necessary for either mark)

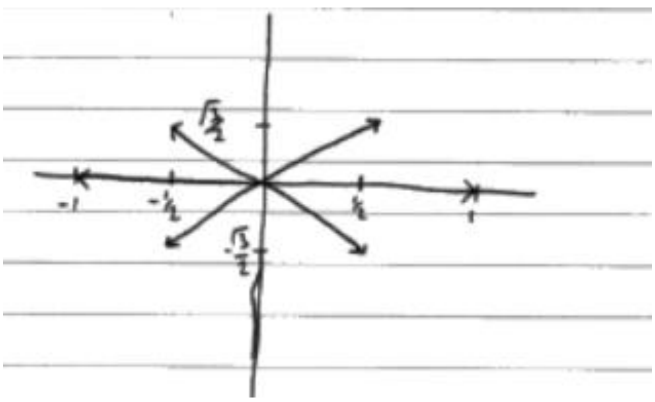
Examples



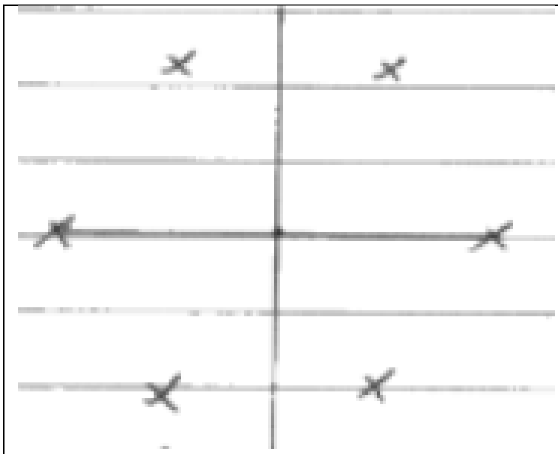
B1dB1



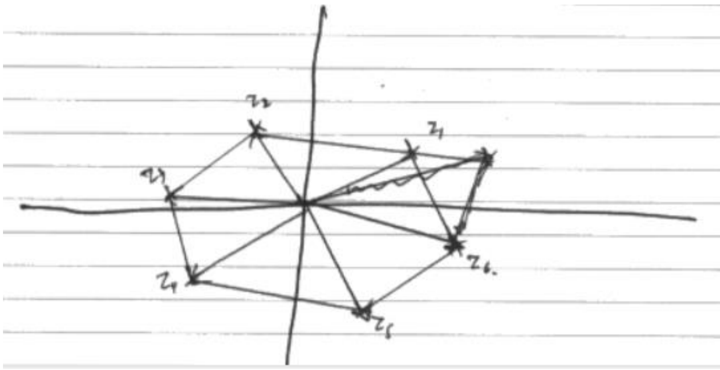
B1dB1



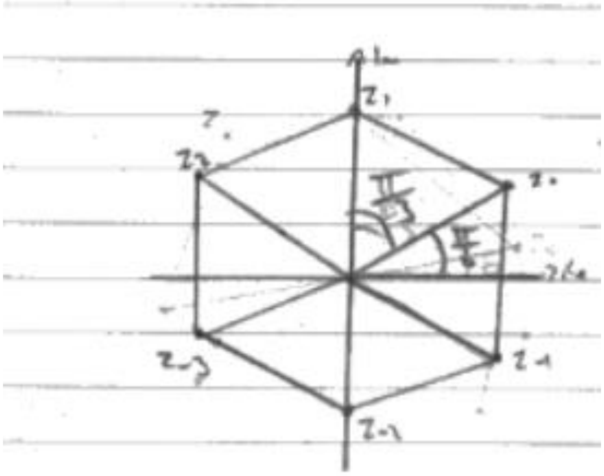
B1dB1



B1dB1



B0dB0



B0dB0

(c)

M1: Converts $\sqrt{3}+\mathrm{i}$ to polar form to obtain $r\mathrm{e}^{\mathrm{i}\theta}$ with at least $r = 2$ or $\theta = \dfrac{\pi}{6}$ and applies the power of 6 correctly to obtain $r^6\mathrm{e}^{6\theta\mathrm{i}}$

A1*: Obtains the given answer with sufficient working shown.

As a minimum need to see $2^6\mathrm{e}^{\frac{6\pi i}{6}} = -64$ or $2^6\mathrm{e}^{\pi\mathrm{i}} = -64$

If r = −2 is seen in their workings withhold this mark.

OR

M1: Converts $\sqrt{3}+\mathrm{i}$ to modulus-argument form $r(\cos\theta + \mathrm{i}\sin\theta)$ with at least $r = 2$ or $\theta = \dfrac{\pi}{6}$ and applies the power of 6 correctly to obtain $r^6(\cos 6\theta + \mathrm{i}\sin 6\theta)$

A1*: Obtains the given answer with sufficient working shown.

OR

M1: Attempts to expand $\left(\sqrt{3}+\text{i}\right)^6$ fully using an attempt at the binomial expansion. Must have 7 terms for $(a+b)^n$ and correct binomial coefficients with $a = \sqrt{3}$, b = i and n = 6

A1*: Obtains the given answer with at least one intermediate line.

OR

M1: Attempts the full expansion of $\left(\sqrt{3}+\text{i}\right)^6 = (\sqrt{3}+\text{i})(\sqrt{3}+\text{i})(\sqrt{3}+\text{i})\ \ldots.\ (\sqrt{3}+\text{i}) =$

There must be no brackets, no irrational numbers and no terms in i in their simplified answer.

A1*: Obtains the given answer with sufficient working shown including correct full expansion, with at least one intermediate line.

(d)

B1: Deduces r = 2 (only)

M1: Obtains at least one value of z in the form $r\text{e}^{\text{i}\theta}$ with their consistent value of r, and θ taking one of

$$\left\{\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{7\pi}{6},\frac{3\pi}{2},\frac{11\pi}{6}\right\}$$

A1: For $2\text{e}^{\frac{\pi}{6}\text{i}}, 2\text{e}^{\frac{\pi}{2}\text{i}}, 2\text{e}^{\frac{5\pi}{6}\text{i}}, 2\text{e}^{\frac{7\pi}{6}\text{i}}, 2\text{e}^{\frac{3\pi}{2}\text{i}}, 2\text{e}^{\frac{11\pi}{6}\text{i}}$ with no incorrect or extra values. Accept unsimplified arguments such as having a solution of $2\text{e}^{\frac{9\pi}{6}\text{i}}$. Ensure i and π are present in each term.

Accept $2\text{e}^{\frac{\pi}{2}\text{i}}$ as 2i and $2\text{e}^{\frac{3\pi}{2}\text{i}}$ as -2i

Question	Scheme	Marks	AOs
7(a)	e.g. $\lvert z_1\rvert = \sqrt{(-4)^2+4^2}$ or $\arg z_1 = \pi - \frac{\pi}{4}$ oe	M1	1.1b
	$(z_1 =)4\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$ or e.g. $(z_1 =)\sqrt{32}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$	A1	1.1b
		(2)	
(b)(i)	$\frac{z_1}{z_2} = \frac{"4\sqrt{2}"}{3}\left(\cos\left("\frac{3\pi}{4}"-\frac{17\pi}{12}\right)+\mathrm{i}\sin\left("\frac{3\pi}{4}"-\frac{17\pi}{12}\right)\right) = \ldots$ **or** $\frac{z_1}{z_2} = \frac{"4\sqrt{2}"\mathrm{e}^{"\frac{3\pi}{4}\mathrm{i}"}}{3\mathrm{e}^{\frac{17\pi}{12}\mathrm{i}}} = \frac{"4\sqrt{2}"}{3}\mathrm{e}^{\left("\frac{3\pi}{4}"-\frac{17\pi}{12}\right)\mathrm{i}}$ **or** $\frac{z_1}{z_2} = \frac{-4+4\mathrm{i}}{3\left(\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)-\mathrm{i}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)} \times \frac{\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)+\mathrm{i}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)}{\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)+\mathrm{i}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)} = \ldots$	M1	3.1a
	$= -\frac{2\sqrt{2}}{3}-\frac{2\sqrt{6}}{3}\mathrm{i}$ or $-\frac{2\sqrt{2}}{3}-\mathrm{i}\frac{2\sqrt{6}}{3}$ or $-\frac{2\sqrt{2}}{3}+\mathrm{i}\left(-\frac{2\sqrt{6}}{3}\right)$	A1	1.1b
		(2)	

Notes

(a) <u>Correct answer with no working scores both marks in (a)</u>

M1: Any correct expression for $\lvert z_1\rvert$ or $\arg z_1$ e.g. $\lvert z_1\rvert = \sqrt{(-4)^2+4^2}$ or $\arg z_1 = \pi - \frac{\pi}{4}$

A1: Correct <u>expression</u>. The "$z_1 =$" is not required.

This mark is not for correct modulus and correct argument it is for the complex number written in the required form. Condone the missing closing bracket e.g. $(z_1 =)\sqrt{32}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right.$

(b)(i) <u>Correct answer with no working scores no marks in (b)(i)</u>

M1: Employs a correct method to find the quotient. E.g.

- uses modulus argument form and divides moduli and subtracts arguments the right way round
- uses exponential form and divides moduli and subtracts arguments the right way round
- converts z_2 to Cartesian form and multiplies numerator and denominator by the complex conjugate of the denominator. Allow if the "3" is missing for this method. Allow with decimals for this method e.g. $\frac{z_1}{z_2} = \frac{-4+4\mathrm{i}}{-0.258\ldots-0.965\ldots\mathrm{i}} \times \frac{-0.258\ldots+0.965\ldots\mathrm{i}}{-0.258\ldots+0.965\ldots\mathrm{i}} = \ldots$

<u>If they convert z_2 to Cartesian form it must be correct as shown or correct decimals.</u>

A1: Correct exact answer in the required form.

Do not allow e.g. $-\frac{2}{3}\left(\sqrt{2}+\sqrt{6}\mathrm{i}\right)$ or $\frac{-2\sqrt{2}-2\sqrt{6}\mathrm{i}}{3}$ unless a correct form is seen previously then apply isw.

Provided a correct method is shown as above, allow to go from the forms in the main scheme to the correct exact answer with no intermediate step.

(ii)	$z_2^4 = 3^4\left(\cos\left(4\times\frac{17\pi}{12}\right)+\mathrm{i}\sin\left(4\times\frac{17\pi}{12}\right)\right)$ **or** $(z_2)^4 = \left(3\mathrm{e}^{\frac{17\pi}{12}\mathrm{i}}\right)^4 = 3^4\mathrm{e}^{\frac{17\pi}{12}\times 4\mathrm{i}}$ **or** $z_2^4 = \left\{3\left(\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)-\mathrm{i}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)\right\}^4 = \ldots$	**M1**	1.1b
	$= \frac{81}{2}-\frac{81\sqrt{3}}{2}\mathrm{i}$ or $\frac{81}{2}-\mathrm{i}\frac{81\sqrt{3}}{2}$ or $\frac{81}{2}+\mathrm{i}\left(-\frac{81\sqrt{3}}{2}\right)$	**A1**	1.1b
		(2)	

(b)(ii) Correct answer with no working scores no marks in (b)(ii)

M1: Applies De Moivre's theorem correctly to z_2. E.g. uses polar form or exponential form and calculates the modulus as 3^4 and the argument as $4\times\frac{17\pi}{12}$

For attempts at $z_2^4 = \left\{3\left(\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)-\mathrm{i}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)\right\}^4$ you would need to see:

- the correct exact form used
- a clear and convincing attempt to expand the brackets e.g. by using a full binomial expansion or a complete attempt to multiply all 4 brackets together but you are not expected to check every detail
- a final answer in the required form with no obvious errors seen

So $z_2^4 = \left\{3\left(\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)-\mathrm{i}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)\right\}^4 = \frac{81}{2}-\frac{81\sqrt{3}}{2}\mathrm{i}$ scores no marks.

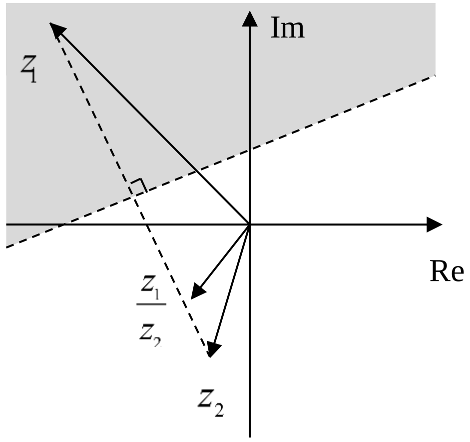
Similar guidance applies if they attempt to expand $\left\{3\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right)\right\}^4$

A1: Correct exact answer in the required form.

Do not allow e.g. $\frac{81}{2}\left(1-\frac{81\sqrt{3}}{2}i\right)$ or $\frac{81-81\sqrt{3}\mathrm{i}}{2}$ unless a correct form is seen previously then apply isw.

Provided a correct method is shown as above, allow to go from the forms in the main scheme to the correct exact answer with no intermediate step.

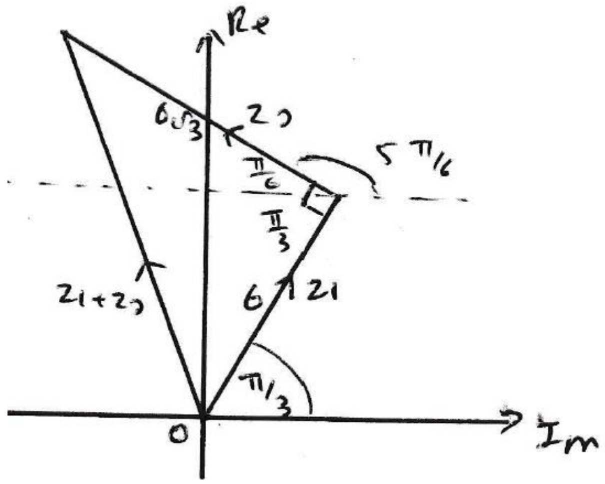
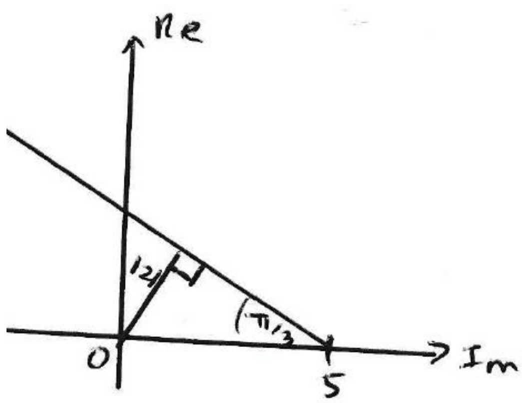
(c)(i) and (ii)



Notes:

(c)(i) **B1:** z_1 and z_2 correctly positioned. Look for <u>correct quadrants</u> with z_1 approximately on $y = -x$ and z_2 below $y = x$ closer to the origin than z_1. Note that the points are usually labelled but mark positively if it is clear which points are which if there is no labelling.	**B1**	1.1b
B1ft: $\frac{z_1}{z_2}$ in the correct quadrant. Follow through their answer to (b)(i). Note that the point is usually labelled but mark positively if it is clear which point it is. It is sometimes labelled as z_3 which is fine.	**B1ft**	1.1b
(ii) **M1:** Draws a line (solid or dashed) that is the perpendicular bisector of z_1z_2 **or** draws a line that crosses z_1z_2 and shades one of the sides of this line.	**M1**	3.1a
A1: A line drawn (solid or dashed) that is the perpendicular bisector of z_1z_2 **<u>with either side shaded</u>** as long as it is clear they are not discounting the upper region. The B1 in part (i) may not have been scored but z_1 must be in quadrant 2 and z_2 in quadrant 3. **Note that some candidates are drawing the region on a separate diagram and this is acceptable. You do not need to see a line joining z_1 to z_2.**	**A1**	1.1b
	(4)	
(10 marks)		

Question	Scheme	Marks	AOs
8(i)	$z_1 = 6\left[cos\left(\frac{\pi}{3}\right) + i\,sin\left(\frac{\pi}{3}\right)\right] = \ldots\{3 + 3\sqrt{3}i\}$ $z_2 = 6\sqrt{3}\left[cos\left(\frac{5\pi}{6}\right) + i\,sin\left(\frac{5\pi}{6}\right)\right] = \ldots\{-9 + 3\sqrt{3}i\}$ $\{z_1 + z_2 =\}(3 + 3\sqrt{3}i) + (-9 + 3\sqrt{3}i) = \ldots\{-6 + 6\sqrt{3}i\}$ Or $\{z_1 + z_2 =\}6\left[cos\left(\frac{\pi}{3}\right) + i\,sin\left(\frac{\pi}{3}\right)\right] + 6\sqrt{3}\left[cos\left(\frac{5\pi}{6}\right) + i\,sin\left(\frac{5\pi}{6}\right)\right] = a + bi$ where a and b are constants, the trig function must be evaluated	M1	3.1a
	Clearly show the method to find modulus **and** argument for $z_1 + z_2$ $arg(z_1 + z_2) = \pi - tan^{-1}\left(\frac{6\sqrt{3}}{6}\right)$ or $tan^{-1}\left(\frac{6\sqrt{3}}{-6}\right) = \ldots\left\{\frac{2\pi}{3}\right\}$ **and** $\lvert z_1 + z_2\rvert = \sqrt{6^2 + (6\sqrt{3})^2} = \ldots\{12\}$ **Alternative 1** $-6 + 6\sqrt{3}i = 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $= 12\left(cos\left(\frac{2\pi}{3}\right) + i\,sin\left(\frac{2\pi}{3}\right)\right)$ **Alternative 2** $12e^{\frac{2\pi}{3}i} = 12\left(cos\frac{2\pi}{3} + i\,sin\frac{2\pi}{3}\right)$ $= \ldots\{-6 + 6\sqrt{3}i\}$	dM1	2.1
	$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$ * $12e^{\frac{2\pi}{3}i} = -6 + 6\sqrt{3}i$ Therefore $z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$*	A1*	1.1b
		(3)	
	Alternative 3 $z_1 + z_2 = 6\boldsymbol{e}^{\frac{\pi}{3}\boldsymbol{i}} + 6\sqrt{3}\boldsymbol{e}^{\frac{5\pi}{6}\boldsymbol{i}}$ $= 12\left[\frac{1}{2}\boldsymbol{cos}\left(\frac{\pi}{3}\right) + \frac{1}{2}\boldsymbol{i\,sin}\left(\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\boldsymbol{cos}\left(\frac{5\pi}{6}\right) + \frac{\sqrt{3}}{2}\boldsymbol{i\,sin}\left(\frac{5\pi}{6}\right)\right]$	M1	3.1a
	$12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 12\left(cos\left(\frac{2\pi}{3}\right) + i\,sin\left(\frac{2\pi}{3}\right)\right)$	dM1	2.1
	$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$*	A1*	1.1b
		(3)	
	Alternative 4 $z_1 + z_2 = 6e^{\frac{\pi}{3}i} + 6\sqrt{3}e^{\frac{5\pi}{6}i} = 6e^{\frac{\pi}{3}i}\left(1 + \sqrt{3}e^{\frac{\pi}{2}i}\right) = 6e^{\frac{\pi}{3}i}(1 + \sqrt{3}i)$	M1	
	Either $r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ **and** $arg = arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$	dM1	

	Or $6e^{\frac{\pi}{3}i}(1+\sqrt{3}i) = 12e^{\frac{\pi}{3}i}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i()e^{\frac{\pi}{3}i}\left(cos\left(\frac{\pi}{3}\right)+i\,sin\left(\frac{\pi}{3}\right)\right)\right)$		
	$z_1 + z_2 = 12e^{\frac{\pi}{3}i}e^{\frac{\pi}{3}i} = 12e^{\frac{2\pi}{3}i}*$	A1*	
		(3)	
	Alternative 5 Uses geometry to show that $z_1,\ z_2$ and $z_1 + z_2$ form a right-angled triangle 	M1	3.1a
	$arg(z_1 + z_2) = \frac{\pi}{3} + tan^{-1}\left(\frac{6\sqrt{3}}{6}\right) = \ldots\left\{\frac{2\pi}{3}\right\}$ $\lvert z_1 + z_2\rvert = \sqrt{(6)^2 + (6\sqrt{3})^2} = \ldots\{12\}$	dM1	1.1b
	$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}*$	A1*	1.1b
		(3)	
(ii)		M1	3.1a
	$sin\left(\frac{\pi}{3}\right) = \frac{\lvert z\rvert}{5} \Rightarrow \lvert z\rvert = \ldots$	M1	1.1b
	$\lvert z\rvert = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	

	Alternative 1 Gradient = $-\tan\left(\frac{\pi}{3}\right)$ $c = 5\tan\left(\frac{\pi}{3}\right)$ leading to $y = -\sqrt{3}x + 5\sqrt{3}$ or $\tan\left(\frac{\pi}{3}\right) = \frac{y}{5-x}$ $\lvert z\rvert^2 = x^2 + y^2 = x^2 + (-\sqrt{3}x + 5\sqrt{3})^2 = 4x^2 - 30x + 75$ $\frac{d\lvert z\rvert^2}{dx} = 8x - 30 = 0 \Rightarrow x = \ldots\{3.75\}$ or $\lvert z\rvert^2 = 4(x - 3.75)^2 + 18.75 \Rightarrow x = \ldots\ldots\{3.75\}$	M1	3.1a
	$\lvert z\rvert = \sqrt{4(\text{their}3.75)^2 - 30(\text{their}3.75) + 75}$	M1	1.1b
	$\lvert z\rvert = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	
	Alternative 2 Gradient = $-\tan\left(\frac{\pi}{3}\right)$ $c = 5\tan\left(\frac{\pi}{3}\right)$ leading to $y = -\sqrt{3}x + 5\sqrt{3}$ Perpendicular line through the origin $y = \frac{1}{\sqrt{3}}x$ and find the point of intersection of the two lines $\left(\frac{15}{4}, \frac{5\sqrt{3}}{4}\right)$	M1	3.1a
	Finds the distance from the origin to their point of intersection $\lvert z\rvert = \sqrt{\left(\text{their } \frac{15}{4}\right)^2 + \left(\text{their } \frac{5\sqrt{3}}{4}\right)^2} = \ldots$	M1	1.1b
	$\lvert z\rvert = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	
			(6 marks)

Notes:

(i)
M1: A complete method to find both z_1 and z_2 in the form $a + bi$ and adds them together.
dM1: Dependent on previous method mark, finds the modulus and argument of $z_1 + z_2$. They must show their method, just stating modulus = 12 and argument = $\frac{2\pi}{3}$ is not sufficient as this is a show question.
Alternative 1: Factorises out 12 and find the argument

Alternative 2: uses $12e^{\frac{2\pi}{3}i} = 12\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \ldots$
A1*: Achieves the correct answer following no errors or omissions.
Alternatively shows that $12e^{\frac{2\pi}{3}i} = -6 + 6\sqrt{3}i$ and concludes therefore $z_1 + z_2 = 12e^{\frac{2\pi}{3}i}*$

Alternative 3

M1: Factorises out 12 and writes in the form $12\left[\ldots \boldsymbol{cos}\left(\frac{\pi}{3}\right)+\ldots \boldsymbol{i\,sin}\left(\frac{\pi}{3}\right)+\ldots \boldsymbol{cos}\left(\frac{5\pi}{6}\right)+\ldots \boldsymbol{i\,sin}\left(\frac{5\pi}{6}\right)\right]$ **dM1:** Dependent on previous mark**.** Writes in the form $12(a+bi)$ leading to the form $12(cos\,\theta + i\,sin\,\theta)$ **A1*:** Achieves the correct answer following no errors or omissions**.** **<u>Alternative 4</u>** **M1:** Factorises out 6 and writes in the form $6e^{\frac{\pi}{3}i}\left(1+\sqrt{3}e^{\frac{\pi}{2}i}\right)=6e^{\frac{\pi}{3}i}(1+ai)$ **dM1:** Dependent on previous method mark, finds the modulus and argument of $(1+ai)$ or $12(a+bi)$ leading to the form $12(cos\,\theta+i\,sin\,\theta)$ **A1*:** Achieves the correct answer following no errors or omissions**.** **<u>Alternative 5</u>** **M1:** Draws a diagram to show that z_1, z_2 and z_1+z_2 form a right-angled triangle. **dM1:** Dependent on previous method mark, finds the modulus and argument of z_1+z_2 **A1*:** Achieves the correct answer following no errors or omissions**.** **Note:** Writing $arg(z_1+z_2)=arctan\left(\frac{6\sqrt{3}}{-6}\right)=-\frac{\pi}{3}$ therefore $arg(z_1+z_2)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$ with no diagram or finding z_1+z_2 is **M0dM0A0**
(ii) **M1**: Draws a diagram and recognises that the shortest distance will form a right-angled triangle. **M1:** Uses trigonometry to find the shortest length. **A1:** Correct exact value. **<u>Alternative 1</u>** **M1**: Finds the equation of the half-line by attempting $m=-tan\left(\frac{\pi}{3}\right)$ $c=5\,tan\left(\frac{\pi}{3}\right)$. Finds x^2+y^2 in terms of *x*, differentiates, sets = 0 and finds the value of *x*. **M1:** Uses their value of *x* to find the minimum value of $\sqrt{x^2+y^2}$ **A1:** Correct exact value. **<u>Alternative 2</u>** **M1**: Finds the equation of the half-line by attempting $m=-tan\left(\frac{\pi}{3}\right)$ $c=5\,tan\left(\frac{\pi}{3}\right)$. Finds the equation of the line perpendicular which passes through the origin. Finds the point of intersection of the lines **M1:** Finds the distance from the origin to their point of intersection **A1:** Correct exact value.

Question	Scheme	Marks	AOs
9(a)	$z^n + \frac{1}{z^n} \equiv e^{in\theta} + \frac{1}{e^{in\theta}} \equiv e^{in\theta} + e^{-in\theta}$	M1	1.1b
	$\equiv \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \equiv 2\cos n\theta$ *	A1*	2.1
		(2)	
(b)	$\left(z + z^{-1}\right)^5 = 32\cos^5\theta$	B1	2.2a
	$\left(z + z^{-1}\right)^5 = z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$	M1 A1	1.1b 1.1b
	$32\cos^5\theta = \left(z^5 + z^{-5}\right) + 5\left(z^3 + z^{-3}\right) + 10\left(z + z^{-1}\right)$ $= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$	M1	2.1
	$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$ *	A1*	1.1b
		(5)	
(c)	$\cos 5\theta + 5\cos 3\theta + 10\cos\theta = -2\cos\theta \Rightarrow 16\cos^5\theta = -2\cos\theta$	B1	3.1a
	$2\cos\theta\left(8\cos^4\theta + 1\right) = 0 \Rightarrow \theta = \ldots$	M1	1.1b
	$8\cos^4\theta + 1 = 0$ has no solution so $\cos\theta = 0$ $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$	A1	2.2a
		(3)	
			(10 marks)

Notes

(a)
M1: Substitutes z into the LHS and simplifies the powers as shown. Allow if they go direct to trigonometric expressions without exponentials. The mark is for sorting out the negative index.
A1*: Converts the exponential form to trigonometric form correctly and correctly completes the proof with no errors seen. The trigonometric expansion must be clearly seen. Condone missing brackets in e.g $\cos -n\theta$ terms if intent is clear. Note the LHS of the identity may be implied.
(b)
B1: Deduces that $\left(z + z^{-1}\right)^5 = 32\cos^5\theta$ Do not accept 2^5 for 32. May be implied.
M1: Attempts to expand $\left(z + z^{-1}\right)^5$. Correct binomial coefficients must be used, terms need not be simplified. Condone at most one slip in powers.
A1: Correct expansion, terms need not be gathered but powers must have been simplified.
M1: Sets their expressions equal and applies the result from (a) – grouping must be shown.
A1*: Reaches the printed answer with no errors and relevant steps all shown.
(c)
B1: Uses the result from (b) to deduce the correct equation.
M1: Must have attempted to use part (b) to obtain $\alpha\cos^5\theta = \beta\cos\theta$ or equivalent. Collects to one side and attempts to factorise and solve. Note dividing through by $\cos\theta$ is M0.
A1: Rejects the inappropriate solution and selects $\cos\theta = 0$ and obtains the correct values only. The equation must have been correct. There must have been some consideration of the $8\cos^4\theta + 1$ e.g. stating $8\cos^4\theta > 0$ so no solutions, or attempting to find complex roots and deducing no answers. May be minimal, but some consideration that no roots arise from this part must have been given.
Note: The correct answer will appear from incorrect attempts – the M must be gained in order to award the A. E.g. assuming the equation reduces to $\cos^5\theta = 0$ will score B0M0A0.
Likewise, answers only scores B0M0A0 (questions says hence so use of (b) must be seen).

Alt (a)	$z^n + \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta} = \frac{\cos^2 n\theta + 2\mathrm{i}\cos n\theta \sin n\theta - \sin^2 n\theta + 1}{\cos n\theta + \mathrm{i}\sin\theta}$	M1	1.1b
	$= \frac{2\cos^2 n\theta + 2\mathrm{i}\cos n\theta\sin n\theta}{\cos n\theta + \mathrm{i}\sin n\theta} = \frac{2\cos n\theta(\cos n\theta + \mathrm{i}\sin n\theta)}{\cos n\theta + \mathrm{i}\sin n\theta} = 2\cos n\theta$ *	A1*	2.1
		(2)	

Notes

Alt: by De Moivre
M1: Applies De Moivre on both terms and puts over a common denominator.
A1*: Complete correctly, using $1 - \sin^2 n\theta = \cos^2 n\theta$ and cancelling $\cos n\theta + \mathrm{i}\sin n\theta$. No errors seen.

Alt (b)	$\cos 5\theta = \mathrm{Re}(\cos 5\theta + \mathrm{i}\sin 5\theta) = \mathrm{Re}(\cos\theta + \mathrm{i}\sin\theta)^5$	B1	2.2a
	$(\cos\theta + \mathrm{i}\sin\theta)^5 = c^5 + 5\mathrm{i}c^4s + 10\mathrm{i}^2c^3s^2 + 10\mathrm{i}^3c^2s^3 + 5\mathrm{i}^4cs^4 + \mathrm{i}^5s^5$	M1	1.1b
	$\mathrm{Re}(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$	A1	1.1b
	$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$ $= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ $= 16\cos^5\theta - \frac{20}{4}(\cos 3\theta + 3\cos\theta) + 5\cos\theta$	M1	2.1
	$\Rightarrow \cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$*	A1*	1.1b
		(5)	

Notes

Alt: by De Moivre
B1: Correctly stated or clearly implied De Moivre statement for $\cos 5\theta$
M1: Attempts to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$ Correct coefficients but allow one slip per main scheme.
The powers of i need not be simplified for the attempt at expansion, accept if only the real terms are shown. Allow *c* and *s* notation.
A1: Correct real terms extracted with the i's removed.
M1: Applies $\sin^2\theta = 1 - \cos^2\theta$ to reduce to an equation in $\cos\theta$ **and** applies $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$ (quoted or derived – allow a slip if derived) to get to an equation without powers of cos terms.
A1*: Reaches the printed answer with no errors and relevant steps all shown.

Question	Scheme	Marks	AOs
10(a)	$\left(z+\frac{1}{z}\right)^6 = 64\cos^6\theta$	B1	2.1
	$\left(z+\frac{1}{z}\right)^6 = z^6 + 6(z^5)\left(\frac{1}{z}\right) + 15(z^4)\left(\frac{1}{z^2}\right) + 20(z^3)\left(\frac{1}{z^3}\right) + 15(z^2)\left(\frac{1}{z^4}\right) + 6(z)\left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$	M1	2.1
	$= \left[z^6 + \frac{1}{z^6}\right] + 6\left[z^4 + \frac{1}{z^4}\right] + 15\left[z^2 + \frac{1}{z^2}\right] + 20$	A1	1.1b
	Uses $z^n + \frac{1}{z^n} = 2\cos n\theta$ $\{64\cos^6\theta\} = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$	M1	2.1
	$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ * cso	A1 *	1.1b
		(5)	
(b)	$\boldsymbol{H = 2}$	B1	3.3
		(1)	
(c)	$\text{vol} = \left\{\frac{1}{2}\right\}\pi\int\left(2\cos^3\left(\frac{x}{4}\right)\right)^2 \text{d}x$	B1ft	3.4
	$\text{vol} = \{2\pi\}\int\cos^6\left(\frac{x}{4}\right)\text{d}x$ $= \{2\pi\}\int\frac{1}{32}\left(\cos\left(\frac{6x}{4}\right) + 6\cos\left(\frac{4x}{4}\right) + 15\cos\left(\frac{2x}{4}\right) + 10\right)\text{d}x = \ldots$	M1	1.1b
	$= \{2\pi\}\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3x}{2}\right) + 6\sin(x) + 30\sin\left(\frac{x}{2}\right) + 10x\right)\right]$	A1	1.1b
	$= 2\times 2\pi\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times 4\right) + 6\sin(4) + 30\sin\left(\frac{4}{2}\right) + (10\times 4)\right) - 0\right] = \ldots$ or $=$ $2\pi\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times 4\right) + 6\sin(4) + 30\sin\left(\frac{4}{2}\right) + (10\times 4)\right) - \frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times -4\right) + 6\sin(-4) + 30\sin\left(-\frac{4}{2}\right) + (10\times -4)\right)\right]$ …	dM1	3.4
	= 24.56	A1	1.1b
		(5)	

(d)	The equation of the curve may not be suitable The measurements may not be accurate The paperweight may not be smooth	B1	3.5b
		(1)	
			(12 marks)

Notes:

(a)
B1: Correct identity or equivalent rearrangement. This can appear anywhere in the proof.
M1: Attempts the expansion of $\left(z+\frac{1}{z}\right)^6$ must have at least 3 correct terms. Combining the powers when expanding is fine.
A1: Correct expansion with z terms simplified, need not be rearranged. (So a correct expansion will score M1A1.)
M1: Uses $z^n + \frac{1}{z^n} = 2\cos n\,\theta$ to write the expression in terms multiple angles of $\cos 6\,\theta,\ \cos 4\,\theta$ and $\cos 2\,\theta$. Pairing of terms must be seen.
A1*: Achieves the printed answer with no errors or omissions. Cso
For approaches using De Moivre B0M1A1M0A0 may be scored if the binomial expansions is attempted (and correct for the A).

(b)
B1: See scheme

(c)
Note: The question instructs use of algebraic integration and part (a), so answer only can score at most B1 for implied correct formula.
B1ft: Correct expression for the volume of the paperweight **or** the solid formed through 360º rotation, stated or implied, ignore limits. No need to expand, but must be applied, not just a formula in *y,* though allow a correct formula followed by correct integral if the π disappears. Follow through their *H*
M1: Uses the result in part part (a) to express the volume in an integrable form and attempts to integrate. Note use of θ instead of x is permissible for this mark. Allow if one term is missing or miscopied.
A1: Correct integration in terms of x. Ignore π, their H^2 and the $\frac{1}{32}$. Note if θ has been used it is A0 unless a correct substitution method has been implied as the coefficients will be incorrect.
dM1: Dependent on previous method mark and must have reached and integral of the correct form -- in terms of x with correct arguments allowing for one slip. Finds the required volume using either$\pi\int_0^4 y^2\,\mathrm{d}x$ or $\frac{1}{2}\pi\int_{-4}^4 y^2\,\mathrm{d}x$ and applies their limits - accept any value following a valid attempt at the integration as an attempt at applying limits.
A1: cao 24.56

(d)
B1: States an appropriate limitation. See scheme for some examples. The limitation should refer to the paperweight, not to paper. Do not accept “it does not take into account thickness of material” as it is a solid, not a shell, being modelled. Award the mark for a correct reason if two reasons are given and one is incorrect.

Question	Scheme	Marks	AOs
11(a)	$(\cos\theta+\mathrm{i}\sin\theta)^7=\cos^7\theta+\binom{7}{1}\cos^6\theta(\mathrm{i}\sin\theta)+\binom{7}{2}\cos^5\theta(\mathrm{i}\sin\theta)^2+...$ Some simplification may be done at this stage e.g. $c^7+7c^6\mathrm{i}s-21c^5s^2-35c^4\mathrm{i}s^3+35c^3s^4+21c^2\mathrm{i}s^5-7cs^6-\mathrm{i}s^7$	M1	1.1b
	$\mathrm{i}\sin 7\theta={}^7\mathrm{C}_1c^6\mathrm{i}s+{}^7\mathrm{C}_3c^4\mathrm{i}^3s^3+{}^7\mathrm{C}_5c^2\mathrm{i}^5s^5+\mathrm{i}^7s^7$ or $=7c^6\mathrm{i}s+35c^4\mathrm{i}^3s^3+21c^2\mathrm{i}^5s^5+\mathrm{i}^7s^7$	M1	2.1
	$\sin 7\theta=7c^6s-35c^4s^3+21c^2s^5-s^7$	A1	1.1b
	$=7(1-s^2)^3s-35(1-s^2)^2s^3+21(1-s^2)s^5-s^7$ $=7(1-3s^2+3s^4-s^6)s-35(1-2s^2+s^4)s^3+21(1-s^2)s^5-s^7$	M1	2.1
	$\{7s-21s^3+21s^5-7s^7-35s^3+70s^5-35s^7+21s^5-21s^7-s^7\}$ leading to $\sin 7\theta=7\sin\theta-56\sin^3\theta+112\sin^5\theta-64\sin^7\theta$ *	A1*	1.1b
		(5)	
(b)	$1+\sin 7\theta=0\Rightarrow\sin 7\theta=-1$	M1	3.1a
	$7\theta=-450,-90,\ 270,\ 630,\ ...$ or $7\theta=-\frac{5\pi}{2},-\frac{\pi}{2},\frac{3\pi}{2},\frac{7\pi}{2},...$	A1	1.1b
	$\theta=-\frac{450}{7},-\frac{90}{7},\frac{270}{7},\frac{630}{7},..\Rightarrow\sin\theta=...$ or $\theta=-\frac{5\pi}{14},-\frac{\pi}{14},\frac{3\pi}{14},\frac{7\pi}{14},...\Rightarrow\sin\theta=...$	M1	2.2a
	$x=\sin\theta=-0.901,\ -0.223,\ 0.623,\ 1$	A1 A1	1.1b 2.3
		(5)	
			(10 marks)

Notes

(a)
M1: Attempts to expand $(\cos\theta+\mathrm{i}\sin\theta)^7$ including a recognisable attempt at binomial coefficients
Some simplification may be done at this stage. (May only see imaginary terms)
M1: Identifies imaginary terms with sin 7θ
A1: Correct expression with coefficients evaluated and i's dealt with correctly
M1: Replaces $\cos^2\theta$ with $1-\sin^2\theta$ and applies the expansions of $(1-\sin^2\theta)^2$ and $(1-\sin^2\theta)^3$ to their expression
A1*: Reaches the printed answer with no errors and expansion of brackets seen.
(b)
M1: Makes the connection with part (a) and realises the need to solve $\sin 7\theta=-1$
A1: At least one correct value for 7θ
M1: Divides by 7 and deduces that x values are found by finding at least one value for $\sin\theta$
A1: Awrt 2 correct values for x
A1: Awrt all 4 x values correct and no extras

Question Number	Scheme	Notes	Marks
12(a)	Condone use of e.g. C + iS for $\cos x+\text{i}\sin x$ if the intention is clear.		
	$(\cos 5x \equiv)\text{Re}(\cos x+\text{i}\sin x)^5 \equiv \cos^5 x+\binom{5}{2}\cos^3 x(\text{i}\sin x)^2+\binom{5}{4}\cos x(\text{i}\sin x)^4$ Identifies the correct terms of the binomial expansion of $(\cos x+\text{i}\sin x)^5$ They may expand $(\cos x+\text{i}\sin x)^5$ completely but there must be an attempt to extract the real terms which must have the correct binomial coefficients combined with the correct powers of sin x and cos x. Condone use of a different variable e.g. θ.		M1
	$(\cos 5x \equiv)\cos^5 x-10\cos^3 x\sin^2 x+5\cos x\sin^4 x$ Correct simplified expression. Condone use of a different variable e.g. θ.		A1
	$\equiv \cos x\left(\cos^4 x-10\cos^2 x\sin^2 x+5\sin^4 x\right)$ $\equiv \cos x\left(\left(1-\sin^2 x\right)^2-10\left(1-\sin^2 x\right)\sin^2 x+5\sin^4 x\right)$ Applies $\cos^2 x=1-\sin^2 x$ to obtain an expression in terms of sin x inside the bracket. Condone use of a different variable e.g. θ.		M1
	$\equiv \cos x\left(16\sin^4 x-12\sin^2 x+1\right)$	Correct expression. Must be in terms of x now. The "cos5x =" is not required.	A1
			(4)
(b)	Allow use of a different variable in (b) e.g. x for <u>all</u> marks.		
	$\cos 5\theta=\sin 2\theta\sin\theta-\cos\theta$ $\Rightarrow \cos\theta\left(16\sin^4\theta-12\sin^2\theta+1\right)=2\sin^2\theta\cos\theta-\cos\theta$ $\Rightarrow \cos\theta\left(16\sin^4\theta-14\sin^2\theta+2\right)=0$ Uses the result from part (a) with $\sin 2\theta=2\sin\theta\cos\theta$ and collects terms		M1
	$16\sin^4\theta-14\sin^2\theta+2=0$ $\Rightarrow \sin^2\theta=\frac{7\pm\sqrt{17}}{16}\Rightarrow \sin\theta=...$ Solves for $\sin^2\theta$ by any method including calculator and takes square root to obtain at least one value for $\sin\theta$. Depends on the first mark. May be implied by their values of $\sin\theta$ or θ. NB $\frac{7\pm\sqrt{17}}{16}=0.69519..., 0.17980...$		dM1
	$\sin\theta=\sqrt{\frac{7\pm\sqrt{17}}{16}}\Rightarrow\theta=...$ NB $\sqrt{\frac{7\pm\sqrt{17}}{16}}=0.833783...,\ \ 0.424035...$ A full method to reach at least one value for θ. Depends on the previous mark. May be implied by their values of θ		ddM1
	$(\theta=)0.986,\ 0.438$	Correct values and no others in range. Allow awrt these values.	A1
			(4)
			Total 8

Note that it is possible to do 7(b) by changing to cos *θ* e.g.

$$\cos\theta\left(16\sin^4\theta-12\sin^2\theta+1\right)=\cos\theta\left(16\left(1-\cos^2\theta\right)^2-12\left(1-\cos^2\theta\right)+1\right)$$

$$\cos\theta\left(16\left(1-\cos^2\theta\right)^2-12\left(1-\cos^2\theta\right)+1\right)=2\sin^2\theta\cos\theta-\cos\theta$$

$$16\cos^4\theta-18\cos^2\theta+4=0$$

$$\cos^2\theta=\frac{9\pm\sqrt{17}}{16}\Rightarrow\cos\theta=\sqrt{\frac{9\pm\sqrt{17}}{16}}$$

$$(\theta=)0.986,\ 0.438$$

This is acceptable as they used part (a) and can be scored as:

M1: Uses part (a) with $\sin^2\theta=1-\cos^2\theta$ and $\sin 2\theta=2\sin\theta\cos\theta$ and collects terms.

dM1: Solves for $\cos^2\theta$ by any method including calculator and takes square root to obtain at least one value for $\cos\theta$. Depends on the first mark. May be implied by their values of $\cos\theta$ or θ.

NB $\frac{9\pm\sqrt{17}}{16}=0.82019...,\ 0.30480...$

dM1: A full method to reach at least one value for θ.
Depends on the previous mark. May be implied by their values of θ

NB $\sqrt{\frac{9\pm\sqrt{17}}{16}}=0.905645...,\ \ 0.552092...$

A1: $(\theta=)0.986,\ 0.438$

Question	Scheme	Marks	AOs
13(a)	$\alpha = \frac{z_1+z_2}{2} = \frac{35-25\text{i}-29+39\text{i}}{2} = \ldots$ $\alpha = z_1 + \frac{1}{2}\overrightarrow{z_1z_2} = 35-25\text{i}+\frac{1}{2}\left(-64+64\text{i}\right) = \ldots$ $\alpha = z_2 + \frac{1}{2}\overrightarrow{z_2z_1} = -29+39+\frac{1}{2}\left(64-64\text{i}\right) = \ldots$	**M1**	1.1b
	$= 3+7\text{i}*$	**A1***	1.1b
		(2)	
(b)	$\beta\left(z_1-\alpha\right) = \left(\frac{1+\text{i}}{64}\right)\left(35-25\text{i}-(3+7\text{i})\right) = \left(\frac{1+\text{i}}{64}\right)\left(32-32\text{i}\right) =$ $= \frac{1}{64}\left(32-32\text{i}+32\text{i}-32\text{i}^2\right) = \frac{1}{64}\left(32-32\text{i}+32\text{i}+32\right)$	**M1**	1.1b
	$= \frac{1}{64}\left(64\right) = 1*$	**A1***	1.1b
		(2)	
(c)(i)	Roots are $\left\{\text{e}^0\left(\text{or } 1 \text{ or } \text{e}^{\text{i}2\pi}\right)\right\}, \text{e}^{\text{i}\frac{\pi}{3}}, \text{e}^{\text{i}\frac{2\pi}{3}}, \text{e}^{\text{i}\pi}, \text{e}^{\text{i}\frac{4\pi}{3}}, \text{e}^{\text{i}\frac{5\pi}{3}}$ or $\text{e}^{\text{i}\frac{k\pi}{3}}, \quad k = 0,1,2,3,4,5$ $\left\{\text{e}^0\left(\text{or } 1 \text{ or } \text{e}^{\text{i}2\pi}\right)\right\}, \text{e}^{\text{i}\frac{\pi}{3}}, \text{e}^{\text{i}\frac{2\pi}{3}}, \text{e}^{\text{i}\pi}, \text{e}^{-\text{i}\frac{\pi}{3}}, \text{e}^{-\text{i}\frac{2\pi}{3}}$ or $\text{e}^{\text{i}\frac{k\pi}{3}}, \quad k = -2,-1,0,1,2,3,$	**B1**	1.1b
		(1)	
(c)(ii)	$w = \beta(z-\alpha) = \text{e}^{\text{i}\frac{k\pi}{3}} \Rightarrow z = \frac{\text{e}^{\text{i}\frac{k\pi}{3}}}{\beta} + \alpha$	**M1**	3.1a
	$\Rightarrow z = \frac{64\left(\cos\frac{k\pi}{3} + \text{i}\sin\frac{k\pi}{3}\right)(1-\text{i})}{(1+\text{i})(1-\text{i})} + 3+7\text{i} = \ldots$	**M1**	1.1b
	Two of $\left(19+16\sqrt{3}\right)+\left(-9+16\sqrt{3}\right)\text{i}$ $\quad \left(-13-16\sqrt{3}\right)+\left(23-16\sqrt{3}\right)\text{i}$ $\left(-13+16\sqrt{3}\right)+\left(23+16\sqrt{3}\right)\text{i}$ $\quad \left(19-16\sqrt{3}\right)-\left(9+16\sqrt{3}\right)\text{i}$ Or four correct decimal answers 46.7 + 18.7i – 40.7 – 4.7i 14.7 + 50.7i – 8.7 – 36.7i	**A1**	2.5

	All four of $\left(19+16\sqrt{3}\right)+\left(-9+16\sqrt{3}\right)\text{i}$ $\left(-13+16\sqrt{3}\right)+\left(23+16\sqrt{3}\right)\text{i}$ $\left(-13-16\sqrt{3}\right)+\left(23-16\sqrt{3}\right)\text{i}$ $\left(19-16\sqrt{3}\right)-\left(9+16\sqrt{3}\right)\text{i}$	**A1**	2.2a
		(4)	
	Alternative 1 $\left(\beta(z-\alpha)\right)^6=1^6 \Rightarrow (z-\alpha)^6=\frac{1}{\beta^6}=8589934459\text{i}$ $r=\sqrt[6]{858993459}=32\sqrt{2}$ or 45.25... and $\theta=\frac{\pi}{12}+\frac{k\pi}{3}$ or $\theta=-\frac{\pi}{4}+\frac{k\pi}{3}$	**M1**	3.1a
	$z=r\left(\cos\theta-\text{i}\sin\theta\right)+3+7\text{i}=...$	**M1**	1.1b
	Two of $\left(19+16\sqrt{3}\right)+\left(-9+16\sqrt{3}\right)\text{i}$ $\quad\left(-13-16\sqrt{3}\right)+\left(23-16\sqrt{3}\right)\text{i}$ $\left(-13+16\sqrt{3}\right)+\left(23+16\sqrt{3}\right)\text{i}$ $\quad\left(19-16\sqrt{3}\right)-\left(9+16\sqrt{3}\right)\text{i}$ Or four correct decimal answers 46.7 + 18.7i – 40.7 – 4.7i 14.7 + 50.7i – 8.7 – 36.7i	**A1**	2.5
	All four of $\left(19+16\sqrt{3}\right)+\left(-9+16\sqrt{3}\right)\text{i}$ $\left(-13+16\sqrt{3}\right)+\left(23+16\sqrt{3}\right)\text{i}$ $\left(-13-16\sqrt{3}\right)+\left(23-16\sqrt{3}\right)\text{i}$ $\left(19-16\sqrt{3}\right)-\left(9+16\sqrt{3}\right)\text{i}$	**A1**	2.2a
		(4)	
	Alternative 2 Rotation matrix $\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$ and $\begin{pmatrix}35\\-25\end{pmatrix}-\begin{pmatrix}3\\7\end{pmatrix}$ or $\begin{pmatrix}-29\\39\end{pmatrix}-\begin{pmatrix}3\\7\end{pmatrix}$ Or find the exponential form for $\begin{pmatrix}35\\-25\end{pmatrix}-\begin{pmatrix}3\\7\end{pmatrix}$ or $\begin{pmatrix}-29\\39\end{pmatrix}-\begin{pmatrix}3\\7\end{pmatrix}$	**M1**	3.1a

	$32\sqrt{2}e^{\frac{\pi}{4}}$ or $32\sqrt{2}e^{-\frac{\pi}{4}}$		
	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 32 \\ -32 \end{pmatrix}+\begin{pmatrix} 3 \\ 7 \end{pmatrix}=...$ or $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -32 \\ 32 \end{pmatrix}+\begin{pmatrix} 3 \\ 7 \end{pmatrix}=...$ Or $32\sqrt{2}e^{\frac{\pi}{4}i}\times e^{\frac{\pi}{3}i}=...$ then applies $r(\cos\theta-i\sin\theta)+3+7i=...$ $32\sqrt{2}e^{-\frac{\pi}{4}i}\times e^{\frac{\pi}{3}i}=...$ then applies $r(\cos\theta-i\sin\theta)+3+7i=...$	**M1**	1.1b
	Two of $(19+16\sqrt{3})+(-9+16\sqrt{3})i$ $\quad(-13-16\sqrt{3})+(23-16\sqrt{3})i$ $(-13+16\sqrt{3})+(23+16\sqrt{3})i$ $\quad(19-16\sqrt{3})-(9+16\sqrt{3})i$ Or four correct decimal answers 46.7 + 18.7i – 40.7 – 4.7i 14.7 + 50.7i – 8.7 – 36.7i Or as coordinates	**A1**	2.5
	All four of $(19+16\sqrt{3})+(-9+16\sqrt{3})i$ $(-13+16\sqrt{3})+(23+16\sqrt{3})i$ $(-13-16\sqrt{3})+(23-16\sqrt{3})i$ $(19-16\sqrt{3})-(9+16\sqrt{3})i$	**A1**	2.2a
		(4)	

(11 marks)

Notes:

(a)

M1: Attempts the midpoint of z_1 and z_2

A1*: Correct point.

(b)

M1: Substitutes into the equation with z_1 and α and β, simplifies and expands and applies $i^2=-1$, this may be implied by their working.

A1*: Completes the proof to find the correct answer with no errors seen, all necessary brackets as required

Mark (c) as one

(c) (i)

B1: Correct roots, accept all 6 listed or given in general form as in the scheme. Need not show the 1.

(c)(ii)

M1: Realises the need to set the roots of unity equal to $\beta(z-\alpha)$ and solve for *z*. Must be attempted at least once with any of their roots.

M1: Finds the Cartesian form for their equation for at least one of the roots other than z_1 and z_2

A1: At least two correct other roots than z_1 and z_2 in Cartesian form.

A1: Deduces all four correct in Cartesian form and no extra solutions

Alternative 1

M1: Finds the modulus and argument of $(z-\alpha)^6$

M1: Finds the Cartesian form for one of their modulus and arguments

A1A1: same as above

Alternative 2

M1: Finds the rotation matrix and subtracts the centre from z_1 or z_2. Or finds the exponential from for $z_1-\alpha$ or $z_2-\alpha$

M1: Finds the Cartesian form by multiplying by the rotation matrix and adding the centre. Or multiplies by $e^{\frac{\pi}{3}i}$ write in Cartesian form and adds on the centre

A1A1: same as above

Note all four correct decimal answers or written as coordinates score **A1A0**

46.7 + 18.7i   – 40.7 – 4.7i   14.7 + 50.7i   – 8.7 – 36.7i

Note: Correct answers implies the method marks

Question Number	Scheme	Marks
14(a)	$(\cos 6\theta =)\text{Re}(\cos\theta + \text{i}\sin\theta)^6 = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$	M1A1
	$\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$ $= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$	M1
	$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta) - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$ $\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$*	A1*
		(4)
(b)	$48x^6 - 72x^4 + 27x^2 - 1 = 0$ $x = \cos\theta \Rightarrow \frac{3}{2}\cos 6\theta + \frac{1}{2} = 0 \Rightarrow \cos 6\theta = ...\left(-\frac{1}{3}\right)$	M1
	$\cos 6\theta = -\frac{1}{3} \Rightarrow 6\theta = ... \Rightarrow \theta = ...$ (pv 6θ =109.5°, 1.91 rad, θ = 18.25°, 0.318 rad)	dM1
	$x = \cos\theta = ...$ (commonly 0.950)	dM1
	$x = 0.204$	A1
		(4)
		Total 8

Notes:
(a)

M1: Attempts the binomial expansion of $(\cos\theta + \text{i}\sin\theta)^6$ and extracts the real terms. Binomial coefficients should be correct but allow if there are sign errors. Ignore errors on imaginary terms. Allow use of $(c+\text{i}s)^6$ notation.
A1: Correct extracted expression. Need not be set equal to cos 6θ at this stage.
M1: Applies $\sin^2\theta = 1 - \cos^2\theta$ to obtain an expression in cos θ only.

A1*: Sets equal to cos 6θ and fully expands the square and cube brackets (must be shown) then completes with no errors to obtain the given answer.

(b)
Note: The question says hence and thus methods via solving the cubic in x^2 and square rooting the smallest root score no marks.
M1: Attempts to use the result from part (a) and obtains $\cos 6\theta = k,\ |k| < 1,\ k \neq 0$
dM1: Solves to obtain at least one value for θ. If the $6\theta = \arccos\left(-\frac{1}{3}\right)$ is seen, accept any value that follows for the attempt, but if this method is not shown answers must be correct for their value of $\cos(6\theta)$. The principal values are shown in the scheme. Other values that may be seen are
In radians: for 6θ 4.33, 8.19, 10.66, 14.48, 16.94 ; for θ 0.73, 1.37, 1.78, 2.41, 2.82
In degrees: for 6θ 251°, 469°, 611°, 829°, 1331°; for θ 41.8°, 78.2°, 102°, 138°, 222°;
dM1: Depends on first M. Reverses the substitution to find at least one value for x.
Full list of values for reference: 0.950, 0.746, 0.204, −0.204, −0.746, −0.950
A1: Awrt 0.204 and must be seen identified as the answer, not just given in a list.

Note those who find $\cos 6\theta = -\frac{1}{3}$ but go on to find $6\theta = \arccos\frac{1}{3} = 1.23$ (70.5°), and so $\theta = 0.205$ (11.8°) etc (0.979 as primary answer) may score the M1M0M1A0 but watch out for those who use $\pi - \arccos\frac{1}{3}$ as these can score full marks.

9(a) Alt	$\left(z+\frac{1}{z}\right)^6 = z^6+6z^4+15z^2+20+\frac{15}{z^2}+\frac{6}{z^4}+\frac{1}{z^6}$ $=\left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20$ $=2\cos 6\theta+6(2\cos 4\theta)+15(2\cos 2\theta)+20$	M1A1
	$=2\cos 6\theta+12\left(2\cos^2 2\theta-1\right)+30\left(2\cos^2\theta-1\right)+20$ $=2\cos 6\theta+12\left(2\left(2\cos^2\theta-1\right)^2-1\right)+30\left(2\cos^2\theta-1\right)+20$	M1
	$64\cos^6\theta=2\cos 6\theta+12\left(8\cos^4\theta-8\cos^2\theta+1\right)+60\cos^2\theta-30+20$ $\Rightarrow \cos 6\theta\equiv 32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1$*	A1*
		(4)

Notes:
(a)

M1: Attempts the binomial expansion of $\left(z+z^{-1}\right)^6$, groups terms (may be implied) and attempts to replace the z^n+z^{-n} to achieve an expression in cosines. Allow if there are errors in the "$2\cos(n\theta)$".

A1: Correct expression in cosines. Need not be set equal to $64\cos^6\theta$ at this stage.

M1: Applies $\cos 2A=2\cos^2 A-1$ repeatedly to write the $\cos 4\theta$ and $\cos 2\theta$ in terms of $\cos\theta$ only.

A1*: Sets equal to $64\cos^6\theta$, expands all brackets and rearranges to the given expression with no errors to obtain the given answer.

Question Number	Scheme	Marks
15 (a)	$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos5\theta+\mathrm{i}\sin5\theta$	B1
	$=\cos^5\theta+5\cos^4(\mathrm{i}\sin\theta)+\frac{5\times4}{2!}\cos^3\theta(\mathrm{i}\sin\theta)^2$ $+\frac{5\times4\times3}{3!}\cos^2\theta(\mathrm{i}\sin\theta)^3+\frac{5\times4\times3\times2}{4!}\cos\theta(\mathrm{i}\sin\theta)^4+(\mathrm{i}\sin\theta)^5$	M1
	$=\cos^5\theta+\underline{5\mathrm{i}\cos^4\theta\sin\theta}-10\cos^3\theta\sin^2\theta\underline{-10\mathrm{i}\cos^2\theta\sin^3\theta}+5\cos\theta\sin^4\theta\underline{+\mathrm{i}\sin^5\theta}$	A1
	$\sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$ $=5(1-\sin^2\theta)^2\sin\theta-10(1-\sin^2\theta)\sin^3\theta+\sin^5\theta\frac{\mathrm{d}y}{\mathrm{d}x}$ $=5(1-2\sin^2\theta+\sin^4\theta)\sin\theta-10(1-\sin^2\theta)\sin^3\theta+\sin^5\theta$	M1
	$\sin5\theta=16\sin^5\theta-20\sin^3\theta+5\sin\theta$ *	A1* (5)
	Alternative: Using "$z-\frac{1}{z}$" $z^5-\frac{1}{z^5}=2\mathrm{i}\sin5\theta$ oe	B1
	Binomial expansion of $\left(z-\frac{1}{z}\right)^5$	M1
	$32\sin^5\theta=2\sin5\theta-10\sin3\theta+20\sin\theta$	A1
	Uses double angle formulae etc to obtain $\sin3\theta=3\sin\theta-4\sin^3\theta$ and then use it in their expansion	M1
	$\sin5\theta=16\sin^5\theta-20\sin^3\theta+5\sin\theta$ *	A1* (5)
(b)	Let $x=\sin\theta$ $16x^5-20x^3+5x=-\frac{1}{5}\Rightarrow\sin5\theta=...$	M1A1
	$\Rightarrow\theta=\frac{1}{5}\sin^{-1}\left(\pm\frac{1}{5}\right)=38.306$ (or $-2.307, 69.692. 110.306, 141.693,182.306$)	dM1
	(or in radians −0.0402…0.6685..., 1.216…,1.925…, 2.473…)	
	Two of (awrt) $x=\sin\theta=-0.963,\ -0.555,\ -0.040,\ 0.620,\ 0.938$	A1
	All of (awrt) $x=\sin\theta=-0.963,\ -0.555,\ -0.040,\ 0.620,\ 0.938$	A1 (5)
(c)	$\int_0^{\frac{\pi}{4}}(4\sin^5\theta-5\sin^3\theta-6\sin\theta)\mathrm{d}\theta=\left(\int_0^{\frac{\pi}{4}}\frac{1}{4}(\sin5\theta-5\sin\theta)-6\sin\theta\right)\mathrm{d}\theta$	M1
	$=\left[\frac{1}{4}\left(-\frac{1}{5}\cos5\theta+5\cos\theta\right)+6\cos\theta\right]_0^{\frac{\pi}{4}}\left(=\left[-\frac{1}{20}\cos5\theta+\frac{29}{4}\cos\theta\right]_0^{\frac{\pi}{4}}\right)$	A1
	$\frac{1}{4}\left[-\frac{1}{5}\cos\frac{5\pi}{4}+5\cos\frac{\pi}{4}-\left(-\frac{1}{5}+5\right)\right]+6\cos\frac{\pi}{4}-6$	
	$=\frac{1}{4}\left[\frac{1}{5}\times\frac{1}{\text{Ö}2}+\frac{5}{\text{Ö}2}-4\frac{4}{5}\right]+\frac{6}{\sqrt{2}}-6$	dM1
	$=\frac{73\sqrt{2}}{20}-\frac{36}{5}$ oe	A1 (4)
		[14]
	Notes	

(a)	
B1	Applies de Moivre correctly. Need not see full statement, but must be correctly applied.
M1	Use binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$ May only show imaginary parts - ignore errors in real parts. Binomial coefficients must be evaluated.
A1	Simplify coefficients to obtain a simplified result with all imaginary terms correct
M1	Equate imaginary parts and obtain an expression for $\sin 5\theta$ in terms of powers of $\sin\theta$ No $\cos\theta$ now
A1*	Correct **given** result obtained from fully correct working with at least one intermediate line wit the $(1-\sin^2\theta)^2$ expanded. Must see both sides of answer (may be split across lines). A0 if equating of imaginary terms is not clearly implied.
(b)	
	Note Answers only with no working score no marks as the "hence" has not been used. But if the first M1A1 gained then dM1 may be implied by a correct answer.
M1	Use substitution $x = \sin\theta$ and attempts to use the result from (a) to obtain a value for $\sin 5\theta$
A1	Correct value for $\sin 5\theta$
dM1	Proceeds to apply arcsin and divide by 5 to obtain at least one value for θ. Note for $\sin 5\theta = \frac{1}{5}$ the values you may see are the negatives of the true answers. FYI: $(5\theta = -11.53..., 191.53..., 348.46..., 551.53..., 708.46..., 911.53...)$ (Or in radians $-0.201\ldots$ 3.3428..., 6.0819..., 9.6260..., 12.365..., 15.909…)
A1	Proceeds to take sin and achieve at least 2 different correct values for x or $\sin\theta$
A1	For all 5 values of x or $\sin\theta$ awrt 3 d.p. (allow 0.62 and – 0.04)
(c)	
M1	Use previous work to change the integrand into a function that can be integrated
A1	Correct result after integrating. Any limits shown can be ignored
dM1	Substitute given limits, subtracts and uses exact numerical values for trig functions
A1	Final answer correct (oe provided in the given form)

Question Number	Scheme	Marks
16(a)	$\left(\cos\theta+\mathrm{i}\sin\theta\right)^4=\cos4\theta+\mathrm{i}\sin4\theta$	
	$\cos^4\theta+4\cos^3\theta(\mathrm{i}\sin\theta)+\frac{4\times3}{2!}\cos^2\theta(\mathrm{i}\sin\theta)^2$ $+\frac{4\times3\times2}{3!}\cos\theta(\mathrm{i}\sin\theta)^3+(\mathrm{i}\sin\theta)^4$	M1
	$=\cos^4\theta+4\mathrm{i}\cos^3\theta\sin\theta+\mathrm{i}^2 6\cos^2\theta\sin^2\theta+4\mathrm{i}^3\cos\theta\sin^3\theta+\mathrm{i}^4\sin^4\theta$	A1
	$\cos4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$	M1
	$\sin4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta$	A1
	$\tan4\theta=\frac{\sin4\theta}{\cos4\theta}=\frac{4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta}{\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta}$	
	$\tan4\theta=\frac{\sin4\theta}{\cos4\theta}=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$ *	M1A1* (6)
7(b)	$x=\tan\theta$ $\quad\frac{2\tan\theta-2\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}=\frac{1}{2}\tan4\theta=1$	
	$\tan4\theta=2$	M1
	$x=\tan\theta=0.284,\ 1.79$	A1A1 (3) [9]
(a)		
M1	Correct use of de Moivre and attempt the complete expansion	
A1	Correct expansion. Coefficients to be single numbers but powers of i may still be present.	
M1	Equate the real and imaginary parts	
A1	Correct expressions for $\cos4\theta$ and $\sin4\theta$	
M1	Use $\tan4\theta=\frac{\sin4\theta}{\cos4\theta}$ and divide numerator and denominator by $\cos^4\theta$ Only tangents now.	
A1*	Correct **given** answer, no errors seen.	
(b)		
M1	Substitute $x=\tan\theta$ and re-arrange to $\tan4\theta=\pm2$ or $\pm\frac{1}{2}$	
A1A1	A1 for either solution; A2 for both. Deduct one mark only for failing to round either or both to 3 sf (One correct answer but not rounded scores A0A0; two correct answers neither rounded scores A1A0; two correct answers, only one rounded, scores A1A0)	

Question Number	Scheme	Marks
	Alternative for first 4 marks of 7(a):	
	$\sin 4\theta = \frac{1}{2\mathrm{i}}\left(z^4 - z^{-4}\right) = \frac{1}{2\mathrm{i}}\left(\left(\cos\theta - \mathrm{i}\sin\theta\right)^4 - \left(\cos\theta + \mathrm{i}\sin\theta\right)^{-4}\right)$	M1
	$= \frac{1}{2\mathrm{i}}\left(\cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta\right)$	M1
	$-\frac{1}{2\mathrm{i}}\left(-\cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta + 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta - \sin^4\theta\right)$	
	$= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$	A1
	Similar work leads to $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$	A1
	Remaining 2 marks as main scheme	

M1	For the expression derived from de Moivre for **either** $\sin 4\theta$ or $\cos 4\theta$
A1	Both shown and correct
M1	Attempt the binomial expansion for either, reaching a simplified expression
A1	Both simplified expressions correct