London Academy of Maths

Complex Numbers: Roots of Unity and De Moivre's Theorem

Question Paper

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1. Solve the equation

$$z^5 - 32i = 0$$

giving each answer in the form $re^{i\theta}$ where $0 < \theta < 2\pi$

(4)

Question 1 continued

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2. (a) Express the complex number $183 - \sqrt{1}8i$ in the form

$$r(\cos\theta + i\sin\theta) \quad -\pi < \theta \leqslant \pi$$

(3)

(b) Solve the equation

$$z^4 = 18\sqrt{3} - 18i$$

giving your answers in the form $re^{i\theta}$ where $-\pi < \theta \leqslant \pi$

(5)

Question 2 continued

Leave blank

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3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express the complex number

$$-4 - 4\sqrt{3}\mathrm{i}$$

in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

(3)

(b) Solve the equation

$$z^3 + 4 + 4\sqrt{3}\mathrm{i} = 0$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

(4)

Question 3 continued

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4.

$$z = 6 - 6\sqrt{3}\,\mathrm{i}$$

(a) (i) Determine the modulus of z

(ii) Show that the argument of z is $-\frac{\pi}{3}$

(3)

Using de Moivre's theorem, and making your method clear,

(b) determine, in simplest form, z^4

(2)

(c) Determine the values of w such that $w^2 = z$, giving your answers in the form $a + \mathrm{i}b$, where a and b are real numbers.

(3)

Leave blank

Question 4 continued

5. The complex number z_1 is defined as

$$z_1 = \frac{\left(\cos\frac{5\pi}{12} + \mathrm{i}\sin\frac{5\pi}{12}\right)^4}{\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)^3}$$

(a) Without using your calculator show that

$$z_1 = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$$

(4)

(b) Shade, on a single Argand diagram, the region R defined by

$$|z - z_1| \leqslant 1 \quad \textbf{and} \quad 0 \leqslant \arg(z - z_1) \leqslant \frac{3\pi}{4}$$

(4)

Given that the complex number z lies in R

(c) determine the smallest possible positive value of $\arg z$

(2)

Question 5 continued

Leave blank

6. (a) Determine the roots of the equation

$$z^6 = 1$$

giving your answers in the form $e^{i\theta}$ where $0 \leqslant \theta < 2\pi$

(2)

(b) Show the roots of the equation in part (a) on a single Argand diagram.

(2)

(c) Show that

$$\left(\sqrt{3} + i\right)^6 = -64$$

(2)

(d) Hence, or otherwise, solve the equation

$$z^6 + 64 = 0$$

giving your answers in the form $re^{i\theta}$ where $0 \leqslant \theta < 2\pi$

(3)

Leave blank

Question 6 continued

7. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$z_1 = -4 + 4\mathrm{i}$$

(a) Express z_1 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r \in \mathbb{R}$, $r > 0$ and $0 \leqslant \theta < 2\pi$

(2)

$$z_2 = 3\left(\cos\frac{17\pi}{12} + \mathrm{i}\sin\frac{17\pi}{12}\right)$$

(b) Determine in the form $a + \mathrm{i}b$, where a and b are exact real numbers,

(i) $\dfrac{z_1}{z_2}$

(2)

(ii) $(z_2)^4$

(2)

(c) Show on a single Argand diagram

(i) the complex numbers z_1, z_2 and $\dfrac{z_1}{z_2}$

(ii) the region defined by $\{z \in \mathbb{C} : |z - z_1| < |z - z_2|\}$

(4)

Question 7 continued

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8. (i) Given that

$$z_1 = 6e^{\frac{\pi}{3}i} \text{ and } z_2 = 6\sqrt{3}e^{\frac{5\pi}{6}i}$$

show that

$$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$$

(3)

(ii) Given that

$$\arg(z - 5) = \frac{2\pi}{3}$$

determine the least value of $|z|$ as z varies.

(3)

Question 8 continued

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9. The complex number $z = e^{i\theta}$, where θ is real.

(a) Show that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta$$

where n is a positive integer.

(2)

(b) Show that

$$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

(5)

(c) Hence, making your reasoning clear, determine all the solutions of

$$\cos 5\theta + 5\cos 3\theta + 12\cos\theta = 0$$

in the interval $0 \leqslant \theta < 2\pi$

(3)

Leave blank

Question 9 continued

10. (a) Given

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad n \in \mathbb{N}$$

show that

$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

(5)

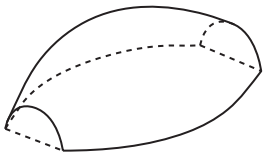

Figure 1

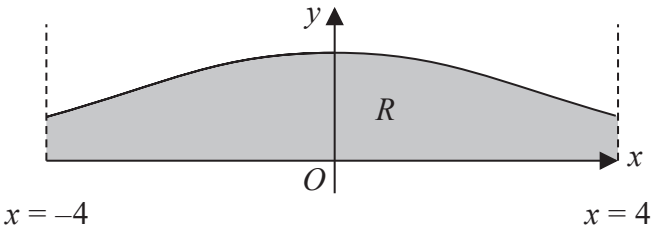

Figure 2

Figure 1 shows a solid paperweight with a flat base.

Figure 2 shows the curve with equation

$$y = H\cos^3\left(\frac{x}{4}\right) \qquad -4 \leqslant x \leqslant 4$$

where H is a positive constant and x is in radians.

The region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = -4$, the line with equation $x = 4$ and the x-axis.

The paperweight is modelled by the solid of revolution formed when R is rotated **180°** about the x-axis.

Given that the maximum height of the paperweight is 2 cm,

(b) write down the value of H.

(1)

(c) Using algebraic integration and the result in part (a), determine, in cm^3, the volume of the paperweight, according to the model. Give your answer to 2 decimal places.

[*Solutions based entirely on calculator technology are not acceptable.*]

(5)

(d) State a limitation of the model.

(1)

Question 10 continued

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11. (a) Use de Moivre's theorem to prove that

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

(5)

(b) Hence find the distinct roots of the equation

$$1 + 7x - 56x^3 + 112x^5 - 64x^7 = 0$$

(5)

giving your answer to 3 decimal places where appropriate.

Leave blank

Question 11 continued

12. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Use de Moivre's theorem to show that

$$\cos 5x \equiv \cos x\left(a\sin^4 x + b\sin^2 x + c\right)$$

where a, b and c are integers to be determined.

(4)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\cos 5\theta = \sin 2\theta \sin\theta - \cos\theta$$

giving your answers to 3 decimal places.

(4)

Leave blank

Question 12 continued

13. The points representing the complex numbers $z_1 = 35 - 25i$ and $z_2 = -29 + 39i$ are opposite vertices of a regular hexagon, H, in the complex plane.

The centre of H represents the complex number α

(a) Show that $\alpha = 3 + 7i$

(2)

Given that $\beta = \dfrac{1+i}{64}$

(b) show that

$$\beta(z_1 - \alpha) = 1$$

(2)

The vertices of H are given by the roots of the equation

$$(\beta(z - \alpha))^6 = 1$$

(c) (i) Write down the roots of the equation $w^6 = 1$ in the form $re^{i\theta}$

(1)

(ii) Hence, or otherwise, determine the position of the other four vertices of H, giving your answers as complex numbers in Cartesian form.

(4)

Question 13 continued

14. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Use De Moivre's theorem to show that

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

(4)

(b) Hence determine the smallest positive root of the equation

$$48x^6 - 72x^4 + 27x^2 - 1 = 0$$

giving your answer to 3 decimal places.

(4)

Question 14 continued

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15. (a) Use de Moivre's theorem to show that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

(5)

(b) Hence determine the five distinct solutions of the equation

$$16x^5 - 20x^3 + 5x + \frac{1}{5} = 0$$

giving your answers to 3 decimal places.

(5)

(c) Use the identity given in part (a) to show that

$$\int_0^{\frac{\pi}{4}} \left(4\sin^5\theta - 5\sin^3\theta - 6\sin\theta\right) \mathrm{d}\theta = a\sqrt{2} + b$$

where a and b are rational numbers to be determined.

(4)

Question 15 continued

Leave blank

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16. (a) Use de Moivre's theorem to show that

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(6)

(b) Use the identity given in part (a) to find the 2 positive roots of

$$x^4 + 2x^3 - 6x^2 - 2x + 1 = 0$$

giving your answers to 3 significant figures.

(3)

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Question 16 continued