London Academy of Maths

Sums of Series involving Complex Numbers

Worked Solutions

1. The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

a) $C + iS = \cos\theta + \frac{1}{2}\cos(5\theta) + \frac{1}{4}\cos(9\theta) + \ldots + i\sin\theta$

$+ \frac{1}{2} i\sin(5\theta) + \frac{1}{4} i\sin(9\theta) + \ldots$

$= \cos\theta + i\sin\theta + \frac{1}{2}(\cos(5\theta) + i\sin(5\theta)) + \frac{1}{4}(\cos(5\theta) + i\sin(5\theta)) + \ldots$

DMT
$= e^{i\theta} + \frac{1}{2}e^{5i\theta} + \frac{1}{4}e^{9i\theta} + \ldots$

$= \dfrac{e^{i\theta}}{1 - \frac{1}{2}e^{4i\theta}} \times \dfrac{2}{2}$ (since $S_\infty = \dfrac{a}{1-r}$)

$= \dfrac{2e^{i\theta}}{2 - e^{4i\theta}}$

P 6 1 1 7 8 A 0 1 0 2 8

Question 1 continued

b) Can't find S just yet since denominator is complex.

$$C+iS = \frac{2e^{i\theta}}{2-e^{4i\theta}} \times \frac{2-e^{-4i\theta}}{2-e^{-4i\theta}}$$

This is the key step. It's slightly different to how you would rationalise $\frac{a+bi}{c+di}$ $\left(\times \frac{c-di}{c-di}\right)$

$$= \frac{4e^{i\theta} - 2e^{-3i\theta}}{4 - 2e^{4i\theta} - 2e^{-4i\theta} + 1}$$

$$= \frac{4\cos\theta + 4i\sin\theta - 2\cos(-3\theta) - 2i\sin(-3\theta)}{5 - 2\left(e^{4i\theta} + e^{-4i\theta}\right)}$$

since $\cos(-\theta) = \cos(\theta)$ & $\sin(-\theta) = -\sin(\theta)$

$$= \frac{4\cos\theta - 2\cos(3\theta) + i\left(4\sin\theta + 2\sin(3\theta)\right)}{5 - 2\left(2\cos(4\theta)\right)}$$

$z^4 + \frac{1}{z^4} = 2\cos(4\theta)$

see textbook.

$$S = \operatorname{Im}(C+iS)$$

$$= \frac{4\sin\theta + 2\sin(3\theta)}{5 - 4\cos(4\theta)} \quad \square$$

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2 Convergent infinite series C and S are defined by

$$C = 1 + \tfrac{1}{2}\cos\theta + \tfrac{1}{4}\cos 2\theta + \tfrac{1}{8}\cos 3\theta + \ldots ,$$
$$S = \quad \tfrac{1}{2}\sin\theta + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{8}\sin 3\theta + \ldots .$$

(i) Show that $C + \mathrm{i}S = \dfrac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}$. **[4]**

(ii) Hence show that $C = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$, and find a similar expression for S. **[4]**

See Q1 if you're not sure about first line.

a) $C + iS = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{8}e^{3i\theta} + \ldots$

$= \dfrac{1}{1 - \frac{1}{2}e^{i\theta}} \times \dfrac{2}{2}$ $\qquad \left(S_\infty = \dfrac{a}{1-r} \right)$

$= \dfrac{2}{2 - e^{i\theta}}$

b) $C + iS = \dfrac{2}{2 - e^{i\theta}} \times \dfrac{2 - e^{-i\theta}}{2 - e^{-i\theta}}$

$= \dfrac{4 - 2e^{-i\theta}}{4 - 2e^{i\theta} - 2e^{-i\theta} + 1}$

$= \dfrac{4 - 2\cos(-\theta) - 2i\sin(-\theta)}{5 - 2(e^{i\theta} + e^{-i\theta})}$

$= \dfrac{4 - 2\cos\theta + 2i\sin(\theta)}{5 - 2(2\cos\theta)}$ $\qquad \left\{ \begin{array}{l} \cos(-\theta) = \cos(\theta) \\ \sin(-\theta) = -\sin(\theta) \end{array} \right.$ $\qquad \left\{ z + \frac{1}{z} = 2\cos\theta \right.$

$\therefore C = \mathrm{Re}(C + iS) = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$

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3 The integrals C and S are defined by

$$C = \int_0^{\frac{1}{2}\pi} e^{2x} \cos 3x \, dx \qquad \text{and} \qquad S = \int_0^{\frac{1}{2}\pi} e^{2x} \sin 3x \, dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\tfrac{1}{13}(2 + 3e^{\pi}),$$

and obtain a similar expression for S. [8]

(You may assume that the standard result for $\int e^{kx}\,dx$ remains true when k is a complex constant, so that $\int e^{(a+ib)x}\,dx = \frac{1}{a+ib}e^{(a+ib)x}$.) (*)

$$C+iS = \int_0^{\frac{1}{2}\pi} e^{2x}\cos(3x)\,dx + i\int_0^{\frac{1}{2}\pi} e^{2x}\sin(3x)\,dx$$

$$= \int_0^{\frac{1}{2}\pi} \left(e^{2x}\cos(3x) + ie^{2x}\sin(3x)\right)\,dx$$

$$= \int_0^{\frac{1}{2}\pi} e^{2x}\left(\cos(3x) + i\sin(3x)\right)\,dx$$

$$= \int_0^{\frac{1}{2}\pi} e^{2x}\,e^{3ix}\,dx$$

$$= \int_0^{\frac{1}{2}\pi} e^{(2+3i)x}\,dx$$

$$= \left[\frac{1}{2+3i}e^{(2+3i)x}\right]_0^{\frac{1}{2}\pi} \qquad (\text{by } *)$$

$$= \frac{1}{2+3i}\left[e^{(2+3i)x}\right]_0^{\frac{1}{2}\pi}$$

$$= \frac{1}{2+3i}\left(e^{(2+3i)\frac{1}{2}\pi} - e^{(2+3i)0}\right)$$

$$= \frac{1}{2+3i}\left(e^{\pi + \frac{3}{2}\pi i} - 1\right)$$

$$= \frac{1}{2+3i}\left(e^{\pi} e^{3/2 \pi i} - 1\right)$$

$$= \frac{1}{2+3i}\left(e^{\pi}\left(\cos\left(\tfrac{3}{2}\pi\right) + i\sin\left(\tfrac{3}{2}\pi\right)\right) - 1\right)$$

$$= \frac{1}{2+3i}\left(e^{\pi}(-i) - 1\right)$$

$$= \frac{2-3i}{13}\left(-e^{\pi}i - 1\right)$$ by rationalising $\frac{1}{2+3i}$

$$= \frac{-2-3e^{\pi} + i(-2e^{\pi}+3)}{13}$$

$$C = \mathrm{Re}(C+iS) = \frac{-2-3e^{\pi}}{13} = \frac{-1}{13}\left(3e^{\pi}+2\right)$$

$$S = \mathrm{Im}(C+iS) = \frac{3-2e^{\pi}}{13}$$

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4. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \dots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots \quad = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \dots$ cannot be purely imaginary, giving a reason for your answer.

(2)

a) $1 + z + z^2 + z^3 + \dots = \dfrac{1}{1-z}$ $\quad \left(S_\infty = \dfrac{a}{1-r}\right)$

b) let $C = 1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos(2\theta) + \dots$

$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin(2\theta) + \dots$

$C + iS = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \dots = \dfrac{1}{1 - \frac{1}{2}e^{i\theta}}$ (by (a))

$= \dfrac{2}{2 - e^{i\theta}}$

$= \dots\dots$ (see Q2)

$= \dfrac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}$

$\frac{1}{2}\sin\theta + \frac{1}{4}\sin(2\theta) + \dots$

$= \mathrm{Im}(C + iS) = \dfrac{2\sin\theta}{5 - 4\cos\theta}$

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Question 4 continued

ii) Suppose $1 + z + z^2 + \ldots$ is purely imaginary.

then $\text{Re}(C + iS) = 0$

i.e. $\dfrac{4 - 2\cos\theta}{5 - 4\cos\theta} = 0$

$4 - 2\cos\theta = 0$

$2 = \cos\theta$

which is not possible since $-1 \leq \cos\theta \leq 1$

$\therefore \text{Re}(C + iS) \neq 0$ and so $1 + z + z^2 + \ldots$

is not purely imaginary

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5 Let $S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{10i\theta}$.

(i) (a) Show that, for $\theta \neq 2n\pi$, where n is an integer,

$$S = \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}.$$ **[4]**

(b) State the value of S for $\theta = 2n\pi$, where n is an integer. **[1]**

(ii) Hence show that, for $\theta \neq 2n\pi$, where n is an integer,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}.$$ **[3]**

(iii) Hence show that $\theta = \frac{1}{11}\pi$ is a root of $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = 0$ and find another root in the interval $0 < \theta < \frac{1}{4}\pi$. **[4]**

i)a

$$S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{10i\theta}$$

$$= \frac{e^{i\theta}\left(1 - \left(e^{i\theta}\right)^{10}\right)}{1 - e^{i\theta}}$$

$$= \frac{e^{i\theta}\left(1 - e^{10i\theta}\right)}{1 - e^{i\theta}} \times \frac{-e^{-\frac{1}{2}i\theta}}{-e^{-\frac{1}{2}i\theta}}$$

$$= \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}\theta}}$$

$$= \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{\cos(\frac{1}{2}\theta) + i\sin(\frac{1}{2}\theta) - \cos(-\frac{1}{2}\theta) + i\sin(-\frac{1}{2}\theta)}$$

$$= \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin(\frac{1}{2}\theta)}$$

since $i\sin(-\frac{1}{2}\theta) = -i\sin(\frac{1}{2}\theta)$

& $\cos(-\frac{1}{2}\theta) = \cos(\frac{1}{2}\theta)$

i b) if $\theta = 2n\pi$

$$S = e^{2n\pi i} + e^{4n\pi i} + \dots + e^{20n\pi i}$$

but $\cos(2n\pi) = 1$ & $\sin(2n\pi) = 0$ for all $n \in \mathbb{Z}$.

$$\therefore S = 1 + 1 + \dots + 1 = 10$$

ii) where $\theta \neq 2n\pi$, $$S = \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)} \times \frac{-i}{-i}$$

$$= \frac{-ie^{\frac{21}{2}i\theta} + ie^{\frac{1}{2}i\theta}}{2\sin\left(\frac{1}{2}\theta\right)}$$

$$= \frac{-i\cos\left(\frac{21}{2}\theta\right) + \sin\left(\frac{21}{2}\theta\right) + i\cos\left(\frac{1}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}$$

$$\cos\theta + \cos(2\theta) + \dots + \cos(10\theta) = \mathrm{Re}(S)$$

$$= \frac{\sin\left(\frac{21}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}$$

$$= \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}$$

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iii) $$\frac{\sin\left(\frac{21}{2} \times \frac{1}{11}\pi\right)}{2\sin\left(\frac{1}{2} \times \frac{1}{11}\pi\right)} - \frac{1}{2} = \frac{\sin\left(\frac{21}{22}\pi\right)}{2\sin\left(\frac{1}{22}\pi\right)} - \frac{1}{2}$$

$$= \frac{\cancel{\sin\left(\frac{1}{22}\pi\right)}}{2\cancel{\sin\left(\frac{1}{22}\pi\right)}} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

$\sin(\pi - \theta) \equiv \sin(\theta)$

to find other root

$$\frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2} = 0$$

$$\Leftrightarrow \quad \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} = \frac{1}{2}$$

$$\sin\left(\frac{21}{2}\theta\right) = \sin\left(\frac{1}{2}\theta\right)$$

$\frac{21}{2}\theta = \frac{1}{2}\theta$ OR $\frac{21}{2}\theta = \pi - \frac{1}{2}\theta$ OR $\frac{21}{2}\theta = \frac{1}{2}\theta + 2\pi$

$\theta = 0$

not valid

$11\theta = \pi$

$\theta = \frac{1}{11}\pi$

used already

$10\theta = 2\pi$

$$\theta = \frac{1}{5}\pi$$

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6 (i) Show that, if $z \neq \pm 1$ and $z \neq 0$,

$$\sum_{r=1}^{n} z^{2r-1} = \frac{1-z^{2n}}{z^{-1}-z}.$$ **[2]**

(ii) Hence show that, if $\sin\theta \neq 0$,

$$\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}.$$ **[6]**

(iii) Hence find the exact value of

$$\int_0^{\frac{1}{6}\pi} \frac{\sin^2 3\theta}{\sin\theta}\,d\theta.$$ **[3]**

i) $\sum_{r=1}^{n} z^{2r-1} = z + z^3 + z^5 + \dots + z^{2n-1}$

$= \frac{z(1-(z^2)^n)}{1-z^2}$

$= \frac{z(1-z^{2n})}{1-z^2} \times \frac{1/z}{1/z}$

$= \frac{1-z^{2n}}{z^{-1}-z}$

ii) let $z = \cos\theta + i\sin\theta$

$C+iS = \sum_{r=1}^{n} z^{2r-1} = \frac{1-z^{2n}}{z^{-1}-z} = \frac{1-e^{2in\theta}}{\cos(-\theta)+i\sin(-\theta)-\cos(\theta)-i\sin(\theta)}$

$= \frac{1-e^{2in\theta}}{-2i\sin(\theta)} \times \frac{i}{i}$ seen similar

$= \frac{i-ie^{2in\theta}}{2\sin\theta}$

$= \frac{i-i\cos(2n\theta)+\sin(2n\theta)}{2\sin(\theta)}$

P.T.O

but $S := \sum_{r=1}^{n} \sin(2r-1)\theta$ is just $\operatorname{Im}(C + iS)$

$$\therefore S = \frac{1-\cos(2n\theta)}{2\sin(\theta)}$$

$$= \frac{1-(1-2\sin^2(n\theta))}{2\sin(\theta)} \quad \left(\text{since } \cos(2A) = 1-2\sin^2(A)\right)$$

$$= \frac{2\sin^2(n\theta)}{2\sin(\theta)}$$

$$= \frac{\sin^2(n\theta)}{\sin(\theta)} \quad \text{as required.}$$

iii) $$\int_0^{\frac{1}{6}\pi} \frac{\sin^2(3\theta)}{\sin(\theta)} = \int_0^{\frac{1}{6}\pi} \sum_{r=1}^{3} \sin((2r-1)\theta)\, d\theta$$

$$= \int_0^{\frac{1}{6}\pi} (\sin\theta + \sin(3\theta) + \sin(5\theta))\, d\theta$$

$$= \left[-\cos\theta - \frac{1}{3}\cos(3\theta) - \frac{1}{5}\cos(5\theta)\right]_0^{\frac{1}{6}\pi}$$

$= \ldots$

$$= \left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{10}\right) - \left(-1 - \frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{1}{15}(23 - 6\sqrt{3}) \quad \text{or} \quad \frac{23}{15} - \frac{2\sqrt{3}}{5}$$