

Coupled Differential Equations

Question Paper

1 The simultaneous differential equations

$$\frac{dx}{dt} + 2x = 4y + e^{-2t}$$
$$\frac{dy}{dt} + 3y = 5x + 2e^{-2t}$$

are to be solved.

(i) Obtain a second order differential equation for x in terms of t . Hence find the general solution for x . **[12]**

(ii) Find the corresponding general solution for y . **[4]**

When $t = 0$, $y = -\frac{2}{3}$ and $\frac{dy}{dt} = 0$.

(iii) Find the particular solutions for x and y . **[5]**

(iv) Find the set of values of t for which $y > x$. **[3]**

2 The simultaneous differential equations

$$\frac{dx}{dt} = -3x - 4y + 23,$$

$$\frac{dy}{dt} = 2x + y - 7$$

are to be solved.

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 5$. **[5]**

(ii) Find the general solution for x . **[7]**

(iii) Find the corresponding general solution for y . **[4]**

When $t = 0$, $x = 8$ and $y = 0$.

(iv) Find the particular solutions for x and y . **[4]**

(v) Show that for sufficiently large t , y is always greater than x . **[4]**

3 The following simultaneous differential equations are to be solved.

$$\frac{dx}{dt} = x - 2y + \sin t$$

$$\frac{dy}{dt} = 4x - 3y + \cos t$$

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 3\sin t - \cos t$. [5]

(ii) Find the general solution for x in terms of t . [10]

(iii) Hence obtain the corresponding general solution for y . [5]

(iv) Obtain approximate expressions for x and y in terms of t , valid for large t . Hence show that, for large t , x is approximately equal to y . Show that, for small t , this is not necessarily the case. [4]

4 The simultaneous differential equations

$$\begin{aligned}\frac{dx}{dt} &= 7x + 6y + 2e^{-3t} \\ \frac{dy}{dt} &= -12x - 10y + 5 \sin t\end{aligned}$$

are to be solved for $t \geq 0$.

(i) Show that

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 14e^{-3t} + 30 \sin t. \quad [5]$$

(ii) Show that this differential equation has a particular integral of the form $x = ae^{-3t} - 9 \cos t + 3 \sin t$, where a is a constant to be determined.

Hence find the general solution for x in terms of t . [8]

(iii) Find the corresponding general solution for y . [4]

(iv) Show that, for large values of t , $x = y$ when $\tan t \approx k$, where k is a constant to be determined. [4]

(v) Find the ratio of the amplitudes of y and x for large values of t . [3]

5. A scientist is studying the effect of introducing a population of type A bacteria into a population of type B bacteria.

At time t days, the number of type A bacteria, x , and the number of type B bacteria, y , are modelled by the differential equations

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 3y - 2x$$

- (a) Show that

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 5x = 0 \tag{3}$$

- (b) Determine a general solution for the number of type A bacteria at time t days. (4)
- (c) Determine a general solution for the number of type B bacteria at time t days. (2)

The model predicts that, at time T hours, the number of bacteria in the two populations will be equal.

Given that $x = 100$ and $y = 275$ when $t = 0$

- (d) determine the value of T , giving your answer to 2 decimal places. (5)
- (e) Suggest a limitation of the model. (1)

6. Two compounds, X and Y , are involved in a chemical reaction. The amounts in grams of these compounds, t minutes after the reaction starts, are x and y respectively and are modelled by the differential equations

$$\frac{dx}{dt} = -5x + 10y - 30$$

$$\frac{dy}{dt} = -2x + 3y - 4$$

- (a) Show that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$$

(3)

- (b) Find, according to the model, a general solution for the amount in grams of compound X present at time t minutes.

(6)

- (c) Find, according to the model, a general solution for the amount in grams of compound Y present at time t minutes.

(3)

Given that $x = 2$ and $y = 5$ when $t = 0$

- (d) find

- (i) the particular solution for x ,
(ii) the particular solution for y .

(4)

A scientist thinks that the chemical reaction will have stopped after 8 minutes.

- (e) Explain whether this is supported by the model.

(1)

7 Solve the simultaneous differential equations

$$\frac{dx}{dt} + 2x = 4y, \quad \frac{dy}{dt} + 3x = 5y,$$

given that when $t = 0$, $x = 0$ and $y = 1$.

[11]

8. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f , and the number of rabbits, r , on the island are modelled by the differential equations

$$\frac{df}{dt} = 0.2f + 0.1r$$

$$\frac{dr}{dt} = -0.2f + 0.4r$$

(a) Show that $\frac{d^2f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0$ (3)

(b) Find a general solution for the number of foxes on the island at time t years. (4)

(c) Hence find a general solution for the number of rabbits on the island at time t years. (3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

- (d) (i) According to this model, in which year are the rabbits predicted to die out?
(ii) According to this model, how many foxes will be on the island when the rabbits die out?
(iii) Use your answers to parts (i) and (ii) to comment on the model. (7)

9 At time t , the quantities x and y are modelled by the simultaneous differential equations

$$\frac{dx}{dt} = 2x - 5y + 9e^{-2t},$$
$$\frac{dy}{dt} = x - 4y + 3e^{-2t}.$$

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = 3e^{-2t}$. [5]

(ii) Find the general solution for x . [8]

(iii) Find the corresponding general solution for y . [4]

Initially $x = 0$ and $y = 2$.

(iv) Find the particular solutions. [4]

(v) Describe the behaviour of the solutions as $t \rightarrow \infty$.

State, with reasons, whether this behaviour is different if the initial value of y is just less than 2, and the initial value of x is still 0. [3]

10 The simultaneous differential equations

$$\frac{dx}{dt} = 4x - 6y - 9 \sin t,$$

$$\frac{dy}{dt} = 3x - 5y - 7 \sin t,$$

are to be solved.

(i) Show that $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = -9 \cos t - 3 \sin t.$ [6]

(ii) Find the general solution for $x.$ [9]

(iii) Hence find the corresponding general solution for $y.$ [3]

It is given that x is bounded as $t \rightarrow \infty.$

(iv) Show that y is also bounded as $t \rightarrow \infty.$ [2]

(v) Given also that $y = 0$ when $t = 0,$ find the particular solutions for x and $y.$ Write down the expressions for x and y as $t \rightarrow \infty.$ [4]

11 The quantities x and y at time t are modelled by the simultaneous differential equations

$$\frac{dx}{dt} = -3x - 2y + 3t,$$

$$\frac{dy}{dt} = 2x + y + t + 2.$$

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = -5t - 1$. [5]

(ii) Find the general solution for x . [8]

(iii) Find the corresponding general solution for y . [4]

When $t = 0$, $x = 9$ and $y = 0$.

(iv) Find the particular solutions. [4]

(v) Find approximate expressions for x and y in terms of t , valid for large positive values of t . [3]

12. A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.

At time t years, the number of white-clawed crayfish, w , and the number of signal crayfish, s , are modelled by the differential equations

$$\frac{dw}{dt} = \frac{5}{2}(w - s)$$
$$\frac{ds}{dt} = \frac{2}{5}w - 90e^{-t}$$

(a) Show that

$$2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t}$$

(3)

(b) Find a general solution for the number of white-clawed crayfish at time t years. (6)

(c) Find a general solution for the number of signal crayfish at time t years. (2)

The model predicts that, at time T years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$

(d) find the value of T , giving your answer to 3 decimal places. (6)

(e) Suggest a limitation of the model. (1)

13. 1 patient is treated by administering an antibiotic intravenously at a constant rate for some time.

Initially there is none of the antibiotic in the patient.

At time t minutes after treatment began

- the concentration of the antibiotic in the blood of the patient is x mg/ml
- the concentration of the antibiotic in the tissue of the patient is y mg/ml

The concentration of antibiotic in the patient is modelled by the equations

$$\frac{dx}{dt} = 0.025y - 0.045x + 2$$

$$\frac{dy}{dt} = 0.032x - 0.025y$$

- (a) Show that

$$40\,000 \frac{d^2y}{dt^2} + 2800 \frac{dy}{dt} + 13y = 2560 \quad (3)$$

- (b) Determine, according to the model, a general solution for the concentration of the antibiotic in the patient's tissue at time t minutes after treatment began. (5)
- (c) Hence determine a particular solution for the concentration of the antibiotic in the tissue at time t minutes after treatment began. (4)

To be effective for the patient the concentration of antibiotic in the tissue must eventually reach a level between 185 mg/ml and 200 mg/ml.

- (d) Determine whether the rate of administration of the antibiotic is effective for the patient, giving a reason for your answer. (2)

14 The simultaneous differential equations

$$\frac{dx}{dt} = x - y + 3 \cos t$$

$$\frac{dy}{dt} = 5x - y - 12 \sin t$$

are to be solved for $t \geq 0$, where t denotes time.

(i) Eliminate y to obtain a second order differential equation for x in terms of t . Hence find the general solution for x . **[12]**

(ii) Find the corresponding general solution for y . **[3]**

When $t = 0$, $y = 0$ and $\frac{dy}{dt} = 5$.

(iii) Find the particular solutions for x and y . **[4]**

(iv) Find the time that elapses between the first two occasions on which $x = y$. **[5]**

- 15** The populations of foxes, x , and rabbits, y , on an island at time t are modelled by the simultaneous differential equations

$$\frac{dx}{dt} = 0.1x + 0.1y,$$
$$\frac{dy}{dt} = -0.2x + 0.3y.$$

(i) Show that $\frac{d^2x}{dt^2} - 0.4\frac{dx}{dt} + 0.05x = 0$. [5]

(ii) Find the general solution for x . [4]

(iii) Find the corresponding general solution for y . [4]

Initially there are x_0 foxes and y_0 rabbits.

(iv) Find the particular solutions. [4]

(v) In the case $y_0 = 10x_0$, find the time at which the model predicts the rabbits will die out. Determine whether the model predicts the foxes die out before the rabbits. [7]

16 The simultaneous differential equations

$$\frac{dx}{dt} = -2x - y + 6,$$

$$\frac{dy}{dt} = x - 2y + 7,$$

are to be solved.

(i) Eliminate y to obtain a second order differential equation for x in terms of t . Hence find the general solution for x . **[12]**

(ii) Find the corresponding general solution for y . **[3]**

Initially $x = 7$ and $y = 0$.

(iii) Find the particular solutions. **[4]**

As $t \rightarrow \infty$, $\frac{y}{x} \rightarrow k$.

(iv) State the value of k and show that $y = kx$ for infinitely many values of t . **[5]**

17 The simultaneous differential equations

$$\frac{dx}{dt} = x - 2y - z$$

$$\frac{dy}{dt} = x + 3y + z$$

$$\frac{dz}{dt} = -z$$

are to be solved. When $t = 0$, $x = 1$, $y = 0$ and $z = 2$.

- (i) Use the third equation to find the particular solution for z in terms of t . [2]
- (ii) Using part (i) eliminate y and z to obtain a second order differential equation for x . Hence find the general solution for x in terms of t . [12]
- (iii) Find the corresponding general solution for y . [3]
- (iv) Find the particular solutions for x and y . [4]
- (v) Show that $x = y$ when $3 \sin t = e^{-3t}$. Deduce that $x = y$ occurs infinitely often. [3]

- 18** Two species of insects, X and Y, compete for survival on an island. The populations of the species are x and y respectively at time t , where t is measured in tens of years. The situation is modelled by the simultaneous differential equations

$$\frac{dx}{dt} = 2x + 2y,$$

$$\frac{dy}{dt} = 6y - 4x.$$

- (i)** Eliminate y to obtain a second order differential equation for x in terms of t . Hence find the general solution for x . [7]

- (ii)** Find the corresponding general solution for y . [4]

When $t = 0$, $\frac{dx}{dt} = 10$ and the population of species Y is k times the population of species X, where k is a positive constant.

- (iii)** Find the particular solutions for x and y , in terms of t and k . [5]

Consider the case $k = 6$.

- (iv)** Determine whether the model predicts that species X or species Y dies out first. State the value of t at which this first species dies out. [7]

- (v)** Comment on why the time predicted by the model for the second species to die out is unreliable. [1]

19 The simultaneous differential equations

$$\frac{dx}{dt} = -\frac{1}{2}x - \frac{3}{2}y + t$$

$$\frac{dy}{dt} = \frac{3}{2}x - \frac{1}{2}y + 2t$$

are to be solved.

(i) Eliminate y to obtain a second order differential equation for x in terms of t . Hence find the general solution for x . **[13]**

(ii) Find the corresponding general solution for y . **[4]**

When $t = 0$, $x = 1$ and $y = 0$.

(iii) Find the particular solutions. **[3]**

(iv) Show that in this case $x + y$ tends to a finite limit as $t \rightarrow \infty$ and state its value. Determine whether $x + y$ is equal to this limit for any values of t . **[4]**

- 20 Two species of small rodent, X and Y, compete for survival in the same environment. The populations of the species, at time t years, are x and y respectively and they are modelled by the simultaneous differential equations

$$\frac{dx}{dt} = 2(x - y),$$
$$\frac{dy}{dt} = \frac{3}{8}(x - 80e^{-\frac{1}{2}t}).$$

- (i) Show that

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + \frac{3}{4}x = 60e^{-\frac{1}{2}t}.$$

Find the general solution for x . [10]

- (ii) Find the corresponding general solution for y . [3]

When $t = 0$, $x = 40$ and $y = 50$.

- (iii) Find the particular solutions for x and y . [4]

- (iv) Find the time T at which the model predicts that the rodents of species X will die out. Find the population of species Y predicted at this time. [6]

- (v) Comment on the suitability of the model for times greater than T . [1]

21 The simultaneous differential equations

$$\frac{dx}{dt} = -x + 2y$$

$$\frac{dy}{dt} = -x - 4y + e^{-2t}$$

are to be solved.

(i) Eliminate y to obtain a second order differential equation for x in terms of t . Hence find the general solution for x . **[14]**

(ii) Find the corresponding general solution for y . **[3]**

Initially $x = 5$ and $y = 0$.

(iii) Find the particular solutions. **[4]**

(iv) Show that $\frac{y}{x} \rightarrow -\frac{1}{2}$ as $t \rightarrow \infty$. Show also that there is no value of t for which $\frac{y}{x} = -\frac{1}{2}$. **[3]**

22 The simultaneous differential equations

$$\frac{dx}{dt} = 7x + 2y + 13e^{4t},$$

$$\frac{dy}{dt} = -9x + y + e^{7t}$$

are to be solved.

- (i) Eliminate x to obtain a second order differential equation for y in terms of t . Hence find the general solution for y . [12]
- (ii) Given that $y = -3$ and $\frac{dy}{dt} = 60$ when $t = 0$, find the particular solution for y . [4]
- (iii) Find the corresponding particular solution for x . [2]
- (iv) Find the smallest positive value of t for which $y = 0$. [4]
- (v) Show that $\frac{y}{x} \rightarrow 0$ as $t \rightarrow \infty$. [2]

- 23** A children's play centre has two rooms, a room full of bouncy castles and a room full of ball pits. At any given instant, each child in the centre is playing either on the bouncy castles or in the ball pits. Each child can see one room from the other room and can decide to change freely between the two rooms. It is assumed that such changes happen instantaneously.

The number of children playing on the bouncy castles at time t hours, is denoted by C and the corresponding number of children playing in the ball pits is P . Because the number of children is large for most of the time, C and P are modelled as being continuous.

When there is a different number of children in each room, some children will move from the room with more children to the room with fewer children. A researcher therefore decides to model C and P with the following coupled differential equations.

$$\frac{dP}{dt} = \alpha(P - C) + \gamma t$$

$$\frac{dC}{dt} = \alpha(C - P)$$

- (a)** Explain why α must be negative. [1]

After examining data, the researcher chooses $\alpha = -2$ and $\gamma = 32$.

- (b)** Show that P satisfies the second order differential equation $\frac{d^2P}{dt^2} + 4\frac{dP}{dt} = 64t + 32$. [2]

- (c) (i)** Find the complementary function for the differential equation from part **(b)**. [1]

- (ii)** Explain why a particular integral of the form $P = at + b$ will not work in this situation. [1]

- (iii)** Using a particular integral of the form $P = at^2 + bt$, find the general solution of the differential equation from part **(b)**. [3]

At a certain time there are 55 children playing in the ball pits and 24 children per hour are arriving at the ball pits.

- (d)** Use the model, starting from this time, to estimate the number of children in the ball pits 30 minutes later. [4]

- (e)** Explain why the model becomes unreliable as t gets very large. [1]

- 24** During an industrial process substance X is converted into substance Z . Some of the substance X goes through an intermediate phase, and is converted to substance Y , before being converted to substance Z . The situation is modelled by

$$\frac{dy}{dt} = 0.3x - 0.2y \quad \text{and} \quad \frac{dz}{dt} = 0.2y + 0.1x$$

where x , y and z are the amounts in kg of X , Y and Z at time t hours after the process starts.

Initially there is 10 kg of substance X and nothing of substances Y and Z . The amount of substance X decreases exponentially. The initial rate of decrease is 4 kg per hour.

- (i) Show that $x = Ae^{-0.4t}$, stating the value of A . [3]
- (ii) (a) Show that $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 0$. [2]
- (b) Comment on this result in the context of the industrial process. [2]
- (iii) Express y in terms of t . [5]
- (iv) Determine the maximum amount of substance Y present during the process. [3]
- (v) How long does it take to produce 9 kg of substance Z ? [2]

25 Two similar species, X and Y, of a small mammal compete for food and habitat. A model of this competition assumes, in a particular area, the following.

- In the absence of the other species, each species would increase at a rate proportional to the number present with the same constant of proportionality in each case.
- The competition reduces the rate of increase of each species by an amount proportional to the number of the other species present.

So if the numbers of species X and Y present at time t years are x and y respectively, the model gives the differential equations

$$\frac{dx}{dt} = kx - ay \quad \text{and} \quad \frac{dy}{dt} = ky - bx,$$

where k , a and b are positive constants.

(a) (i) Show that the general solution for x is $x = Ae^{(k+n)t} + Be^{(k-n)t}$, where $n = \sqrt{ab}$ and A and B are arbitrary constants. [6]

(ii) Hence find the general solution for y in terms of A , B , k , n , a and t . [2]

Observations suggest that suitable values for the model are $k = 0.015$, $a = 0.04$ and $b = 0.01$. You should use these values in the rest of this question.

(b) When $t = 0$, the numbers present of species X and Y in this area are x_0 and y_0 respectively.

(i) Show that $x = \frac{1}{2}(x_0 - 2y_0)e^{0.035t} + \frac{1}{2}(x_0 + 2y_0)e^{-0.005t}$. [3]

(ii) Hence show that $y = \frac{1}{4}(x_0 + 2y_0)e^{-0.005t} - \frac{1}{4}(x_0 - 2y_0)e^{0.035t}$. [1]

(c) Use initial values $x_0 = 500$ and $y_0 = 300$ with the results in part (b) to determine what the model predicts for each of the following questions.

(i) What numbers of each species will be present after 25 years? [2]

(ii) **In this question you must show detailed reasoning.**

When will the numbers of the two species be equal? [4]

(iii) Does either species ever disappear from the area? Justify your answer. [3]

(d) Different initial values will apply in other areas where the two species compete, but previous studies indicate that one species or the other will eventually dominate in any given area.

(i) Identify a relationship between x_0 and y_0 where the model does **not** predict this outcome. [1]

(ii) Explain what the model predicts in the long term for this exceptional case. [2]

- 26** A biologist is studying the effect of pesticides on crops. On a certain farm pesticide is regularly applied to a particular crop which grows in soil. Over time, pesticide is transferred between the crop and the soil at a rate which depends on the amount of pesticide in both the crop and the soil. The amount of pesticide in the crop after t days is x grams. The amount of pesticide in the soil after t days is y grams. Initially, when $t = 0$, there is no pesticide in either the crop or the soil.

At first it is assumed that no pesticide is lost from the system. The biologist further assumes that pesticide is added to the crop at a constant rate of k grams per day, where $k > 6$.

After collecting some initial data, the biologist suggests that for $t \geq 0$, this situation can be modelled by the following pair of first order linear differential equations.

$$\frac{dx}{dt} = -2x + 78y + k$$

$$\frac{dy}{dt} = 2x - 78y$$

(a) (i) Show that $\frac{d^2x}{dt^2} + 80\frac{dx}{dt} = 78k$. **[2]**

(ii) Determine the particular solution for x in terms of k and t . **[7]**

If more than 250 grams of pesticide is found in the crop, then it will fail food safety standards.

- (iii)** The crop is tested 50 days after the pesticide is first added to it.

Explain why, according to this model, the crop will fail food safety standards as a result of this test. **[1]**

Further data collection suggests that some pesticide decays in the soil and so is lost from the system. The model is refined in light of this data. The particular solution for x for this refined model is

$$x = k \left(20 - e^{-41t} \left(20 \cosh(\sqrt{1677}t) + \frac{819}{\sqrt{1677}} \sinh(\sqrt{1677}t) \right) \right)$$

- (b)** Given now that $k < 12$, determine whether the crop will fail food safety standards in the long run according to this refined model. **[2]**

In the refined model, it is still assumed that pesticide is added to the crop at a constant rate.

- (c)** Suggest a reason why it might be more realistic to model the addition of pesticide as not being at a constant rate. **[1]**