London Academy of Maths

Hyperbolic Equations and Identities

Mark Scheme

Question Number	Scheme	Notes	Marks
1	$7\cosh x + 3\sinh x = 2e^x + 7 \Rightarrow$ $7\left(\dfrac{e^x + e^{-x}}{2}\right) + 3\left(\dfrac{e^x - e^{-x}}{2}\right) = 2e^x + 7$ $\left\{\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7\right\}$	Substitutes at least one correct exponential form for either of the hyperbolic terms and achieves an **equation** in exponentials and constants alone	M1
	$\Rightarrow 7\left(e^{2x}+1\right) + 3\left(e^{2x}-1\right) = 4e^{2x} + 14e^x$ $\left\{\Rightarrow 5e^{2x} + 2 = 2e^{2x} + 7e^x\right\}$	Multiplies through by e^x to obtain any equation that would form a 3TQ in e^x if like terms were collected	M1
	$\Rightarrow 6e^{2x} - 14e^x + 4 = 0 \quad \left\{3e^{2x} - 7e^x + 2 = 0\right\}$	A correct three term quadratic in e^x. Could be implied by a correct root even if terms have not been collected.	A1
	$\Rightarrow \left(3e^x - 1\right)\left(e^x - 2\right) = 0 \Rightarrow e^x = \ldots$	Solves their 3TQ - usual rules. One correct root for their quadratic if no working. Ignore labelling of the roots even if e.g.,"x" is used.	M1
	$x = \ln 2,\ \ln\dfrac{1}{3}$	Both correct and simplified but do not isw if there are **other answers.** Allow $-\ln\dfrac{1}{2}$ for ln 2 and –ln 3 or ln 3^{-1} for $\ln\dfrac{1}{3}$	A1
	Answer only is 0/5		**Total 5**
	Note that it is possible to multiply through by e^{-x} to form an equation in e^{-2x}, e^{-x} and constants. Score as main scheme, e.g., $\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7$ $\Rightarrow \frac{7}{2} + \frac{7}{2}e^{-2x} + \frac{3}{2} - \frac{3}{2}e^{-2x} = 2 + 7e^{-x}$ (M1) $\Rightarrow 2e^{-2x} - 7e^{-x} + 3 = 0$ (A1) $\left(2e^{-x}-1\right)\left(e^{-x}-3\right) = 0 \Rightarrow e^{-x} = \frac{1}{2},\ 3$ (M1) $\Rightarrow e^x = 2,\ \frac{1}{3} \Rightarrow x = \ln 2,\ \ln\frac{1}{3}$ (A1)		

Question Number	Scheme	Notes	Marks
2	$18\cosh x + 14\sinh x = 11 + e^x$		
	$18\left(\frac{e^x + e^{-x}}{2}\right) + 14\left(\frac{e^x - e^{-x}}{2}\right) = 11 + e^x$	Uses the correct exponential forms	M1
	$9e^{2x} + 9 + 7e^{2x} - 7 = 11e^x + e^{2x}$		
	$15e^{2x} - 11e^x + 2 \ (= 0)$ or $15e^x - 11 + 2e^{-x} \ (= 0)$	M1: Collects terms to obtain a 3 term equation A1:Correct equation in either of the forms shown	M1A1
	$(5e^x - 2)(3e^x - 1) = 0 \Rightarrow e^x = ...$ or $\left(5e^{\frac{x}{2}} - 2e^{\frac{-x}{2}}\right)\left(3e^{\frac{x}{2}} - e^{\frac{-x}{2}}\right)$ or $(5e^x - 2)(3 - e^{-x})$	Attempt to solve their 3TQ Depends on the second M mark	dM1
	$x = \ln\frac{2}{5}, \ \ln\frac{1}{3}$	Both; $\ln\frac{2}{5}$ or $\ln 0.4$; $\ln\frac{1}{3}$ or ln0.3rec -ln3 scores A0	A1
			(5)
			Total 5

Question Number	Scheme		Marks
3.(a)	$\{\frac{1}{2}(e^{x}+e^{-x})\}^{2}-\{\frac{1}{2}(e^{x}-e^{-x})\}^{2}=\{\frac{1}{4}(e^{2x}+2+e^{-2x})\}-\{\frac{1}{4}(e^{2x}-2+e^{-2x})\}$		M1
	M1: Uses the correct exponential forms for cosh and sinh and squares both brackets obtaining 3 terms each time		
	$\frac{1}{2}+\frac{1}{2}=1$	**At least one line of intermediate working** (e.g. combines fractions with a common denominator) with no errors seen and concludes = 1	A1
			(2)
(b)	$(e^{x}-e^{-x})+7\times\frac{1}{2}(e^{x}+e^{-x})=9$ $\Rightarrow \frac{9}{2}e^{x}+\frac{5}{2}e^{-x}-9=0$	M1: Uses exponential forms **and** collects terms A1: Any correct form with terms collected	M1A1
	$\Rightarrow 9e^{2x}-18e^{x}+5=0$ so $e^{x}=\ldots$	Solves their three term quadratic in e^x as far as $e^x =$	M1
	$e^{x}=\frac{1}{3}$ or $\frac{5}{3}$	Both values correct	A1
	$x=\ln\frac{1}{3}$ and $\ln\frac{5}{3}$	Both values correct (accept equivalents)	A1
			(5)
			Total 7
	Alternatives for (b) – Special Cases		
Way 2	$2\sinh x=9-7\cosh x\Rightarrow 45\cosh^{2}x-126\cosh x+85=0$		M1A1
	M1: Attempt to square both sides A1: Correct quadratic in cosh$x$		
	$(15\cosh x-17)(3\cosh x-5)=0\Rightarrow \cosh x=\frac{17}{15}$ or $\cosh x=\frac{5}{3}$		
	$\frac{e^{x}+e^{-x}}{2}=\frac{17}{15}\Rightarrow 15e^{2x}-34e^{x}+15=0,\ \frac{e^{x}+e^{-x}}{2}=\frac{5}{3}\Rightarrow 3e^{2x}-10e^{x}+3=0$		
	$(5e^{x}-3)(3e^{x}-5)=0\Rightarrow e^{x}=\frac{3}{5},\ e^{x}=\frac{5}{3}$ $(3e^{x}-1)(e^{x}-3)=0\Rightarrow e^{x}=\frac{1}{3},\ e^{x}=3$ $e^{x}=\frac{5}{3}$ and $e^{x}=\frac{1}{3}$	M1: Solves at least one of their three term quadratics in e^x as far as $e^x =$…., having used the correct exponential form for cosh$x$ A1: $e^{x}=\frac{5}{3}$ and $e^{x}=\frac{1}{3}$ seen	M1A1
	$x=\ln\frac{1}{3}$ and $\ln\frac{5}{3}$	These values only with $\ln 3$ and $\ln\frac{3}{5}$ rejected	A1
Way 3	$7\cosh x=9-2\sinh x\Rightarrow 45\sinh^{2}x+36\sinh x-32=0$		
	M1: Attempt to square both sides A1: Correct quadratic in sinh$x$		M1A1
	$(15\sinh x-8)(3\sinh x+4)=0\Rightarrow \sinh x=\frac{8}{15}$ or $\sinh x=-\frac{4}{3}$		
	$\frac{e^{x}-e^{-x}}{2}=\frac{8}{15}\Rightarrow 15e^{2x}-16e^{x}-15=0,\ \frac{e^{x}-e^{-x}}{2}=-\frac{4}{3}\Rightarrow 3e^{2x}+8e^{x}-3=0$		
	$(3e^{x}-5)(5e^{x}+3)=0\Rightarrow e^{x}=\frac{5}{3},\ e^{x}=-\frac{3}{5}$ $(3e^{x}-1)(e^{x}+3)=0\Rightarrow e^{x}=\frac{1}{3},\ e^{x}=-3$ $e^{x}=\frac{5}{3}$ and $e^{x}=\frac{1}{3}$	M1: Solves at least one of their three term quadratics in e^x as far as $e^x =$…., having used the correct exponential form for sinh$x$ A1: $e^{x}=\frac{5}{3}$ and $e^{x}=\frac{1}{3}$ seen	M1A1
	$x=\ln\frac{1}{3}$ and $\ln\frac{5}{3}$	These values only	A1
	Note: For these special cases, if they use the ln form of arcosh or arsinh from their cosh = … or sinh = … then only the first 2 marks are available as they are not using exponentials.		

Question Number	Scheme	Notes	Marks
4	$15\text{sech}^2 x + 7\tanh x = 13$		
	$15\left(1-\tanh^2 x\right)+7\tanh x = 13$	Uses $\text{sech}^2 x = 1-\tanh^2 x$	M1
	$15\tanh^2 x - 7\tanh x - 2 = 0$	Correct 3 term quadratic, terms in any order	A1
	$\left(5\tanh x+1\right)\left(3\tanh x-2\right)=0$ $\Rightarrow \tanh x = -\frac{1}{5},\ \frac{2}{3}$	M1: Solves their 3 term quadratic to obtain at least one value for tanh$x$ Correct answers implies method A1: Both correct values If solved by formula accept $\frac{7\pm 13}{30}$	M1A1
	$x=\frac{1}{2}\ln\frac{2}{3},\ \ \frac{1}{2}\ln 5$	A1: One correct exact answer A1: Both exact answers correct Allow equivalent answers e.g. $x=\frac{1}{2}\ln 2-\frac{1}{2}\ln 3, \ln\frac{\sqrt{6}}{3}, \ln\sqrt{\frac{2}{3}}, \ln\sqrt{5}$ etc	A1, A1
			(6)
			Total 6
	Alternative Using Exponentials		
	$15\left(\frac{2}{e^x+e^{-x}}\right)^2+7\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)=13$	Substitutes the correct exponential forms The equation may have been re-arranged before substitution. ½s may have been cancelled.	M1
	$6e^{2x}-34+20e^{-2x}=0$	Correct 3 term quadratic in e^{2x}	A1
	$3e^{4x}-17e^{2x}+10=0$		
	$\left(3e^{2x}-2\right)\left(e^{2x}-5\right)=0$ or $\left(3e^x-2e^{-x}\right)\left(e^x-5e^{-x}\right)=0$ $\Rightarrow e^{2x}=\frac{2}{3}$ or 5	M1: Solves their 3 term quadratic to obtain at least one value for e^{2x} A1: Both correct values	M1A1
	$x=\frac{1}{2}\ln\frac{2}{3},\ \ \frac{1}{2}\ln 5$	A1: One correct answer A1: Both answers correct Allow equivalent answers e.g. $x=\frac{1}{2}\ln 2-\frac{1}{2}\ln 3$	A1, A1

Solving quadratics by calculator: check their solutions if the equation is incorrect. If the solution is correct for their equation, award M1

Question Number	Scheme	Notes	Marks
5 **Way 1** **Converts to sinh and cosh**	$4\tanh x - \text{sech}\, x = 1$ $4\dfrac{\sinh x}{\cosh x} - \dfrac{1}{\cosh x} = 1$ $4\sinh x - 1 - \cosh x = 0$ $4\dfrac{e^x - e^{-x}}{2} - 1 - \dfrac{e^x + e^{-x}}{2} = 0$	Replaces **one** hyperbolic function with its correct exponential equivalent. Allow for correct replacement of just e.g., $\sinh x$ after using $\tanh x = \dfrac{\sinh x}{\cosh x}$ May follow errors but do not allow any further marks if the original equation was reduced to one in a single hyperbolic function.	M1
	$3e^{2x} - 2e^x - 5 = 0$	M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in e^x A1: Correct 3TQ	M1 A1
	$e^x = \dfrac{2 \pm \sqrt{4+60}}{6} \left(\Rightarrow \dfrac{2+8}{6} = \dfrac{5}{3} \right)$	M1: Solves 3TQ (or 2TQ with no constant) in e^x. Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation. A1: Any correct unsimplified expression for e^x that includes the positive root. Must be exact	M1 A1
	$x = \ln\dfrac{5}{3}$	$\ln\dfrac{5}{3}$, $\ln 1\dfrac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = \ldots$ No unrejected extra solutions	A1
			Total 6
Way 2 **Straight to e^x**	$4\dfrac{e^x - e^{-x}}{e^x + e^{-x}} - \dfrac{2}{e^x + e^{-x}} = 1$	Replaces **one** hyperbolic function with its correct exponential equivalent	M1
	$3e^{2x} - 2e^x - 5 = 0$	M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in e^x A1: Correct 3TQ	M1 A1
	$e^x = \dfrac{2 \pm \sqrt{4+60}}{6} \left(\Rightarrow \dfrac{2+8}{6} = \dfrac{5}{3} \right)$	M1: Solves 3TQ (or 2TQ with no constant) in e^x. Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation. A1: Any correct unsimplified expression for e^x that includes the positive root. Must be exact	M1 A1
	$x = \ln\dfrac{5}{3}$	$\ln\dfrac{5}{3}$, $\ln 1\dfrac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = \ldots$ No unrejected extra solutions	A1
			Total 6
	In Ways 1 & 2, if they form an equation which is not a quadratic in e^x they must achieve the correct exact root of $\dfrac{5}{3}$ to access the middle four marks		

Question Number	Scheme	Notes	Marks
5 **Way 3a** **Squaring (sinh)**	$4\sinh x - 1 = \cosh x$ $16\sinh^2 x - 8\sinh x + 1 = \cosh^2 x$ $16\sinh^2 x - 8\sinh x + 1 = 1 + \sinh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in sinh x	M1
	$15\sinh^2 x - 8\sinh x = 0$	M1: Obtains a 2TQ with no constant or 3TQ in sinh x A1: Correct 2TQ	M1 A1
	$\sinh x = \frac{8}{15}$	Solves 2TQ (with no constant) or 3TQ in sinh x. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \text{arsinh}\frac{8}{15} = \ln\left(\frac{8}{15} + \sqrt{\left(\frac{8}{15}\right)^2 + 1}\right)$ or $15e^{2x} - 16e^x - 15 = 0 \Rightarrow$ $e^x = \frac{16 \pm \sqrt{256 + 900}}{30}$	A correct unsimplified expression for x as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = ...$ No unrejected extra solutions	A1
			Total 6
Way 3b **Squaring (sech)**	$4\tanh x = 1 + \text{sech}\, x$ $16\tanh^2 x = 1 + 2\text{sech}\, x + \text{sech}^2 x$ $16\left(1 - \text{sech}^2 x\right) = 1 + 2\text{sech}\, x + \text{sech}^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in sech x	M1
	$17\text{sech}^2 x + 2\text{sech}\, x - 15 = 0$	M1: Obtains a 2TQ (with no constant) or 3TQ in sech x A1: Correct 3TQ	M1 A1
	$\left(17\text{sech}\, x - 15\right)(\text{sech}\, x + 1) = 0$ $\text{sech}\, x = \frac{15}{17}$	Solves 2TQ with no constant or 3TQ in sech x. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \text{arcosh}\frac{17}{15} = \ln\left(\frac{17}{15} + \sqrt{\left(\frac{17}{15}\right)^2 - 1}\right)$ or $15e^{2x} - 34e^x + 15 = 0 \Rightarrow$ $e^x = \frac{34 \pm \sqrt{1156 - 900}}{30}$	A correct unsimplified expression for x as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = ...$ No unrejected extra solutions	A1
			Total 6

Question Number	Scheme	Notes	Marks
5 **Way 3c** **Squaring (tanh)**	$4\tanh x-1=\text{sech}\,x$ $16\tanh^2 x-8\tanh x+1=\text{sech}^2 x$ $16\tanh^2 x-8\tanh x+1=1-\tanh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in tanh *x*	M1
	$17\tanh^2 x-8\tanh x=0$	M1: Obtains a 2TQ with no constant or 3TQ in tanh *x* A1: Correct 2TQ	M1 A1
	$\tanh x=\frac{8}{17}$	Solves 2TQ with no constant or 3TQ in tanh *x*. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x=\text{artanh}\frac{8}{17}=\frac{1}{2}\ln\left(\frac{1+\frac{8}{17}}{1-\frac{8}{17}}\right)$ or $9e^{2x}-25=0\Rightarrow$ $e^x=\frac{5}{3}$	A correct unsimplified expression for *x* as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x=\ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow *k* = ... No unrejected extra solutions	A1
			Total 6
Way 3d **Squaring (cosh)**	$4\sinh x=1+\cosh x$ $16\sinh^2 x=1+2\cosh x+\cosh^2 x$ $16\cosh^2 x-16=1+2\cosh x+\cosh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in cosh *x*	M1
	$15\cosh^2 x-2\cosh x-17=0$	M1: Obtains a 2TQ with no constant or 3TQ in cosh *x* A1: Correct 3TQ	M1 A1
	$(15\cosh x-17)(\cosh x+1)=0$ $\cosh x=\frac{17}{15}$	Solves 2TQ (with no constant) or 3TQ in cosh *x*. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x=\text{arcosh}\frac{17}{15}=\ln\left(\frac{17}{15}+\sqrt{\left(\frac{17}{15}\right)^2-1}\right)$ or $15e^{2x}-34e^x+15=0\Rightarrow$ $e^x=\frac{34\pm\sqrt{1156-900}}{30}$	A correct unsimplified expression for *x* as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x=\ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow *k* = ... No unrejected extra solutions	A1
			Total 6

Question Number	Scheme	Notes	Marks
6	$\coth^2 x + 5\text{cosech}x + 5 = 0$		
Way 1	$\text{cosech}^2 x + 1 + 5\text{cosech}x + 5$	Uses $\coth^2 x = \pm\,\text{cosech}^2 x \pm 1$	M1
	$\text{cosech}^2 x + 5\text{cosech}x + 6 = 0$	Correct quadratic	A1
	$(\text{cosech}x + 3)(\text{cosech}x + 2) = 0$ $\Rightarrow \text{cosech}x = -3,\ -2$	M1: Solves their 3 term quadratic A1: Both correct values	M1A1
	$x = \ln\left(\dfrac{\sqrt{10}-1}{3}\right),\ \ln\left(\dfrac{\sqrt{5}-1}{2}\right)$	A1: One correct answer A1: Both answers correct with no errors.	A1, A1
			(6)
			Total 6
Way 2	$\coth^2 x + 5\text{cosech}x + 5 = 0$ $\Rightarrow \cosh^2 x + 5\sinh^2 x + 5\sinh x = 0$	M1: $\times\sinh^2 x$ and uses $\cosh^2 x = \pm 1 \pm \sinh^2 x$	M1
	$\Rightarrow 6\sinh^2 x + 5\sinh x + 1 = 0$	A1: Correct quadratic	A1
	$(3\sinh x + 1)(2\sinh x + 1) = 0$ $\Rightarrow \sinh x = -\dfrac{1}{3},\ -\dfrac{1}{2}$	M1: Solves their 3 term quadratic A1: Both correct values	M1A1
	$x = \ln\left(\dfrac{\sqrt{10}-1}{3}\right),\ \ln\left(\dfrac{\sqrt{5}-1}{2}\right)$	A1: One correct answer A1: Both answers correct with no errors.	A1A1
Way 3	$\dfrac{(e^x + e^{-x})^2}{(e^x - e^{-x})^2} + \dfrac{5\times 2}{e^x - e^{-x}} + 5 = 0$	Substitutes the correct exponential forms	M1
	$3e^{4x} + 5e^{3x} - 4e^{2x} - 5e^x + 3 = 0$	Correct quartic in e^x	A1
	$(3e^{2x} + 2e^x - 3)(e^{2x} + e^x - 1) = 0$ $e^x = \dfrac{-2\pm\sqrt{40}}{6},\ \dfrac{-1\pm\sqrt{5}}{2}$	M1: Solves their 3 term quadratic which must have come from their quartic. A1: Both correct values	M1A1
	$x = \ln\left(\dfrac{\sqrt{10}-1}{3}\right),\ \ln\left(\dfrac{\sqrt{5}-1}{2}\right)$	A1: One correct answer A1: Both answers correct with no errors.	A1A1

Notes

If more than two solutions are found then final A0.

Question Number	Scheme	Notes	Marks
7(a)	$1-\tanh^2 x \equiv \text{sech}^2 x$		
	$1-\tanh^2 x = 1-\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2$	Replaces the tanh x on the lhs with a **correct** expression in terms of exponentials.	B1
	$=\dfrac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}=\dfrac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}$ or e.g. $\dfrac{2e^{2x}\times 2e^{-2x}}{(e^x+e^{-x})^2}$ Attempts to find common denominator and expand numerator		M1
	$=\left(\dfrac{4}{(e^x+e^{-x})^2}\right)=\text{sech}^2 x*$	Obtains the rhs with no errors.	A1cso
			(3)
ALT 1	$1-\tanh^2 x=(1-\tanh x)(1+\tanh x)$ $=\left(1-\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)\right)\left(1+\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)\right)$	Uses the difference of 2 squares on the lhs and replaces the tanh x with a **correct** expression in terms of exponentials.	B1
	$=\left(\dfrac{2e^{-x}}{e^x+e^{-x}}\right)\left(\dfrac{2e^x}{e^x+e^{-x}}\right)$	Attempt to find common denominators and simplify numerators.	M1
	$=\left(\dfrac{4}{(e^x+e^{-x})^2}\right)=\text{sech}^2 x*$	Obtains the rhs with no errors.	A1cso
ALT 2	$\text{sech}^2 x=\dfrac{4}{(e^x+e^{-x})^2}$	Replaces the sech x on the rhs with a **correct** expression in terms of exponentials.	B1
	$=\dfrac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}=\dfrac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ Attempts to express the "4" in terms of the denominator.		M1
	$=1-\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2=1-\tanh^2 x*$	Obtains the lhs with no errors.	A1cso

(b)	$2\operatorname{sech}^2 x + 3\tanh x = 3 \Rightarrow 2(1-\tanh^2 x) + 3\tanh x = 3$ $\Rightarrow 2\tanh^2 x - 3\tanh x + 1 = 0$ Uses $\operatorname{sech}^2 x = 1-\tanh^2 x$ and forms a 3 term quadratic in tanh x		M1
	$(2\tanh x - 1)(\tanh x - 1) = 0 \Rightarrow \tanh x = ...$	Solves 3TQ by any valid method including calculator.	M1
	$\tanh x = \frac{1}{2} \to x = \ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			(3)
ALT	$2\operatorname{sech}^2 x + 3\tanh x = 3 \Rightarrow 2\left(\frac{4}{(e^x+e^{-x})^2}\right) + 3\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right) = 3$ $\Rightarrow 8 + 3(e^{2x} - e^{-2x}) = 3(e^{2x} + 2 + e^{-2x}) \Rightarrow ...$ Substitutes the correct exponential forms, attempts to eliminate fractions and collect terms		M1
	$6e^{-2x} = 2 \Rightarrow e^{-2x} = \frac{1}{3}$	Rearranges to reach $e^{-2x} = ...$	M1
	$x = \ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			Total 6

Question Number	Scheme		Marks	
Q8	(a) $2\sinh x\cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$			
	$= \frac{e^{2x} - e^{-2x}}{2}$		M1	
	$= \sinh 2x$ *	cso	A1	(2)
	(b) $x = 0$		B1	
	$2\sinh x\cosh x = 6\sinh^2 x + 7\sinh x$		M1	
	$2\cosh x = 6\sinh x + 7$			
	$e^x + e^{-x} = 3e^x - 3e^{-x} + 7$		M1	
	$2e^{2x} + 7e^x - 4 = 0$		A1	
	$(2e^x - 1)(e^x + 4) = 0$		M1	
	$e^x = \frac{1}{2}$		A1	
	$x = -\ln 2$	accept $\ln\frac{1}{2}$	A1	(7)
				[9]

Question Number	Scheme	Notes	Marks
9.	$1+2\sinh^2 x-7\sinh x=5$	Replaces cosh2x by $1+2\sinh^2 x$ or replaces cosh2x with $\cosh^2 x+\sinh^2 x$ and then $\cosh^2 x$ with $1+\sinh^2 x$. There must be no incorrect identities used.	M1
	$2\sinh^2 x-7\sinh x-4=0$	Correct quadratic	A1
	$(2\sinh x+1)(\sinh x-4)=0\Rightarrow \sinh x=$	Attempt to solve 3TQ in sinh$x$ (usual rules)	M1
	$\sinh x=-\frac{1}{2},4$	Both (allow un-simplified e.g. $\frac{7\pm 9}{4}$)	A1
	$\text{arsinh}x=\ln\left(x+\sqrt{x^2+1}\right)$	Use of the **correct** log form of arsinh	M1
	This mark may also be gained by using the exponential form of sinh$x$ and attempting to solve to give x in terms of ln		
	$x=\ln\left(-\frac{1}{2}+\sqrt{\frac{5}{4}}\right),\ \ln\left(4+\sqrt{17}\right)$	A1: One correct exact value of x. Allow equivalent exact answers which may be un-simplified. A1: Both values correct and exact and no incorrect values. Allow equivalent exact answers which may be un-simplified. **Condone missing brackets.**	A1, A1
	Correct work giving $x=\ln\left(-\frac{1}{2}\pm\sqrt{\frac{5}{4}}\right),\ \ln\left(4\pm\sqrt{17}\right)$ would generally lose the final mark		
			(7)
			Total 7
	Alternative:		
	$\left(\frac{e^{2x}+e^{-2x}}{2}\right)-7\left(\frac{e^{x}-e^{-x}}{2}\right)=5$	M1: Substitutes the correct exponential definitions for cosh2x and sinh$x$ A1: Correct expression	M1A1
	$e^{4x}-7e^{3x}-10e^{2x}-7e^{x}+1=0$	M1: Multiplies by e^{2x} A1: Correct quartic in e^x	M1A1
	$\left(e^{2x}+e^{x}-1\right)\left(e^{2x}-8e^{x}-1\right)=0\Rightarrow e^{x}=...$ $\Rightarrow x=...$	Solves their quartic as far as $e^x=$ ... and then converts to give x in terms of ln. There must be a recognisable attempt to solve a quartic with at least 4 terms as e.g. the product of two 3TQ's in e^x.	M1
	$x=\ln\left(-\frac{1}{2}+\sqrt{\frac{5}{4}}\right),\ \ln\left(4+\sqrt{17}\right)$	A1: One correct exact value of x. Allow equivalent exact answers which may be un-simplified. A1: Both values correct and exact and no incorrect values. Allow equivalent exact answers which may be un-simplified. **Condone missing brackets**	A1, A1

Question Number	Scheme	Notes	Marks
10(a)	$4\sinh^3 x + 3\sinh x = 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$ $= 4\left(\frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{8}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right)$ Uses $\sinh x = \frac{e^x - e^{-x}}{2}$ on both sinh terms and attempts to cube the bracket (min accepted is a linear x a quadratic bracket)		M1
	$= \frac{1}{2}e^{3x} - \frac{3}{2}e^x + \frac{3}{2}e^{-x} - \frac{1}{2}e^{-3x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x}$ $= \frac{e^{3x} - e^{-3x}}{2} = \sinh 3x$*		A1*
			(2)
(b)	$\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x + 3\sinh x = 19\sinh x$ $\Rightarrow 4\sinh^3 x - 16\sinh x = 0$ Uses the result from (a) and combines terms		M1
	($\sinh x = 0$ or) $\sinh^2 x = 4$	$\sinh^2 x = 4$ or $\sinh x = (\pm)2$	A1
	(0, 0)	States the origin as one intersection	B1
	$\ln\left(2+\sqrt{5}\right)$ **and** $-\ln\left(2+\sqrt{5}\right)$	Two correct non-zero x values(allow e.g. $\ln\left(-2+\sqrt{5}\right)$ for $-\ln\left(2+\sqrt{5}\right)$)	A1
	$\left(\ln\left(2+\sqrt{5}\right), 38\right)$ **and** $\left(-\ln\left(2+\sqrt{5}\right), -38\right)$	Two correct **points** (allow e.g. $\ln\left(-2+\sqrt{5}\right)$ for $-\ln\left(2+\sqrt{5}\right)$)	A1
			(5)
	Alternative for (b) using exponentials		
	$\sinh 3x = 19\sinh x \Rightarrow \frac{e^{3x} - e^{-3x}}{2} = \frac{19\left(e^x - e^{-x}\right)}{2} \Rightarrow \ldots$ Substitutes the correct exponential forms and collects terms to one side		M1
	$\Rightarrow e^{6x} - 19e^{4x} + 19e^{2x} - 1 = 0$	Correct equation (or equivalent)	A1
	(0, 0)	States the origin as one intersection	B1
	$\frac{1}{2}\ln\left(9+4\sqrt{5}\right)$ **or** $\frac{1}{2}\ln\left(9-4\sqrt{5}\right)$	Two correct non-zero x values (oe)	A1
	$\left(\frac{1}{2}\ln\left(9+4\sqrt{5}\right), 38\right)$ **and** $\left(\frac{1}{2}\ln\left(9-4\sqrt{5}\right), -38\right)$	Two correct **points** (oe)	A1
			Total 7

Question Number	Scheme		Marks
11 (a)(i)	$2\cosh^2 x - 1 = 2\frac{\left(e^x + e^{-x}\right)^2}{4} - 1 = \frac{\left(e^{2x} + 2e^x \times e^{-x} + e^{-2x}\right)}{2} - 1$ Substitutes the correct definition for cosh$x$ into the rhs and squares - full expansion must be seen but allow 2 for $2e^x \times e^{-x}$		M1
	$= \frac{\left(e^{2x} + e^{-2x}\right)}{2} + 1 - 1 = \cosh 2x*$	Correct completion with no errors seen.	A1
	Working from left to right:		
	$\cosh 2x = \frac{\left(e^{2x} + e^{-2x}\right)}{2} = \frac{\left(e^x + e^{-x}\right)^2 - 2}{2}$ Uses the correct definition for cosh2x on lhs and expresses in terms of $\left(e^x + e^{-x}\right)^2$.		M1
	$2\cosh^2 x - 1*$	Correct completion with no errors seen.	A1
(ii)	$2\sinh x \cosh x = 2\frac{\left(e^x - e^{-x}\right)}{2} \times \frac{\left(e^x + e^{-x}\right)}{2} = ...$ Use both correct definitions on rhs and attempts to multiply $2\sinh x \cosh x = \frac{1}{2}\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right) = ...\text{scores M0}$ as the definitions for sinh$x$ and cosh$x$ have not been seen		M1
	$\frac{\left(e^{2x} - e^{-2x}\right)}{2} = \sinh 2x*$	Correct completion with no errors seen.	A1
	Working from left to right:		
	$\sinh 2x = \frac{\left(e^{2x} - e^{-2x}\right)}{2} = \frac{\left(e^x + e^{-x}\right)\left(e^x - e^{-x}\right)}{2}$ Uses the correct definition for sinh2x on lhs and uses the difference of 2 squares.		M1
	$2\sinh x \cosh x*$	Correct completion with no errors seen.	A1

If they work from both ends then a clear link must be established as a conclusion e.g. lhs = rhs, tick QED etc.

(b)	$2\cosh^2 x - 1 - 7\cosh x + 7 = 0$	Use the identity for cosh2x	M1
	$2\cosh^2 x - 7\cosh x + 6 = 0 \Rightarrow (2\cosh x - 3)(\cosh x - 2) = 0 \Rightarrow \cosh x = ...$ Solve their 3TQ in cosh x (the usual rules for solving can be applied if necessary)		M1
	$\cosh x = \frac{3}{2},\ 2$	Correct answers, both needed	A1
	$\cosh x = \alpha \Rightarrow x = \ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ **or** $\frac{e^x + e^{-x}}{2} = 2 \Rightarrow e^{2x} - 4e^x + 1 = 0$ or $\frac{e^x + e^{-x}}{2} = \frac{3}{2} \Rightarrow e^{2x} - 3e^x + 1 = 0$ $\Rightarrow e^x = \frac{4 \pm \sqrt{12}}{2}$ or $e^x = \frac{3 \pm \sqrt{5}}{2}$ $\Rightarrow x = \ln...$ Changes at least one arcosh to ln form either using the correct ln form of arcosh or by returning to the correct exponential form of cosh and solving a quadratic in e^x. (Note that returning to exponentials is more likely to give all 4 answers below)		M1
	$x = \ln\left(\frac{3}{2} + \sqrt{\frac{5}{4}}\right),\ -\ln\left(\frac{3}{2} + \sqrt{\frac{5}{4}}\right)\left(\text{or } \ln\left(\frac{3}{2} - \sqrt{\frac{5}{4}}\right)\right),$ $\ln\left(2 + \sqrt{3}\right), -\ln\left(2 + \sqrt{3}\right)\left(\text{or } \ln\left(2 - \sqrt{3}\right)\right)$ All 4 correct, must be exact logarithms but can be any equivalent to those shown **with brackets**. Allow unsimplified if necessary and apply isw e.g. allow $\ln\left(\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 - 1}\right)$ for $\ln\left(\frac{3}{2} + \sqrt{\frac{5}{4}}\right)$		A1
			(5)
			[Total 9]

	Alternative for 11(b) using exponentials:	
	$\cosh 2x - 7\cosh x = -7 \Rightarrow \dfrac{e^{2x}+e^{-2x}}{2} - 7\left(\dfrac{e^{x}+e^{-x}}{2}\right) = -7$ $\Rightarrow e^{4x} - 7e^{3x} + 14e^{2x} - 7e^{x} + 1 = 0$ Substitutes the correct exponential forms and forms quartic in e^x	M1
	$e^{4x} - 7e^{3x} + 14e^{2x} - 7e^{x} + 1 = 0 \Rightarrow \left(e^{2x} - 4e^{x} + 1\right)\left(e^{2x} - 3e^{x} + 1\right) = 0$ $\Rightarrow e^{x} = \dfrac{4 \pm \sqrt{12}}{2}$ or $e^{x} = \dfrac{3 \pm \sqrt{5}}{2}$ M1: Attempts to solve one of their quadratics in e^x which has come from their quartic in e^x to **obtain exact values** for e^x A1: For at least 2 **exact** values of e^x	M1A1
	$\Rightarrow e^{x} = \dfrac{4 \pm \sqrt{12}}{2}$ or $e^{x} = \dfrac{3 \pm \sqrt{5}}{2}$ $\Rightarrow x = \ln ...$ Change at least one exponential form to ln form	M1
	$\Rightarrow x = \ln\left(\dfrac{4 \pm \sqrt{12}}{2}\right)$ and $x = \ln\left(\dfrac{3 \pm \sqrt{5}}{2}\right)$ All 4 correct, must be exact logarithms but can be any equivalent to those shown with brackets if necessary but e.g. they would not be required in the above forms.	A1

Question	Scheme	Marks	AOs
12	Solves the quadratic equation for $cosh^2 x$ e.g. $(8\,cosh^2 x - 9)(8\,cosh^2 + 1) = 0 \Rightarrow cosh^2 x = \ldots$	M1	3.1a
	$cosh^2 x = \frac{9}{8}\ \left\{-\frac{1}{8}\right\}$	A1	1.1b
	$cosh\,x = \frac{3}{4}\sqrt{2} \Rightarrow x = ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\left(\frac{3}{4}\sqrt{2}\right)^2 - 1}\right]$ **Alternatively** $cosh\,x = \frac{3}{4}\sqrt{2} \Rightarrow \frac{1}{2}(e^x + e^{-x}) \Rightarrow e^{2x} - \frac{3}{2}\sqrt{2}e^x + 1 = 0$ $\Rightarrow e^x = \sqrt{2}$ or $\frac{\sqrt{2}}{2} \Rightarrow x = \ldots$	M1	1.1b
	$x = \pm\frac{1}{2}ln\,2$	A1	2.2a
		(4)	
			(4 marks)

Notes:

M1: Solves the quadratic equation for $cosh^2 x$by any valid means. If by calculator accept for reaching the positive value for $cosh^2 x$ (negative may be omitted or incorrect) but do not allow for going directly to a value for $cosh\,x$ Alternatively score a correct process leading to a value for sinh $2x$ or its square (Alt 1) or use of correct exponential form for cosh x to form and expand to an equation in e^{4x} and e^{2x} (Alt 2)

A1: Correct value for $cosh^2 x$ (ignore negative or incorrect extra roots.). In Alt 1 score for a correct value for $\sinh^2 2x$ or sinh $2x$. In Alt 2 score for a correct simplified equation in e^{4x}.

M1: For a correct method to achieve at least one value for x (from cosh ^{2} x). In the main scheme or Alt 1, takes positive square root (if appropriate) and uses the correct formula for arcosh x or arsinh x to find a value for x. (No need to see negative square root rejected.) In Alt 2 it is for solving the quadratic in e^{4x} and proceeding to find a value for x.

Alternatively uses the exponential definition for cosh x, forms and solves a quadratic for e^x leading to a value for x

A1: Deduces (both) the correct values for x and no others. Must be in the form specified.

SC Allow M0A0M1A1 for cases where a calculator was used to get the value for cosh x with no evidence if a correct method for find both values is shown.

12 Alt 1	$64\cosh^2 x\left(\cosh^2 x-1\right)-9=0 \Rightarrow 64\cosh^2 x\sinh^2 x-9=0$ $\Rightarrow 16\sinh^2 2x=9 \Rightarrow \sinh^2 2x=\dfrac{9}{16}$ Or $(8\sinh x\cosh x-3)(8\sinh x\cosh x+3)=0 \Rightarrow \sinh 2x=\pm\frac{3}{4}$	M1 A1	3.1a 1.1b
	$\sinh 2x=\pm\frac{3}{4} \Rightarrow x=\frac{1}{2}\ln\left[\pm\frac{3}{4}+\sqrt{\frac{9}{16}+1}\right]$ (or use exponentials, or proceed via cosh4x)	M1	1.1b
	$x=\pm\dfrac{1}{2}\ln 2$	A1	2.2a
		(4)	
12 Alt 2	$64\left(\dfrac{e^x+e^{-x}}{2}\right)^4-64\left(\dfrac{e^x+e^{-x}}{2}\right)^2-9=0 \Rightarrow$ $4(e^{4x}+4e^{2x}+6+4e^{-2x}+e^{-4x})-16(e^{2x}+2+e^{-2x})-9=0$	M1	3.1a
	$4e^{4x}-17+4e^{-4x}=0$	A1	1.1b
	$(4e^{4x}-1)(1-4e^{-4x})=0 \Rightarrow e^{4x}=\ldots \Rightarrow x=\ldots$	M1	1.1b
	$x=\pm\dfrac{1}{2}\ln 2$	A1	2.2a
		(4)	

Question Number	Scheme	Notes	Marks
13(a)	$8\cosh^4 x = 8\left(\frac{e^x + e^{-x}}{2}\right)^4 = \frac{8}{16}\left(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}\right)$ Applies $\cosh x = \frac{e^x + e^{-x}}{2}$ and attempts to expand the bracket to at least 4 different and no more than 5 different terms of the correct form but they may be "uncollected" depending on how they do the expansion. Allow unsimplified terms e.g. $\left(e^x\right)^3 e^{-x}$. May see $8\left(\frac{e^x + e^{-x}}{2}\right)^2\left(\frac{e^x + e^{-x}}{2}\right)^2$ but must attempt to expand as above		M1
	$= \frac{1}{2}\left(e^{4x} + e^{-4x}\right) + 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 3 = ...$	Collects appropriate terms and reaches the form $\cosh 4x + p\cosh 2x + q$ or obtains values of p and q.	M1
	$= \cosh 4x + 4\cosh 2x + 3$	Correct expression or values e.g. $p = 4$ and $q = 3$	A1
	No marks are available in (a) if exponentials are not used but note that they may appear in combination with the use of hyperbolic identities e.g.: $8\cosh^4 x = 8\left(\cosh^2 x\right)^2 = 8\left(\frac{\cosh 2x + 1}{2}\right)^2 = 2\left(\frac{e^{2x} + e^{-2x}}{2} + 1\right)^2$ $= 2\left(\frac{e^{4x} + 2 + e^{-4x}}{4} + e^{2x} + e^{-2x} + 1\right) = \frac{e^{4x} + e^{-4x}}{2} + 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 2$ $= \cosh 4x + 4\cosh 2x + 3$ **Allow to "meet in the middle" e.g. expands as above and compares with** $\frac{1}{2}\left(e^{4x} + e^{-4x}\right) + p\left(\frac{e^{2x} + e^{-2x}}{2}\right) + q \Rightarrow p = ..., q = ...$ **but to score any marks the expansion must be attempted.**		
			(3)

(b) Way 1	$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 8\cosh^4 x - 4\cosh 2x - 3 - 17\cosh 2x + 9 = 0$ $\Rightarrow 8\cosh^4 x - 21\cosh 2x + 6 = 0 \Rightarrow 8\cosh^4 x - 21\left(2\cosh^2 x - 1\right) + 6 = 0$ Uses **their** result from part (a) and $\cosh 2x = \pm 2\cosh^2 x \pm 1$ to obtain a quadratic equation in $\cosh^2 x$ **or** $\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 2\left(2\cosh^2 x - 1\right)^2 - 1 - 17\left(2\cosh^2 x - 1\right) + 9 = 0$ Uses $\cosh 4x = \pm 2\cosh^2 2x \pm 1$ and $\cosh 2x = \pm 2\cosh^2 x \pm 1$ to obtain a quadratic equation in $\cosh^2 x$		M1
	$\Rightarrow 8\cosh^4 x - 42\cosh^2 x + 27 = 0$	Correct 3TQ in $\cosh^2 x$	A1
	$\Rightarrow 8\cosh^4 x - 42\cosh^2 x + 27 = 0$ $\Rightarrow \cosh^2 x = \frac{9}{2}\left(, \frac{3}{4}\right)$	Solves 3TQ in $\cosh^2 x$ (apply usual rules if necessary) to obtain $\cosh^2 x = k \ \left(k \in \mathbb{R} \text{ and } > 1\right)$. May be implied by their values – check if necessary.	M1
	$\cosh^2 x = \frac{9}{2} \Rightarrow \cosh x = \frac{3}{\sqrt{2}} \Rightarrow x = \pm\ln\left(\frac{3}{\sqrt{2}} + \sqrt{\frac{9}{2} - 1}\right)$ or $\cosh x = \frac{3}{\sqrt{2}} \Rightarrow \frac{e^x + e^{-x}}{2} = \frac{3}{\sqrt{2}} \Rightarrow \sqrt{2}e^{2x} - 6e^x + \sqrt{2} = 0 \Rightarrow e^x = ... \Rightarrow x = ...$ or $\cosh^2 x = \frac{9}{2} \Rightarrow \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{9}{2} \Rightarrow e^{4x} - 16e^{2x} + 1 = 0 \Rightarrow e^{2x} = ... \Rightarrow x = ...$ Takes square root to obtain $\cosh x = k \ \left(k > 1\right)$ and applies the correct logarithmic form for arcosh or uses the correct exponential form for cosh x to obtain at least one value for x The root(s) must be real to score this mark.		M1
	$x = \pm\ln\left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2}\right)$ Both correct and exact including brackets. Accept simplified equivalents e.g. $x = \ln\left(\frac{3}{\sqrt{2}} \pm \frac{\sqrt{7}}{\sqrt{2}}\right)$ but withhold this mark if additional answers are given unless they are the same e.g. allow $x = \pm\ln\left(\frac{3\sqrt{2}}{2} \pm \frac{\sqrt{14}}{2}\right)$		A1
			(5)

(b) Way 2	$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 2\cosh^2 2x - 1 - 17\cosh 2x + 9 = 0$ Applies $\cosh 4x = \pm 2\cosh^2 2x \pm 1$ to obtain a quadratic equation in cosh $2x$		M1
	$2\cosh^2 2x - 17\cosh 2x + 8 = 0$	Correct 3TQ in cosh $2x$	A1
	$2\cosh^2 2x - 17\cosh 2x + 8 = 0$ $\Rightarrow \cosh 2x = 8\left(, \frac{1}{2}\right)$	Solves 3TQ in cosh $2x$ (apply usual rules if necessary) to obtain $\cosh 2x = k \ \ (k \in \mathbb{R}$ and $> 1)$	M1
	$\cosh 2x = 8 \Rightarrow 2x = \pm\ln\left(8 + \sqrt{8^2 - 1}\right)$ or $\cosh 2x = 8 \Rightarrow \frac{e^{2x} + e^{-2x}}{2} = 8 \Rightarrow e^{4x} - 16e^{2x} + 1 = 0 \Rightarrow e^{2x} = ... \Rightarrow 2x = ...$ Applies the correct logarithmic form for arcosh from $\cosh 2x = k \ \ (k > 1)$ or uses the correct exponential form for cosh $2x$ to obtain at least one value for $2x$ The root(s) must be real to score this mark.		M1
	$x = \pm\frac{1}{2}\ln\left(8 + 3\sqrt{7}\right)$ or e.g. $x = \pm\ln\left(8 + 3\sqrt{7}\right)^{\frac{1}{2}}$	Both correct and exact with brackets. Accept simplified equivalents e.g. $x = \frac{1}{2}\ln\left(8 \pm \sqrt{63}\right)$ but withhold this mark if additional answers are given unless they are the same as above.	A1
(b) Way 3	$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow \frac{e^{4x} + e^{-4x}}{2} - \frac{17}{2}\left(e^{2x} + e^{-2x}\right) + 9 = 0$ $\Rightarrow e^{8x} - 17e^{6x} + 18e^{4x} - 17e^{2x} + 1 = 0$ M1: Applies the correct exponential forms and attempts a quartic equation in e^{2x} A1: Correct equation		M1A1
	$e^{8x} - 17e^{6x} + 18e^{4x} - 17e^{2x} + 1 = 0$ $\Rightarrow e^{2x} = 8 \pm 3\sqrt{7}, ...$	Solves and proceeds to a value for e^{2x} where $e^{2x} > 1$ and real.	M1
	$\Rightarrow e^{2x} = 8 \pm 3\sqrt{7} \Rightarrow 2x = \ln\left(8 \pm 3\sqrt{7}\right)$	Takes ln's to obtain at least one value for $2x$ The root(s) must be real to score this mark.	M1
	$x = \frac{1}{2}\ln\left(8 \pm 3\sqrt{7}\right)$ or e.g. $x = \ln\left(8 \pm 3\sqrt{7}\right)^{\frac{1}{2}}$	Both correct and exact with brackets. Accept simplified equivalents e.g. $x = \pm\frac{1}{2}\ln\left(8 + 3\sqrt{7}\right)$ but withhold this mark if additional answers are given unless they are the same as above.	A1
			Total 8

Question	Scheme	Marks
14.	$\cosh y = x, y < 0 \Rightarrow y = \ln\left[x - \sqrt{x^2-1}\right]$	
	$\cosh y = x \Rightarrow x = \dfrac{e^y + e^{-y}}{2}$	**B1**
	$\Rightarrow 2xe^y = e^{2y} + 1$	**M1**
	$\Rightarrow e^{2y} - 2xe^y + 1 = 0 \Rightarrow e^y = \dfrac{2x \pm \sqrt{(2x)^2 - 4\times1\times1}}{2}$ or $\Rightarrow e^{2y} - 2xe^y + 1 = 0 \Rightarrow (e^y - x)^2 + 1 - x^2 = 0 \Rightarrow e^y = ...$	**M1**
	$= x \pm \sqrt{x^2-1}$	**A1**
	So $y = \ln\left[x - \sqrt{x^2-1}\right]$ *	**A1***
	since $y < 0 \Rightarrow e^y < 1$ so need $x - \sqrt{x^2-1}$ (as $x > 1$ so must subtract)	**B1**
		(6)
		(6 marks)

Notes:

B1: Correct statement for x in terms of exponentials. $\cosh y = \dfrac{e^x + e^{-x}}{2}$ scores B0.

M1: Multiplies through by e^y to achieve a quadratic in e^y. (Terms need not be gathered.)

M1: Uses the quadratic formula or other valid method (e.g. completing the square) to solve for e^y.

A1: Correct solution(s) for e^y. Accept if only the negative one is given. Accept $\dfrac{2x \pm \sqrt{4x^2-4}}{2}$

A1*: Completely correct work leading to the given answer regardless of the justification why the negative root is taken (correct or incorrect). Must be no errors seen.

B1: Suitable justification for taking the negative root given.

E.g. $y < 0$ so $y = \ln\left[x - \sqrt{x^2-1}\right]$. Condone $x \pm \sqrt{x^2-1} < 1$ so $y = \ln\left[x - \sqrt{x^2-1}\right]$.

Note that the B1 can only be awarded if all previous marks have been awarded.

But the reason may be given before or after ln has been taken.

E.g. $(e^y - x)^2 + 1 - x^2 = 0 \Rightarrow e^y - x = \pm\sqrt{x^2-1}$ but $y < 0$ so $e^y - x = -\sqrt{x^2-1}$

Working backwards:

$$y = \ln\left[x - \sqrt{x^2-1}\right] \Rightarrow e^y = x - \sqrt{x^2-1}\,(\text{B1}) \Rightarrow e^y + e^{-y} = x - \sqrt{x^2-1} + \frac{1}{x - \sqrt{x^2-1}}\,(\text{M1})$$

$$x - \sqrt{x^2-1} + \frac{1}{x - \sqrt{x^2-1}} = \frac{2x\left(x - \sqrt{x^2-1}\right)}{x - \sqrt{x^2-1}}\,(\text{M1}) = 2x\,(\text{A1}) \Rightarrow x = \frac{e^y + e^{-y}}{2} = \cosh y\,(\text{A1})$$

Final B1 unlikely to be available.

Question Number	Scheme/Notes	Marks
15(a)	$\sinh(A+B)=\sinh A\cosh B+\cosh A\sinh B$	
	There is no credit for proofs that do not use exponential definitions	
	$\{\sinh A\cosh B+\cosh A\sinh B=\}$ $\frac{e^A-e^{-A}}{2}\times\frac{e^B+e^{-B}}{2}+\frac{e^A+e^{-A}}{2}\times\frac{e^B-e^{-B}}{2}$ or e.g., $\frac{(e^A-e^{-A})(e^B+e^{-B})+(e^A+e^{-A})(e^B-e^{-B})}{4}$ Replaces two of the four hyperbolic functions with correct exponential expressions. Condone poor bracketing. If they immediately start expanding this mark is only implied by completely correct work (i.e., with exponential definitions correct) and not just the fractions shown in the A1* note	**M1**
	$=\frac{e^{A+B}-e^{B-A}+e^{A-B}-e^{-A-B}+e^{A+B}+e^{B-A}-e^{A-B}-e^{-A-B}}{4}$ Expands numerator (or numerators if 2 separate fractions). Allow for sign errors only with coefficients and indices **and must see at least four terms** (but count terms which have been crossed out by cancelling) Allow this mark for: $=\frac{e^Ae^B-e^{-A}e^B+e^Ae^{-B}-e^{-A}e^{-B}+e^Ae^B+e^{-A}e^B-e^Ae^{-B}-e^{-A}e^{-B}}{4}$ Must see at least four terms as before but the last mark will not be available unless the requirements shown below are satisfied.	**M1**
	$=\frac{2e^{A+B}-2e^{-(A+B)}}{4}$ or $\frac{2\left(e^{A+B}-e^{-(A+B)}\right)}{4}$ or $\frac{e^{A+B}-e^{-(A+B)}}{2}$ or $\frac{1}{2}\left(e^{A+B}-e^{-(A+B)}\right)$ or $\frac{e^{A+B}}{2}-\frac{e^{-(A+B)}}{2}$ $=\sinh(A+B)$ * Reaches $\sinh(A+B)$ with no errors. Condone if the "$\sinh A\cosh B+\cosh A\sinh B=$" is missing at the start but the "$=\sinh(A+B)$" or "= LHS" must be seen. All bracketing correct where required but condone an unclosed bracket. One of the expressions shown or similar must be seen and allow $-A-B$ used for $-(A+B)$. Allow a "meet in the middle" proof and condone a "1=1" style approach provided it is complete. In both these cases a minimal conclusion is required e.g., "shown" but allow if both "LHS = ..." and "...=RHS" are seen. Do not condone sinh and/or cosh written as sin/cos for this mark	**A1***
	Attempts that start with the LHS and do not revert to a "meet in the middle" approach: Score the second M provided an **eight** term expanded numerator is achieved. The first M is for two explicitly clear correct replacements of hyperbolic expressions with two of $\sinh A, \cosh B, \cosh A$ and $\sinh B$. Condone if the $\sinh(A+B)=$ is missing at the start in these cases but the RHS or "...=RHS" must be seen.	
		(3)

Question Number	Scheme	Notes	Marks
15(b)	Condone the use of e.g., B for α or k for R for the first three marks but allow the A mark if recovered which may be via a correct expression which might be in (c)		
	$10\sinh x+8\cosh x=R\sinh x\cosh\alpha+R\cosh x\sinh\alpha$ $\Rightarrow R\sinh\alpha=8,\quad R\cosh\alpha=10$ Equates coefficients to obtain the two correct equations. This mark could be implied by either **correct** elimination, i.e., $R^2=10^2-8^2$ or $\tanh\alpha=\frac{8}{10}$ **provided incorrect equations are not seen.**		**B1 (M1 on ePen)**
	A complete attempt at finding a positive value for R: **By elimination**: $R^2\left(\cosh^2\alpha-\sinh^2\alpha\right)=10^2-8^2\Rightarrow R^2=36\Rightarrow R=6$ Allow this mark for $R=\sqrt{10^2+8^2}=2\sqrt{41}$ or $\sqrt{164}$. May just see e.g., $R=2\sqrt{41}$ **Following a positive value obtained for α where $\alpha=k\ln p,\ k>0,\ p>1$:** $\alpha=\frac{1}{2}\ln 9=\ln 3\Rightarrow R\cosh(\ln 3)=10\Rightarrow R\left(\frac{e^{\ln 3}+e^{-\ln 3}}{2}\right)=10\Rightarrow R=...\quad\left\{\frac{5}{3}R=10\Rightarrow R=6\right\}$ or $R\sinh(\ln 3)=8\Rightarrow R\left(\frac{e^{\ln 3}-e^{-\ln 3}}{2}\right)=8\Rightarrow R=...\quad\left\{\frac{4}{3}R=8\Rightarrow R=6\right\}$ Correct exponential definitions must be used but can be implied by correct work. Allow if the 10 and 8 are mixed up and allow slips in solving		**1st M1**
	A complete attempt at finding a positive value for α where $\alpha=k\ln p,\ k>0,\ p>1$: **By elimination:** $\tanh\alpha=\frac{8}{10}\Rightarrow\alpha=\text{artanh}\left(\frac{4}{5}\right)=\frac{1}{2}\ln\left(\frac{1+\frac{4}{5}}{1-\frac{4}{5}}\right)=...\left\{=\frac{1}{2}\ln 9=\ln 3\right\}$ A correct logarithmic form must be used with a valid value for artanh (<1) **Following a positive value obtained for R:** $\sinh\alpha=\frac{8}{"6"}\Rightarrow\alpha=\text{arsinh}\left(\frac{8}{"6"}\right)=\ln\left(\frac{8}{"6"}+\sqrt{\left(\frac{8}{"6"}\right)^2+1}\right)\ \{=\ln 3\}$ $\cosh\alpha=\frac{10}{"6"}\Rightarrow\alpha=\text{arcosh}\left(\frac{10}{"6"}\right)=\ln\left(\frac{10}{"6"}+\sqrt{\left(\frac{10}{"6"}\right)^2-1}\right)\ \{=\ln 3\}$ A correct logarithmic form must be used with a valid value if using arcosh (>1) The appropriate logarithmic forms could be implied by correct values. Allow this mark if e.g., $\frac{8}{10}$ is erroneously simplified but the value must be valid for the inverse hyperbolic function. If an exponential form is used to evaluate an inverse hyperbolic the form must be correct and the solving of any resulting 3TQ (most likely in e^{α} or e^{x}) must satisfy usual rules with one root correct if no working. Note that using tanh leads to a 2TQ which they must get one correct root for. They must also proceed to $\alpha=k\ln p,\ k>0,\ p>1$		**2nd M1**
	$6\sinh(x+\ln 3)$ or $R=6$ and $\alpha=\ln 3$ (or $p=3$) Correct expression but allow values for R and α (or p). If all the values are not seen in (b) then allow if they are seen in (c) and they could be seen embedded in a correct expression. A0 for additional solutions e.g., $6\sinh(x\pm\ln 3)$		**A1**

Question Number	Scheme/Notes	Marks
15(c)	There is no credit for attempts that do not use part (b) so e.g., do not award marks for attempts that apply exponential definitions to $10\sinh x+8\cosh x=18\sqrt{7}$ but note that it is acceptable to use exponential definitions with $6\sinh\left(x+\ln 3\right)=18\sqrt{7}$. Allow work with "made up" values for R and p provided $R > 0$, $p\in\mathbb{Z}$, $p>1$	
	$6\sinh\left(x+\ln 3\right)=18\sqrt{7}$ $\Rightarrow x=\text{arsinh}\left(3\sqrt{7}\right)-\ln 3$ $\Rightarrow x=\ln\left(3\sqrt{7}+\sqrt{\left(3\sqrt{7}\right)^2+1}\right)-\ln 3$ Obtains $x=\text{arsinh}\left(\frac{18\sqrt{7}}{"6"}\right)\pm\ln"3"$ or $x\pm\ln"3"=\text{arsinh}\left(\frac{18\sqrt{7}}{"6"}\right)$ from $"6"\sinh\left(x\pm\ln"3"\right)=18\sqrt{7}$ **and** uses the correct logarithmic form to obtain an expression for, or equation in x in "ln"s only but condone loss of the –ln "3" or +ln"3" after it has been seen. If the –ln "3" or +ln"3" is immediately incorporated to make a single logarithm the subtraction/addition law must be applied correctly. Work must be exact and not in decimals. If e.g., C = arsinh$\left(3\sqrt{7}\right)$ is found using $\frac{e^C-e^{-C}}{2}=3\sqrt{7}$, the exponential definition must be correct and they must solve a 3TQ in e^C satisfying usual rules (or one root correct if no working) and proceed to a valid C =... (e.g., not ln(negative)). This also applies to attempts via $6\frac{e^{x+\ln 3}-e^{-x-\ln 3}}{2}=18\sqrt{7}$ $\left\{\Rightarrow 3e^x-\frac{1}{3}e^{-x}=6\sqrt{7}\Rightarrow 9e^{2x}-18\sqrt{7}e^x-1=0\Rightarrow x=\ln\left(\frac{8+3\sqrt{7}}{3}\right)\right\}$ Note that $e^{2(x+\ln 3)}-6\sqrt{7}e^{x+\ln 3}-1=0\Rightarrow e^{x+\ln 3}=8+3\sqrt{7}\Rightarrow x=\ln\left(\frac{8+3\sqrt{7}}{3}\right)$ is also possible and in such cases the x + ln "3" must be handled correctly	**M1**
	$\left\{x=\ln\left(\frac{3\sqrt{7}+8}{3}\right)=\right\}\ln\left(\sqrt{7}+\frac{8}{3}\right)$ Correct answer in correct form. Accept e.g., $\ln\left(2\frac{2}{3}+\sqrt{7}\right)$. Must be fully bracketed correctly. Accept $q=\frac{8}{3}$ if $\ln\left(\sqrt{7}+q\right)$ is seen. No additional answers.	**A1**
		(2)
		Total 9

Question	Scheme	Marks	AOs
16(a)	$y=\tanh^{-1}(x)\Rightarrow \tanh y = x \Rightarrow x=\frac{\sinh y}{\cosh y}=\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}$	M1 A1	2.1 1.1b
	Note that some candidates only have one variable and reach e.g. $x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ or $\tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ Allow this to score M1A1		
	$x\left(e^{2y}+1\right)=e^{2y}-1\Rightarrow e^{2y}(1-x)=1+x\Rightarrow e^{2y}=\frac{1+x}{1-x}$	M1	1.1b
	$e^{2y}=\frac{1+x}{1-x}\Rightarrow 2y=\ln\left(\frac{1+x}{1-x}\right)\Rightarrow y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$*	A1*	2.1
	Note that $e^{2y}(x-1)+x+1=0$ can be solved as a quadratic in e^{y}: $e^{y}=\frac{-\sqrt{0-4(x-1)(x+1)}}{2(x-1)}=\frac{-\sqrt{4(1-x)(x+1)}}{2(x-1)}=\frac{2\sqrt{(1-x)(x+1)}}{2(1-x)}$ $=\frac{\sqrt{(x+1)}}{\sqrt{(1-x)}}\Rightarrow y=\frac{1}{2}\ln\frac{(x+1)}{(1-x)}$* Score **M1** for an attempt at the quadratic formula to make e^{y} the subject (condone ± √…) and **A1*** for a correct solution that rejects the positive root at some point and deals with the $(x-1)$ bracket correctly		
	$k = 1$ or $-1 < x < 1$	B1	1.1b
		(5)	
(a) Way 2	$\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\Rightarrow x=\tanh\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right)=\frac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}$	M1 A1	2.1 1.1b
	$x=\frac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}=\frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}=x$ Hence true, QED, tick etc.	M1 A1	1.1b 2.1
(b)	$2x=\tanh\left(\ln\sqrt{2-3x}\right)\Rightarrow \tanh^{-1}(2x)=\ln\sqrt{2-3x}$	M1	3.1a
	$\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right)=\frac{1}{2}\ln(2-3x)\Rightarrow\frac{1+2x}{1-2x}=2-3x$	M1	2.1
	$6x^{2}-9x+1=0$	A1	1.1b
	$6x^{2}-9x+1=0\Rightarrow x=...$	M1	1.1b
	$x=\frac{9-\sqrt{57}}{12}$	A1	3.2a
		(5)	

	Alternative for first 2 marks of (b)		
	$2x = \tanh\left(\ln\sqrt{2-3x}\right) \Rightarrow 2x = \dfrac{e^{2\ln\sqrt{2-3x}}-1}{e^{2\ln\sqrt{2-3x}}+1}$	M1	3.1a
	$\Rightarrow \dfrac{2-3x-1}{2-3x+1} = 2x$	M1	2.1
			(10 marks)

Notes

(a)

<u>If you come across any attempts to use calculus to prove the result – send to review</u>

M1: Begins the proof by expressing tanh in terms of exponentials and forms an equation in exponentials.

The exponential form can be any of $\dfrac{\left(e^{y}-e^{-y}\right)/2}{\left(e^{y}+e^{-y}\right)/2},\ \dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}},\ \dfrac{e^{2y}-1}{e^{2y}+1}$

Allow any variables to be used **but the final answer must be in terms of *x***. Allow alternative notation for $\tanh^{-1}x$ e.g. artanh, arctanh.

A1: Correct expression for "x" in terms of exponentials

M1: Full method to make $e^{2"y"}$ the subject of the formula. This must be correct algebra so allow sign errors only.

A1*: Completes the proof by using logs correctly and reaches the printed answer with no errors.

Allow e.g. $\dfrac{1}{2}\ln\left(\dfrac{x+1}{1-x}\right),\ \dfrac{1}{2}\ln\dfrac{x+1}{1-x},\ \dfrac{1}{2}\ln\left|\dfrac{x+1}{1-x}\right|$. Need to see $\tanh^{-1}x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ as a conclusion

but allow if the proof concludes that $y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ with y defined as $\tanh^{-1}x$ earlier.

B1: Correct value for k or writes $-1 < x < 1$

Way 2

M1: Starts with result, takes tanh of both sides and expresses in terms of exponentials

A1: Correct expression

M1: Eliminates exponentials and logs and simplifies

A1: Correct result (i.e. $x = x$) with conclusion

B1: Correct value for k or writes $-1 < x < 1$

(b)

M1: Adopts a correct strategy by taking $\tanh^{-1}$ of both sides

M1: Makes the link with part (a) by replacing artanh($2x$) with $\dfrac{1}{2}\ln\left(\dfrac{1+2x}{1-2x}\right)$ and demonstrates the use of the power law of logs to obtain an equation with logs removed **correctly**.

A1: Obtains the correct 3TQ

M1: Solves their 3TQ using a correct method (see General Guidance – if no working is shown (calculator) and the roots are correct for their quadratic, allow M1)

A1: Correct value with the other solution rejected (accept rejection by omission) so $x = \dfrac{9 \pm \sqrt{57}}{12}$ scores A0 unless the positive root is rejected

Alternative for first 2 marks of (b)

M1: Adopts a correct strategy by expressing tanh in terms of exponentials

M1: Demonstrates the use of the power law of logs to obtain an equation with logs removed correctly

Question Number	Scheme	Marks
17.	(a)	
	$\cosh A\cosh B-\sinh A\sinh B=\left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right)-\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$	M1
	$=\frac{1}{4}\left(e^{A+B}+e^{-A+B}+e^{A-B}+e^{-A-B}-e^{A+B}+e^{-A+B}+e^{A-B}-e^{-A-B}\right)$	
	$=\frac{1}{4}\left(2e^{-A+B}+2e^{A-B}\right)=\frac{e^{A-B}+e^{-(A-B)}}{2}=\cosh(A-B)$ ∗ cso	M1 A1 **(3)**
	(b) $\cosh x\cosh 1-\sinh x\sinh 1=\sinh x$	M1
	$\cosh x\cosh 1=\sinh x(1+\sinh 1)\quad\Rightarrow\quad \tanh x=\frac{\cosh 1}{1+\sinh 1}$	M1
	$\tanh x=\frac{\frac{e+e^{-1}}{2}}{1+\frac{e-e^{-1}}{2}}=\frac{e+e^{-1}}{2+e-e^{-1}}=\frac{e^{2}+1}{e^{2}+2e-1}$ ∗ cso	M1 A1 **(4)**
		[7]
	Alternative for (b)	
	$\frac{e^{x-1}+e^{-(x-1)}}{2}=\frac{e^{x}-e^{-x}}{2}$	M1
	Leading to $e^{2x}=\frac{e^{2}+e}{e-1}$	M1
	$\tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{e^{2x}-1}{e^{2x}+1}=\frac{e^{2}+e-(e-1)}{e^{2}+e+(e-1)}=\frac{e^{2}+1}{e^{2}+2e-1}$ ∗ cso	M1 A1 **(4)**