London Academy of Maths

Integration Techniques 1

Mark Scheme

Question Number	Scheme	Notes	Marks
1(i)	$(8)\int \frac{1}{16+x^2}\,dx = (8)\left(\frac{1}{4}\arctan\left(\frac{x}{4}\right)\right)$	Obtains …arctan (kx) Allow k = 1	M1
	$2\left[\arctan\left(\frac{x}{4}\right)\right]_4^{4\sqrt{3}} = 2(\arctan\sqrt{3} - \arctan 1) = ...$	Substitutes the given limits, subtracts either way round and obtains a value (could be a decimal). The substitution does not need to be seen explicitly and may be implied by their value.	**d**M1
	$\frac{\pi}{6}$ or $p = \frac{1}{6}$ Correct exact value (or value for p) Accept equivalent exact expressions e.g. $\frac{2\pi}{12}$ or $p = \frac{2}{12}$ and isw if necessary.		A1
			(3)
(ii)	$2\int \frac{1}{\sqrt{9-4x^2}}\,dx = 2\left(\frac{1}{2}\arcsin\frac{2x}{3}\right)$ $\left(\text{or e.g. } \arcsin\frac{x}{3/2}\right)$ **M1**: Obtains ...arcsin (kx). Allow k = 1 so allow just arcsin x. **A1**: Fully correct integration but allow unsimplified as above		M1 A1
	$\left[\arcsin\left(\frac{2x}{3}\right)\right]_{\frac{3}{4}}^{k} = \arcsin\left(\frac{2k}{3}\right) - \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{12}$ $\Rightarrow \arcsin\left(\frac{2k}{3}\right) = \frac{\pi}{12} + \frac{\pi}{6} \Rightarrow \frac{2k}{3} = \sin\left(\frac{\pi}{4}\right) \Rightarrow \frac{2k}{3} = \frac{\sqrt{2}}{2} \Rightarrow k = ...$ Substitutes the given limits, subtracts either way round, sets = $\frac{\pi}{12}$, uses $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and the correct **order** of operations condoning sign errors only to reach a value for k e.g. $\pm\alpha\left(\arcsin\left(\frac{2k}{3}\right) - \frac{\pi}{6}\right) = \frac{\pi}{12} \Rightarrow \arcsin\left(\frac{2k}{3}\right) = \frac{\pi}{12\alpha} \pm \frac{\pi}{6} \Rightarrow k = \frac{3\sin\left(\frac{\pi}{12\alpha} \pm \frac{\pi}{6}\right)}{2}$ Note that k may be inexact (decimal) or may be in terms of "sin" but must have a simplified argument e.g. $k = \frac{3\sin\left(\frac{\pi}{4}\right)}{2}$		**d**M1
	$k = \frac{3\sqrt{2}}{4}$ or exact equivalent e.g., $\frac{3}{2\sqrt{2}}$ Note that a common incorrect answer is $k = \frac{3}{2}\sin\left(\frac{5\pi}{24}\right)(= 0.913...)$ which comes from an incorrect integral of $2\arcsin\left(\frac{2x}{3}\right)$ (generally scoring 1010) Condone $x = \frac{3\sqrt{2}}{4}$		A1
			(4)
			Total 7

Question Number	Scheme	Marks
Q2	(a) $\dfrac{dy}{dx} = 3\sinh 3x - 4$	M1 A1
	$3\sinh 3x - 4 = 0$	
	$\sinh 3x = \dfrac{4}{3}$	M1
	$3x = \operatorname{arsinh}\dfrac{4}{3}$	
	$x = \dfrac{1}{3}\operatorname{arsinh}\dfrac{4}{3}$	M1
	$= \dfrac{1}{3}\ln\left[\frac{4}{3} + \sqrt{\left(\left(\frac{4}{3}\right)^2 + 1\right)}\right] = \dfrac{1}{3}\ln 3$	A1 (5)
	(b) $\displaystyle\int(\cosh 3x - 4x)\,dx = \frac{\sinh 3x}{3} - 2x^2$	M1 A1
	$A = \left[\dfrac{\sinh 3x}{3} - 2x^2\right]_0^{\frac{1}{3}\ln 3} = \dfrac{\sinh(\ln 3)}{3} - \dfrac{2(\ln 3)^2}{9}$	M1 A1
	$= \dfrac{e^{\ln 3} - e^{-\ln 3}}{6} - \ \ldots = \dfrac{3 - \frac{1}{3}}{6} - \ \ldots$	M1
	$= \dfrac{4}{9} - \dfrac{2(\ln 3)^2}{9}$	
	$= \dfrac{2}{9}\left[2 - (\ln 3)^2\right]$ * cso	A1 (6)
		[11]

Question Number	Scheme	Marks
3.	$x^2+4x-5=(x+2)^2-9$	B1
	$\int \frac{1}{\sqrt{\left((x+2)^2-9\right)}}\,\mathrm{d}x = \text{arcosh}\,\frac{x+2}{3}$	M1 A1ft
	ft their completing the square, requires arcosh	
	$\left[\text{arcosh}\,\frac{x+2}{3}\right]_1^3 = \text{arcosh}\,\frac{5}{3}\ \left(-\text{arcosh}\,1\right)$	
	$=\ln\left(\frac{5}{3}+\sqrt{\frac{25}{9}-1}\right)=\ln\left(\frac{5}{3}+\frac{4}{3}\right)=\ln 3$	M1 A1 **(5)**
		[5]
	Alternative	
	$x^2+4x-5=(x+2)^2-9$	B1
	Let $x+2=3\sec\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\sec\theta\tan\theta$	
	$\int \frac{1}{\sqrt{\left((x+2)^2-9\right)}}\,\mathrm{d}x = \int \frac{3\sec\theta\tan\theta}{\sqrt{\left(9\sec^2\theta-9\right)}}\,\mathrm{d}\theta$	M1
	$=\int \sec\theta\,\mathrm{d}\theta$	A1ft
	$\left[\ln\left(\sec\theta+\tan\theta\right)\right]_{\text{arcsec}1}^{\text{arcsec}\frac{5}{3}}=\ln\left(\frac{5}{3}+\frac{4}{3}\right)=\ln 3$	M1 A1 **(5)**

Question Number	Scheme	Notes	Marks
4(i)	$x^2-3x+5=\left(x-\frac{3}{2}\right)^2+\frac{11}{4}$	Correct completion of the square	B1
	$\int\frac{1}{\sqrt{x^2-3x+5}}\,dx=\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2+\frac{11}{4}}}\,dx=\sinh^{-1}\frac{2x-3}{\sqrt{11}}(+c)$ M1: Use of $\sinh^{-1}$ A1: Fully correct expression (condone omission of $+c$) Allow equivalent correct expressions e.g. $\sinh^{-1}\frac{x-\frac{3}{2}}{\sqrt{\frac{11}{4}}}(+c)$, $\sinh^{-1}\frac{x-\frac{3}{2}}{\frac{\sqrt{11}}{2}}(+c)$ Allow equivalents for $\sinh^{-1}$ e.g. arsinh, arcsinh but **not** arsin or arcsin		M1A1
	You may see logarithmic forms for the answer: e.g. $\ln\left(\frac{2x-3}{\sqrt{11}}+\sqrt{\left(\frac{2x-3}{\sqrt{11}}\right)^2+1}\right)$, $\ln\left(x-\frac{3}{2}+\sqrt{\left(x-\frac{3}{2}\right)^2+\frac{11}{4}}\right)$ but apply isw once a correct answer is seen.		
			(3)
(ii)	$63+4x-4x^2=-4\left(x^2-x-\frac{63}{4}\right)$ $=-4\left(\left(x-\frac{1}{2}\right)^2-\frac{64}{4}\right)$	Obtains $-4\left(\left(x-\frac{1}{2}\right)^2\pm...\right)$ or $-4\left(x-\frac{1}{2}\right)^2\pm...$ or $...-(2x-1)^2$	M1
	$-4\left(\left(x-\frac{1}{2}\right)^2-16\right)$ or $64-4\left(x-\frac{1}{2}\right)^2$ or $64-(2x-1)^2$	Correct completion of the square	A1
	$\int\frac{1}{\sqrt{63+4x-4x^2}}\,dx=\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{8}\right)(+c)$ M1: Use of $\sin^{-1}$ A1: Fully correct expression (condone omission of $+c$) Allow equivalent correct expressions e.g. $\frac{1}{2}\sin^{-1}\frac{x-\frac{1}{2}}{4}(+c)$, $-\frac{1}{2}\sin^{-1}\frac{\frac{1}{2}-x}{4}(+c)$ Allow equivalents for $\sin^{-1}$ e.g. arsin, arcsin but not arsinh or arcsinh		M1A1
			(4)
	In (ii) there are no marks for using $\int\frac{1}{\sqrt{63+4x-4x^2}}\,dx=-\int\frac{1}{\sqrt{4x^2-63-4x}}\,dx$ But if completion of square attempted first allow M1A1 e.g. for $\int\frac{1}{\sqrt{63+4x-4x^2}}\,dx=\int\frac{1}{\sqrt{64-(2x-1)^2}}\,dx$ but then M0 for $=\int\frac{-1}{\sqrt{(2x-1)^2-64}}\,dx$		
			Total 7

Question Number	Scheme	Notes	Marks
5(a)	$\int \frac{1}{\sqrt{9x^2+16}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{x^2+\frac{16}{9}}}\,dx$ $= \frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)$ or $\frac{1}{3}\text{arsinh}\left(\frac{x}{\frac{4}{3}}\right) \quad (+c)$ or $\frac{1}{3}\ln\left(x+\sqrt{x^2+\left(\frac{4}{3}\right)^2}\right) \quad (+c)$	M1: Obtains $p\,\text{arsinh}(qx)$ or $r\ln\left\{x+\sqrt{x^2+s}\right\}$ or $t\ln\left(ux+\sqrt{vx^2+w}\right)$ $p, q, r, s, t, u, v, w > 0$ A1: Any correct expression. Could be unsimplified and isw. The "$+c$" is not required. Allow $\sinh^{-1}$ and condone "arcsinh". "arcsin" or "arsin" is M0	M1 A1
			(2)
(b)	$\int_{-2}^{2} \frac{1}{\sqrt{9x^2+16}}\,dx$ $= \left[\frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)\right]_{-2}^{2}$ or $\left[\frac{2}{3}\text{arsinh}\left(\frac{3x}{4}\right)\right]_{0}^{2}$ $= \frac{1}{3}\text{arsinh}\left(\frac{3\times 2}{4}\right) - \frac{1}{3}\text{arsinh}\left(\frac{3\times -2}{4}\right)$ or $\frac{2}{3}\text{arsinh}\left(\frac{3}{2}\right)$ OR $\left[\frac{1}{3}\ln\left(x+\sqrt{x^2+\frac{16}{9}}\right)\right]_{-2}^{2}$ $= \frac{1}{3}\ln\left(2+\sqrt{2^2+\frac{16}{9}}\right) - \frac{1}{3}\ln\left(-2+\sqrt{(-2)^2+\frac{16}{9}}\right)$ or $\frac{2}{3}\left(\ln\left(2+\sqrt{2^2+\frac{16}{9}}\right) - \ln\left(0+\sqrt{0^2+\frac{16}{9}}\right)\right)$	Substitutes the limits 2 and -2 into an expression of the form $p\,\text{arsinh}(qx)$ or $r\ln\left\{x+\sqrt{x^2+s}\right\}$ or $t\ln\left(ux+\sqrt{vx^2+w}\right)$ $p, q, r, s, t, u, v, w > 0$ and subtracts either way round or obtains an expression for $2[...]_0^{\pm 2}$ The expression does not have to be consistent with their answer to (a). No rounded decimals unless exact values recovered. Any f(0) = 0 can be implied by omission. Condone poor bracketing.	M1
	$\frac{1}{3}\ln\left(\frac{11}{2}+\frac{3\sqrt{13}}{2}\right)$ or $\frac{1}{3}\ln\frac{11+3\sqrt{13}}{2}$ or $\frac{2}{3}\ln\left(\frac{3}{2}+\frac{\sqrt{13}}{2}\right)$ or $\frac{2}{3}\ln\frac{3+\sqrt{13}}{2}$	**d**M1: Obtains an expression of the form $a\ln\left(b+c\sqrt{13}\right)$ *or* $a\ln\left(\frac{d+e\sqrt{13}}{f}\right)$ where a,b,c,d,e,f are exact and >0. Condone poor bracketing. **Requires previous M mark.** A1: Any correct equivalent in an appropriate form (fractions may not be in simplest form) with correct bracketing if necessary and isw. **Must come from correct work**. Allow e.g., $a=\frac{2}{3}, b=\frac{3}{2}, c=\frac{1}{2}$	**d**M1 A1
	For information the decimal answer is 0.7965038115		**(3)**
			Total 5

Question Number	Scheme	Notes	Marks
6(i)	$3x^2+12x+24=3\left(x^2+4x+8\right)$ $=3\left(\left(x+2\right)^2+4\right)$	Obtains $3\left(\left(x+2\right)^2+...\right)$ or $3\left(x+2\right)^2+...$ Must include 3 now or later	M1
	$3\left(\left(x+2\right)^2+4\right)$ or $3\left(x+2\right)^2+12$		A1
	$\int \frac{1}{3x^2+12x+24}\,\mathrm{d}x=\frac{1}{3}\int \frac{1}{\left(x+2\right)^2+4}\,\mathrm{d}x=\frac{1}{6}\arctan\frac{x+2}{2}(+c)$ M1: Use of arctan A1: Fully correct expression (condone omission of + c)		M1A1
			(4)
(ii)	$27-6x-x^2=-\left(x^2+6x-27\right)$ $=-\left(\left(x+3\right)^2-36\right)$	Obtains $-\left(\left(x+3\right)^2+...\right)$ or $-\left(x+3\right)^2+...$	M1
	$-\left(\left(x+3\right)^2-36\right)$ or $36-\left(x+3\right)^2$		A1
	$\int \frac{1}{\sqrt{27-6x-x^2}}\,\mathrm{d}x=\int \frac{1}{\sqrt{36-\left(x+3\right)^2}}\,\mathrm{d}x=\arcsin\left(\frac{x+3}{6}\right)(+c)$ (Or $=-\arccos\left(\frac{x+3}{6}\right)(+c)$) M1: Use of arcsin (or – arccos) A1: Fully correct expression (condone omission of + c)		M1A1
			(4)
			Total 8

Question	Scheme	Marks	AOs
7(a)	$\dfrac{2x^2+3x+6}{(x+1)(x^2+4)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+4} \Rightarrow 2x^2+3x+6$ $= A(x^2+4)+(Bx+C)(x+1)$	M1	1.1b
	e.g. $x=-1 \Rightarrow A=\ldots,\ x=0 \Rightarrow C=\ldots,$ coeff $x^2 \Rightarrow B=\ldots$ or Compares coefficients and solves to find values for A, B and C $2=A+B,\ 3=B+C,\ 6=4A+C$	dM1	1.1b
	$A=1,\quad B=1,\quad C=2$	A1	1.1b
		(3)	
(b)	$\displaystyle\int_0^2 \frac{1}{x+1}+\frac{x+2}{x^2+4}\,dx = \int_0^2 \frac{1}{x+1}+\frac{x}{x^2+4}+\frac{2}{x^2+4}\,dx$ $= \left[\alpha\ln(x+1)+\beta\ln(x^2+4)+\lambda\arctan\left(\dfrac{x}{2}\right)\right]_0^2$	M1	3.1a
	$= \left[ln(x+1)+\dfrac{1}{2}ln(x^2+4)+arctan\left(\dfrac{x}{2}\right)\right]_0^2$	A1	2.1
	$= \left[ln(3)+\dfrac{1}{2}ln(8)+arctan\ 1\right]-\left[ln(1)+\dfrac{1}{2}ln(4)+arctan(0)\right]$ $=$ $= \left[ln(3)+\dfrac{1}{2}ln(8)+arctan(1)\right]-\left[\dfrac{1}{2}ln\ 4\right] = \underline{ln\left(\dfrac{3\sqrt{8}}{2}\right)}+\dfrac{\pi}{4}$	dM1	2.1
	$ln\left(3\sqrt{2}\right)+\dfrac{\pi}{4}$	A1	2.2a
		(4)	
		(7 marks)	

Notes:

(a)
M1: Selects the correct form for partial fractions and multiplies through to form suitable identity or uses a method to find at least one value (e.g. cover up rule).
dM1: Full method for finding values for all three constants. Dependent on first M. Allow slips as long as the intention is clear.
A1: Correct constants or partial fractions.

(b)
M1: Splits the integral into an integrable form and integrates at least two terms to the correct form. They may use a substitution on the arctan term
A1: Fully correct Integration.
dM1: Uses the limits of 0 and 2 (or appropriate for a substitution), subtracts the correct way round and combines the ln terms from separate integrals to a single term with evidence of correct ln laws at least once.
A1: Correct answer

Question Number	Scheme	Notes	Marks
8(i)	Throughout both parts of this question do not penalise omission of $\mathrm{d}x$ or $\mathrm{d}\theta$		
	$5+4x-x^2=9-(x-2)^2$ oe	Correct completion of the square Any correct result	B1
	$\int \frac{1}{\sqrt{5+4x-x^2}}\,\mathrm{d}x=\int \frac{1}{\sqrt{9-(x-2)^2}}\,\mathrm{d}x=\sin^{-1}\left(\frac{x-2}{3}\right)(+c)$ M1: Obtains $k\sin^{-1}\mathrm{f}(x)$ A1: Correct integration (+ c not needed)		M1A1
			(3)
(ii)	$x=6\Rightarrow\theta=\frac{\pi}{3}$ $x=2\sqrt{3}\Rightarrow\theta=\frac{\pi}{6}$	Correct θ limits in radians	B1
	$\int \frac{18}{(x^2-9)^{\frac{3}{2}}}\,\mathrm{d}x=\int \frac{18\times 3\sec\theta\tan\theta}{(9\sec^2\theta-9)^{\frac{3}{2}}}\,\mathrm{d}\theta$ M1: For $\int \frac{18}{((3\sec\theta)^2-9)^{\frac{3}{2}}}\times\left(\text{their }\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\mathrm{d}\theta$		M1
	$\int \frac{54\sec\theta\tan\theta}{(9\sec^2\theta-9)^{\frac{3}{2}}}\,\mathrm{d}\theta=54\int \frac{\sec\theta\tan\theta}{27\tan^3\theta}\,\mathrm{d}\theta=2\int \frac{\sin\theta\cos^3\theta}{\cos^2\theta\sin^3\theta}\,\mathrm{d}\theta$ $2\int \frac{\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta$ oe eg $2\int \frac{\sec\theta}{\tan^2\theta}\,\mathrm{d}\theta$ Correct simplified integral		A1
	$2\int \frac{\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta=2\int \operatorname{cosec}\theta\cot\theta\,\mathrm{d}\theta=-2\operatorname{cosec}\theta(+c)$ Obtains $k\operatorname{cosec}\theta(+c)$		M1
	$[-2\operatorname{cosec}\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=-2\operatorname{cosec}\frac{\pi}{3}+2\operatorname{cosec}\frac{\pi}{6}$	Uses changed limits correctly. **Depends on all previous method marks**.	**d**M1
	$=4-\frac{4}{3}\sqrt{3}$	Cao Allow these 2 marks if limits have been given in degrees	A1
			(6)
			Total 9

Question Number	Scheme	Notes	Marks
9(a)	$4x^2+4x+17=4\left(x^2+x+\frac{17}{4}\right)=4\left[\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+\frac{17}{4}\right]=(2x+1)^2+16$ or $4x^2+4x+17=4x^2+4px+q\Rightarrow 4px=4x\Rightarrow \underline{p=1},\ q+p^2=17\Rightarrow \underline{q=16}$ B1: Either p or q correct B1: Both correct values in part (a). Allow from any/no work. Values may be embedded within expression $(2x+p)^2+q$.		**B1** **B1**
			(2)
(b)	$A=8,\ B=4$	Both correct values (accept if embedded)	**B1**
			(1)
(c)	Note that this is a Hence question and there is no credit for work on the original fraction		
	$\int\frac{8x+5}{\sqrt{4x^2+4x+17}}\mathrm{d}x=\int\frac{1}{\sqrt{(2x+1)^2+16}}\mathrm{d}x+\int\frac{8x+4}{\sqrt{4x^2+4x+17}}\mathrm{d}x$ $=\frac{1}{2}\text{arsinh}\left(\frac{2x+1}{4}\right)+2\left(4x^2+4x+17\right)^{\frac{1}{2}}$ or $\frac{1}{2}\ln\left(\frac{2x+1}{4}+\sqrt{\left(\frac{2x+1}{4}\right)^2+1}\right)+2\left(4x^2+4x+17\right)^{\frac{1}{2}}$ or $\frac{1}{2}\ln\left(2x+1+\sqrt{(2x+1)^2+16}\right)+2\left((2x+1)^2+16\right)^{\frac{1}{2}}$	M1: For ...$\text{arsinh}(\text{f}(x))$ $\text{f}(x)\neq k$ or logarithmic equivalent i.e., $...\ln\left(\text{f}(x)+\sqrt{(\text{f}(x))^2+c}\right)$ $c\neq 0$ M1: For $...\left(4x^2+4x+17\right)^{\frac{1}{2}}$ or $...\left((2x+1)^2+16\right)^{\frac{1}{2}}$ A1: Fully correct integration	**M1** **M1** **A1**
	Allow for equivalents in e.g., u if substitutions are used e.g., $u=2x+1\Rightarrow\frac{1}{2}\int\frac{1}{\sqrt{u^2+16}}\mathrm{d}u\Rightarrow\frac{1}{2}\text{arsinh}\left(\frac{u}{4}\right)$ $u=4x^2+4x+17\Rightarrow\int\frac{1}{\sqrt{u}}\mathrm{d}u\Rightarrow 2\sqrt{u}$ $4\sinh u=2x+1\Rightarrow\int\frac{2\cosh u}{\sqrt{16\cosh^2 u}}\mathrm{d}u\Rightarrow\frac{1}{2}\int\mathrm{d}u=\frac{1}{2}u$ Score the M marks for appropriate forms (sign/coefficient errors only). If they continue working in terms of u the limits applied for the **dd**M1 must be correct for their substitution which for the above examples would be 3 & $\frac{5}{3}$, 25 & $\frac{169}{9}$ and $\text{arsinh}\left(\frac{3}{4}\right)$ & $\text{arsinh}\left(\frac{5}{12}\right)$		
	$\int_{\frac{1}{3}}^{1}\frac{8x+5}{\sqrt{4x^2+4x+17}}\mathrm{d}x=\frac{1}{2}\text{arsinh}\left(\frac{3}{4}\right)-\frac{1}{2}\text{arsinh}\left(\frac{5}{12}\right)+2\sqrt{25}-2\sqrt{\frac{169}{9}}$ $\Rightarrow\frac{1}{2}\ln\left(\frac{3}{4}+\sqrt{\left(\frac{3}{4}\right)^2+1}\right)-\frac{1}{2}\ln\left(\frac{5}{12}+\sqrt{\left(\frac{5}{12}\right)^2+1}\right)+2\sqrt{25}-\frac{26}{3}$ Condone replacement of $\text{arsinh}\left(\frac{x}{a}\right)$ with $\ln\left(x+\sqrt{x^2+a^2}\right)$ with $a\neq 1$ instead of using $\text{arsinh}x=\ln\left(x+\sqrt{x^2+1}\right)$	Substitutes and subtracts with the given limits and uses the appropriate form for arsinh twice (if required). Results from separate integrals must be combined. Allow slips but the $\text{f}\left(\frac{1}{3}\right)$ terms (and no others) must be subtracted. Not implied by just the final answer. **Requires both previous M marks.**	**ddM1**
	arsinh(...) may be evaluated using correct exp definition & solving a exponential 3TQ		
	$\left\{=\frac{1}{2}\ln 2-\frac{1}{2}\ln\frac{3}{2}+10-\frac{26}{3}\right\}=\frac{4}{3}+\frac{1}{2}\ln\frac{4}{3}$ Correct answer in correct form. May be no further work following substitution but there must be nothing incorrect. Allow $k=\frac{4}{3}$ if $k+\frac{1}{2}\ln k$ is seen. Allow $\frac{1}{2}\ln\frac{4}{3}+\frac{4}{3}$		**A1**
	Algebraic integration must be used. Answer or 1.47717... only scores no marks		
			(5)
			Total 8

Question	Scheme	Marks
10(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{1}{\sqrt{1-k\sqrt{x}^2}}\times\ldots x^{-\frac{1}{2}}$ or $\cos y=2x^{\frac{1}{2}}\Rightarrow\pm\sin y\frac{\mathrm{d}y}{\mathrm{d}x}=..x^{-\frac{1}{2}}$	M1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{1}{\sqrt{1-4x}}\times\left(Kx^{-\frac{1}{2}}\right)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{Kx^{-\frac{1}{2}}}{\sqrt{1-\left(2\sqrt{x}\right)^2}}$	dM1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\sqrt{x}\sqrt{1-4x}}$ oe e.g. $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\sqrt{x-4x^2}}$	A1
		(3)
(b) Way 1	$\int y\,\mathrm{d}x=\int 1\times\arccos\left(2\sqrt{x}\right)\mathrm{d}x=x\arccos\left(2\sqrt{x}\right)-\int x\frac{-1}{\sqrt{x}\sqrt{1-4x}}\mathrm{d}x$	M1
	$=x\arccos\left(2\sqrt{x}\right)+\int\frac{\sqrt{x}}{\sqrt{1-4x}}\mathrm{d}x$ *	A1*
		(2)
Way 2	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\arccos\left(2\sqrt{x}\right)\right)=1.\arccos\left(2\sqrt{x}\right)+x.\frac{-1}{\sqrt{x}\sqrt{1-4x}}$	M1
	$\Rightarrow\int\arccos\left(2\sqrt{x}\right)\mathrm{d}x=x\arccos\left(2\sqrt{x}\right)+\int\frac{\sqrt{x}}{\sqrt{1-4x}}\mathrm{d}x$ *	A1*
		(2)
(c)	$\frac{1}{2\sqrt{x}}\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\frac{1}{2}\sin\theta$, $\mathrm{d}x=-\sqrt{x}\sin\theta\,\mathrm{d}\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\frac{1}{2}\sin\theta\cos\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\frac{1}{4}\sin2\theta$	B1
	$\int\frac{\sqrt{x}}{\sqrt{1-4x}}\mathrm{d}x=\int\frac{-\left(\frac{1}{2}\cos\theta\right)^2\sin\theta}{\sqrt{1-4\left(\frac{1}{2}\cos\theta\right)^2}}\mathrm{d}\theta$	M1
	$=-\frac{1}{4}\int\frac{\cos^2\theta\sin\theta}{\sqrt{1-\cos^2\theta}}\mathrm{d}\theta=-\frac{1}{4}\int\cos^2\theta\,\mathrm{d}\theta$	A1
	$x=0\Rightarrow\theta=\frac{\pi}{2}$ $x=\frac{1}{8}\Rightarrow\theta=\frac{\pi}{4}$ So $\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}\mathrm{d}x=\frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$	A1
		(4)

(d)	$\frac{1}{4}\int\frac{1}{2}(1+\cos 2\theta)\,d\theta = K\left(\theta \pm \frac{1}{2}\sin 2\theta\right)$	**M1**
	$\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}\,dx = \frac{1}{8}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = ...\left(=\frac{\pi}{32}-\frac{1}{16}\right)$ or e.g. $\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}\,dx = \frac{1}{8}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{1}{8}\left[\arccos 2\sqrt{x}+\frac{1}{2}\sin 2\arccos 2\sqrt{x}\right]_0^{\frac{1}{8}}$ $=...\left(=-\frac{1}{8}\left(\frac{\pi}{4}+\frac{1}{2}-\frac{\pi}{2}\right)\right)$	**dM1**
	$\Rightarrow\int_0^{\frac{1}{8}}\arccos\left(2\sqrt{x}\right)dx = \left[x\arccos 2\sqrt{x}\right]_0^{\frac{1}{8}}+\frac{\pi}{32}-\frac{1}{16} = \frac{1}{8}\arccos\frac{1}{\sqrt{2}}-0+\frac{\pi}{32}-\frac{1}{16}$	**dM1**
	$=\frac{\pi}{16}-\frac{1}{16}$ oe	**A1**
		(4)
		(13 marks)

Notes:

(a)

M1: Attempts to apply the arccos derivative formula together with chain rule. Look for $\frac{dy}{dx} = \pm\frac{1}{\sqrt{1-k\sqrt{x}^2}}\times f(x)$ where f (*x*) is an attempt at differentiating $2\sqrt{x}$ where $f(x)\neq\alpha\sqrt{x}$

Note that *k* may be 1 for this mark.

Alternatively, takes cosine of both sides and differentiates to the form shown in the scheme.

dM1: Correct form for the overall derivative achieved, may be errors in sign or constants with $k \neq 1$

Alternatively, divides through by sin *y* and applies Pythagorean identity to achieve derivative in terms of *x*.

A1: Correct derivative, but need not be simplified. Award when first seen and isw.

(b) Way 1

M1: Attempts to apply integration by parts to $1\times\arccos\left(2\sqrt{x}\right)$.

Look for $x\arccos\left(2\sqrt{x}\right)-\int x\text{"their (a)"}\,dx$ or $u=\arccos\left(2\sqrt{x}\right)\Rightarrow\frac{du}{dx}=\text{part}(a),\ \frac{dv}{dx}=1\Rightarrow v=x$

A1*: Correct work leading to the printed answer. There must be a clear statement for the integration by parts before the given answer is stated.

So e.g. $u=\arccos\left(2\sqrt{x}\right)\Rightarrow\frac{du}{dx}=\text{part}(a),\ \frac{dv}{dx}=1\Rightarrow v=x$

$\Rightarrow\int\arccos\left(2\sqrt{x}\right)dx = x\arccos\left(2\sqrt{x}\right)+\int\frac{\sqrt{x}}{\sqrt{1-4x}}\,dx*$ scores M1A0

You can condone $\int\arccos\left(2\sqrt{x}\right)dx = x\arccos\left(2\sqrt{x}\right)+\int\frac{x^{\frac{1}{2}}}{\sqrt{1-4x}}\,dx*$

Way 2

M1: Applies the product rule to $x\arccos\left(2\sqrt{x}\right)$, look for $1.\arccos\left(2\sqrt{x}\right)+x.$"their (a)".

A1*: Rearranges and integrates to achieve the given result, with no errors seen.

(c)

B1: Any correct expression involving $\mathrm{d}x$ and $\mathrm{d}\theta$, see examples in scheme.

M1: Makes a complete substitution in the integral $\int \frac{\sqrt{x}}{\sqrt{1-4x}}\,\mathrm{d}x$ to achieve an integral in θ only.

Ignore attempts at substitution into the $x\arccos\left(2\sqrt{x}\right)$.

A1: A correct simplified integral aside from limits. May be implied by e.g. $\frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$

Note that this mark depends on the B mark.

A1: Finds correct limits for θ and applies to the integral by reversing the sign – i.e. correct answer with limits and sign all correct. Accept equivalent limits e.g. $-\frac{\pi}{4}$ to $-\frac{\pi}{4}$ or $\frac{\pi}{2}$ to $\frac{3\pi}{4}$

Note that this mark depends on the B mark.

(d)

M1: Applies double angle identity to get the integral in a suitable form and attempts to integrate. Accept $\cos^2\theta=\frac{1}{2}(\pm1\pm\cos2\theta)$ used as identity and look for $1\to\theta$ and $\cos2\theta\to\pm\frac{1}{2}\sin2\theta$

dM1: Applies their limits (either way round) to their integral in θ or reverse substitution and applies limits 0 and $\frac{1}{8}$.

Depends on the previous method mark.

dM1: Applies limits of 0 and $\frac{1}{8}$ to the $x\arccos\left(2\sqrt{x}\right)$ to obtain a value (or their limits either way round if they applied the substitution to this to obtain a value) and combines with the result of the other integral.

Depends on both previous method marks.

A1: Correct final answer.

Question Number	Scheme	Marks
Q11	(a) $x=\frac{a}{u} \Rightarrow \frac{dx}{du}=-\frac{a}{u^2}$	B1
	$\int \frac{1}{x\sqrt{(a^2-x^2)}}dx=\int \frac{1}{\frac{a}{u}\sqrt{\left(a^2-\frac{a^2}{u^2}\right)}}\left(-\frac{a}{u^2}\right)du$	M1 A1
	$=-\frac{1}{a}\int \frac{1}{\sqrt{(u^2-1)}}du$	M1
	$=-\frac{1}{a}\text{arcosh}\, u \quad (+A)$	M1
	$=-\frac{1}{a}\text{arcosh}\left(\frac{a}{x}\right) \quad (+A)$	A1 (6)
	(b) $\int \frac{1}{x\sqrt{(25-x^2)}}dx=-\frac{1}{5}\text{arcosh}\left(\frac{5}{x}\right)$	M1
	$\left[-\frac{1}{5}\text{arcosh}\left(\frac{5}{x}\right)\right]_3^4=\frac{1}{5}\left(\text{arcosh}\left(\frac{5}{3}\right)-\text{arcosh}\left(\frac{5}{4}\right)\right)$	M1 A1
	$=\frac{1}{5}[\ln 3-\ln 2]$	
	$=\frac{1}{5}\ln\frac{3}{2}$	M1 A1 (5)
		[11]
	Alternative for part (a)	
	$x=\frac{a}{\cosh u} \Rightarrow \frac{dx}{du}=-\frac{\sinh u}{\cosh^2 u}$	B1
	$\int \frac{1}{x\sqrt{(a^2-x^2)}}dx=\int \frac{1}{\frac{a}{\cosh u}\sqrt{\left(a^2-\left(\frac{a}{\cosh u}\right)^2\right)}}\left(-\frac{a\sinh u}{\cosh^2 u}\right)du$	M1 A1
	$=-\frac{1}{a}\int 1du=-\frac{1}{a}u \quad (+A)$	M1 M1
	$=-\frac{1}{a}\text{arcosh}\left(\frac{a}{x}\right) \quad (+A)$	A1 (6)

Question Number	Scheme	Notes	Marks
12(a)	$\left\{\frac{dy}{dx}=\right\}\text{arcosh}\,5x+\frac{ax}{\sqrt{bx^2-1}}$ or $\text{arcosh}\,5x+\frac{cx}{\sqrt{x^2-d}}$ (M1) $\Rightarrow \text{arcosh}(5x)+\frac{5x}{\sqrt{25x^2-1}}$ (A1) M1: Differentiates to obtain expression of the correct form $a, b, c, d\neq 0$ A1: Correct differentiation. Any equivalent form.		M1 A1
			(2)
(b)	$\frac{d}{dx}\left(x\text{arcosh}(5x)\right)=\text{arcosh}(5x)+"\frac{5x}{\sqrt{25x^2-1}}"\Rightarrow\int\text{arcosh}(5x)\,dx=x\,\text{arcosh}(5x)-\int"\frac{5x}{\sqrt{25x^2-1}}"dx$ M1: Rearranges **their** answer to (a) correctly and integrates or uses the correct formula to apply parts to $1\times\text{arcosh}\,5x$ to obtain the above.		M1
	$\int\text{arcosh}(5x)\,dx=x\,\text{arcosh}(5x)-\int\frac{5x}{\sqrt{25x^2-1}}dx$ A1: **Correct** expression – but **see note below on limited ft**		A1 **(limited ft)**
	$=x\,\text{arcosh}(5x)-\frac{1}{5}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$	M1: $\int\frac{Ax}{\sqrt{Bx^2-1}}dx\to C\left(Bx^2-1\right)^{\frac{1}{2}}$ A1: Fully correct expression with $x$arcosh($5x$) - see note below for limited ft	M1 A1 **(limited ft)**
	Note: Substitutions: $u=5x\Rightarrow\left(u^2-1\right)^{\frac{1}{2}}\Rightarrow\left[\frac{1}{5}\sqrt{u^2-1}\right]_{\frac{5}{4}}^{3}$ $\quad u=25x^2-1\Rightarrow\left[\frac{1}{5}\sqrt{u}\right]_{\frac{9}{16}}^{8}$ M1: Correct form A1: Fully correct expression with $x$arcosh($5x$)		
	A limited ft for **one** of the errors in (a) shown below applies for the first two A marks. **However also allow the following if this error occurs in part (b)** which is most likely to come from not rearranging and effectively restarting by using parts. Note that substitutions could be used. $a=1\Rightarrow x\,\text{arcosh}(5x)-\int\frac{x}{\sqrt{25x^2-1}}dx\Rightarrow x\,\text{arcosh}(5x)-\frac{1}{25}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$ $b=5\Rightarrow x\,\text{arcosh}(5x)-\int\frac{5x}{\sqrt{5x^2-1}}dx\Rightarrow x\,\text{arcosh}(5x)-\left(5x^2-1\right)^{\frac{1}{2}}\ (+c)$ $a=-5\Rightarrow x\,\text{arcosh}(5x)+\int\frac{5x}{\sqrt{25x^2-1}}dx\Rightarrow x\,\text{arcosh}(5x)+\frac{1}{5}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$		
	$\int_{\frac{1}{4}}^{\frac{3}{5}}\text{arcosh}\,5x\,dx=\frac{3}{5}\text{arcosh}(3)-\frac{1}{5}\sqrt{25\times\frac{9}{25}-1}-\left(\frac{1}{4}\text{arcosh}\left(\frac{5}{4}\right)-\frac{1}{5}\sqrt{25\times\frac{1}{16}-1}\right)$ Applies appropriate limits (note substitutions above) with subtraction the right way round seen to obtain an expression of the form $x\,\text{arcosh}(5x)\pm f(x)$ **where f(x) has come from integration**		M1
	$=\frac{3}{5}\text{arcosh}(3)-\frac{2\sqrt{2}}{5}-\frac{1}{4}\text{arcosh}\left(\frac{5}{4}\right)+\frac{3}{20}$	Correct answer seen in any form. **Must not follow clearly incorrect work.**	A1
	$\text{arcosh}3=\ln\left(3+\sqrt{3^2-1^2}\right)$ or $\text{arcosh}\left(\frac{5}{4}\right)=\ln\left(\frac{5}{4}+\sqrt{\left(\frac{5}{4}\right)^2-1^2}\right)$ $\left\{\Rightarrow\frac{3}{5}\ln\left(3+\sqrt{8}\right)-\frac{2\sqrt{2}}{5}-\frac{1}{4}\ln 2+\frac{3}{20}\right\}$	Converts $\text{arcosh}(3)$ or $\text{arcosh}\left(\frac{5}{4}\right)$ to any correct log form. Independent mark but must have obtained $x\,\text{arcosh}(5x)\pm f(x)$ **where f(x) has come from integration**	M1
	$=\frac{3}{20}-\frac{2\sqrt{2}}{5}+\ln\left(3+2\sqrt{2}\right)^{\frac{3}{5}}-\frac{1}{4}\ln 2$ **Must not follow clearly incorrect work.**	Correct answer. Terms in any order but otherwise written as shown. Allow values for p, q, r & k	A1
			(8)
			Total 10

Question Number	Scheme	Notes	Marks
13(i)	$x^2-4x+5=(x-2)^2+1$	Attempts to complete the square. Allow for $(x-2)^2+c,\ c>0$	M1
	$\int \frac{1}{(x-2)^2+1}\,dx = \arctan(x-2)$	Allow for $k\arctan f(x)$.	M1
	$\left[\arctan(x-2)\right]_1^2 = 0-\left(-\frac{\pi}{4}\right)=\frac{\pi}{4}$	$\frac{\pi}{4}$ cao	A1
			(3)
13(ii)	$\int \frac{\sqrt{x^2-3}}{x^2}\,dx = -\frac{\sqrt{x^2-3}}{x}+\int\frac{1}{\sqrt{x^2-3}}\,dx$ Uses integration by parts and obtains $A\frac{\sqrt{x^2-3}}{x}+B\int\frac{1}{\sqrt{x^2-3}}\,dx$		M1
	$=-\frac{\sqrt{x^2-3}}{x}+\text{arcosh}\frac{x}{\sqrt{3}}$	$B\int\frac{1}{\sqrt{x^2-3}}\,dx = k\text{arcosh } f(x)$	M1
		All correct	A1
	$\int_{\sqrt{3}}^{3}\frac{\sqrt{x^2-3}}{x^2}\,dx=\left[-\frac{\sqrt{x^2-3}}{x}+\text{arcosh}\frac{x}{\sqrt{3}}\right]_{\sqrt{3}}^{3}=\left(-\frac{\sqrt{6}}{3}+\text{arcosh}\sqrt{3}\right)-(0+\text{arcosh}1)$ Applies the limits 3 and $\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\text{arcosh}\sqrt{3}-\frac{1}{3}\sqrt{6}=\ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1
			(5)
13(ii) ALT 1	$\int\frac{\sqrt{x^2-3}}{x^2}\,dx=\int\frac{\sqrt{3\cosh^2 u-3}}{3\cosh^2 u}\sqrt{3}\sinh u\,du$	A complete substitution using $x=\sqrt{3}\cosh u$	M1
	$=\int\tanh^2 u\,du$	Obtains $k\int\tanh^2 u\,du$	M1
	$=\int(1-\text{sech}^2 u)\,du = u-\tanh u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3}\frac{\sqrt{x^2-3}}{x^2}\,dx = \left[u-\tanh u\right]_0^{\text{arcosh}\sqrt{3}} = \text{arcosh}\sqrt{3}-\tanh\left(\text{arcosh}\sqrt{3}\right)-0$ Applies the limits 0 and arcosh$\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\text{arcosh}\sqrt{3}-\frac{1}{3}\sqrt{6}=\ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1

13(ii) ALT 2	$\int \frac{\sqrt{x^2-3}}{x^2}\,dx = \int \frac{\sqrt{3\sec^2 u - 3}}{3\sec^2 u}\sqrt{3}\sec u \tan u\,du$	A complete substitution using $x = \sqrt{3}\sec u$	M1
	$= \int \frac{\tan^2 u}{\sec u}\,du$	Obtains $k\int \frac{\tan^2 u}{\sec u}\,du$	M1
	$= \ln(\sec u + \tan u) - \sin u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\,dx = \left[\ln(\sec u + \tan u) - \sin u\right]_0^{\text{arcsec}\sqrt{3}}$ $= \ln\left(\sec\left(\text{arcsec}\sqrt{3}\right) + \tan\left(\text{arcsec}\sqrt{3}\right)\right) - \ln\left(\sec(0) + \tan(0)\right) - \sin\left(\text{arcsec}\sqrt{3}\right)$ Applies the limits 0 and arcsec√3 **Depends on both previous M marks**		**d**M1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\,dx = \ln(\sqrt{2}+\sqrt{3}) - \frac{1}{3}\sqrt{6}$	Correct answer.	A1
			Total 8

Note that there may be other ways to perform the integration in part (ii) e.g. subsequent substitutions. Marks can be awarded if the method leads to something that is integrable and should be awarded as in the main scheme e.g. M1 for a complete method, M2 for simplifying and reaching an expression that itself can be integrated or can be integrated after rearrangement, A1 for correct integration, dM3 for using appropriate limits and A2 as above.

Alternative approach:

$$\int \frac{\sqrt{x^2-3}}{x^2}\,dx = \int \frac{x^2-3}{x^2\sqrt{x^2-3}}\,dx = \int \frac{1}{\sqrt{x^2-3}}\,dx - \int \frac{3}{x^2\sqrt{x^2-3}}\,dx = \text{arcosh}\frac{x}{\sqrt{3}} - ...$$

Can score **M0M1A0dM0A0** if there is no creditable attempt at the second integral.

If the second integral is attempted, it must be using a suitable method
e.g. with either $x = \sqrt{3}\cosh u$ or $x = \sqrt{3}\sec u$:

$$\int \frac{3}{x^2\sqrt{x^2-3}}\,dx = \int \frac{3}{3\cosh^2 u\sqrt{3\cosh^2 u - 3}}\sqrt{3}\sinh u\,du = \int \text{sech}^2 u\;du = \tanh u + c$$

or

$$\int \frac{3}{x^2\sqrt{x^2-3}}\,dx = \int \frac{3}{3\sec^2 u\sqrt{3\sec^2 u - 3}}\sqrt{3}\sec u \tan u\,du = \int \cos u\;du = \sin u + c$$

In these cases the first M can then be awarded and the other marks as defined with the appropriate limits used.

Question	Scheme	Marks	AOs
14(a)	$\int \frac{x^2}{\sqrt{x^2-1}}\,\mathrm{d}x \to \int \mathrm{f}(u)\,\mathrm{d}u$ Uses the substitution $x = \cosh u$ fully to achieve an integral in terms of u only, including replacing the dx	M1	3.1a
	$\int \frac{\cosh^2 u}{\sqrt{\cosh^2 u - 1}} \sinh u\,(\mathrm{d}u)$	A1	1.1b
	Uses correct identities $\cosh^2 u - 1 = \sinh^2 u$ and $\cosh 2u = 2\cosh^2 u - 1$ to achieve an integral of the form $A\int(\cosh 2u \pm 1)\,\mathrm{d}u \qquad A > 0$	M1	3.1a
	Integrates to achieve $A\left(\pm\frac{1}{2}\sinh 2u \pm u\right)(+c) \qquad A > 0$	M1	1.1b
	Uses the identity $\sinh 2u = 2\sinh u\cosh u$ and $\cosh^2 u - 1 = \sinh^2 u$ $\to \sinh 2u = 2x\sqrt{x^2-1}$	M1	2.1
	$\frac{1}{2}\left[x\sqrt{x^2-1} + \mathrm{ar}\cosh x\right] + k$ * cso	A1*	1.1b
		(6)	
(b)	Uses integration by parts the correct way around to achieve $\int \frac{4}{15} x \,\mathrm{arcosh}x\mathrm{d}x = Px^2\mathrm{arcosh}x - Q\int \frac{x^2}{\sqrt{x^2-1}}\,\mathrm{d}x$	M1	2.1
	$= \frac{4}{15}\left(\frac{1}{2}x^2\mathrm{arcosh}x - \frac{1}{2}\int \frac{x^2}{\sqrt{x^2-1}}\,\mathrm{d}x\right)$	A1	1.1b
	$= \frac{4}{15}\left(\frac{1}{2}x^2\mathrm{arcosh}x - \frac{1}{2}\left(\frac{1}{2}\left[x\sqrt{x^2-1} + \mathrm{arcosh}x\right]\right)\right)$	B1ft	2.2a
	Uses the limits x =1 and x = 3 the correct way around and subtracts $= \frac{4}{15}\left(\frac{1}{2}(3)^2\,\mathrm{arcosh}3 - \frac{1}{2}\left(\frac{1}{2}\left[3\sqrt{(3)^2-1} + \mathrm{arcosh}3\right]\right)\right) - \frac{4}{15}(0)$	dM1	1.1b
	$= \frac{4}{15}\left(\frac{9}{2}\ln\left(3+\sqrt{8}\right) - \frac{3\sqrt{8}}{4} - \frac{1}{4}\ln\left(3+\sqrt{8}\right)\right)$ $= \frac{1}{15}\left[17\ln\left(3+2\sqrt{2}\right) - 6\sqrt{2}\right]$ *	A1*	1.1b
		(5)	
			(11 marks)

Notes:

(a)

M1: Uses the substitution $x = \cosh u$ fully to achieve an integral in terms of u only. Must have replaced the dx but allow if the du is missing.

A1: Correct integral in terms of u. (Allow if the du is missing.)

M1: Uses correct identities $\cosh^2 u - 1 = \sinh^2 u$ and $\cosh 2u = 2\cosh^2 u - 1$ to achieve an integrand of the required form

M1: Integrates to achieve the correct form, may be sign errors.

M1: Uses the identities $\sinh 2u = 2\sinh u \cosh u$ and $\cosh^2 u - 1 = \sinh^2 u$ to attempt to find $\sinh 2u$ in terms of x. If using exponentials there must be a full and complete method to attempt the correct form.

A1*: Achieves the printed answer with no errors seen, cso

NB attempts at integration by parts are not likely to make progress – to do so would need to split the integrand as $x\frac{x}{\sqrt{x^2-1}}$. If you see any attempts that you feel merit credit, use review.

(b)

M1: Uses integration by parts the correct way around to achieve the required form.

A1: Correct integration by parts

B1ft: Deduces the integral by using the result from part (a). Follow through on their ‘uv’

dM1: Dependent on previous method mark. Uses the limits $x = 1$ and $x = 3$ the correct way around and subtracts

A1*cso: Achieves the printed answer with at least one intermediate step showing the evaluation of the arcosh 3, and no errors seen.

Question number	Scheme	Marks
15.	$\int \frac{3}{\sqrt{x^2-9}}\mathrm{d}x + \int \frac{x}{\sqrt{x^2-9}}\mathrm{d}x$	B1
	$= \left[3\text{arcosh}\frac{x}{3} + \sqrt{x^2-9}\right]$	M1 A1 A1
	$= \left[3\ln\left(\frac{x+\sqrt{x^2-9}}{(3)}\right) + \sqrt{x^2-9}\right]_5^6$	
	$= \left(3\ln(\frac{6+\sqrt{27}}{3}) + \sqrt{27}\right) - \left(3\ln(\frac{5+4}{3}) + 4\right)$	M1 A1
	$= 3\ln\frac{6+\sqrt{27}}{9} + \sqrt{27} - 4 = 3\ln\frac{2+\sqrt{3}}{3} + 3\sqrt{3} - 4$ (*)	A1 (7)
	Notes	**7**
	B1 Correctly changing to an integrable form. **1M1** Complete attempt to integrate at least one bit. **1A1** One term correct **2A1** All correct **2DM1** Substituting limits in all. **Must have got first M1** **3A1** Correctly (no follow through) **4A1** c.s.o.	

Further Pure FP2
Mark Scheme

Question number	Scheme	Marks
16.	(a) $\int \cosh x \arctan(\sinh x)\,dx = \sinh x \arctan(\sinh x) - \int \sinh x \frac{\cosh x}{1+\sinh^2 x}\,dx$	M1 A1 A1
	$= \sinh x \arctan(\sinh x) - \frac{1}{2}\ln(1+\sinh^2 x)\ (+C)$	M1 A1 (5)
	Or: $\ldots\ldots\ldots\ldots - \int \tanh x\,dx$	
	$= \sinh x \arctan(\sinh x) - \ln(\cosh x)\ (+C)$ M1 A1	
	Alternative:	
	Let $t = \sinh x$, $\frac{dt}{dx} = \cosh x$, $\int \arctan t\,dt = t\arctan t - \int \frac{t}{1+t^2}\,dt$ M1 A1 A1	
	$= \ldots\ldots - \frac{1}{2}\ln(1+t^2)$ M1	
	$= \sinh x \arctan(\sinh x) - \frac{1}{2}\ln(1+\sinh^2 x)\ (+C)$ (or equiv.) A1	
	(b) $\frac{1}{10}\left[\sinh x \arctan(\sinh x) - \ln(\cosh x)\right]_0^2 = \ldots\ldots,$ 0.34 (*)	M1, A1 (2)
		7
	(a) Alternative:	
	Let $\tan t = \sinh x$, $\sec^2 t \frac{dt}{dx} = \cosh x$, $\int t\sec^2 t\,dt = t\tan t - \int \tan t\,dt$ M1 A1 A1	
	$= \ldots\ldots - \ln(\sec t)$ M1	
	$= \sinh x \arctan(\sinh x) - \ln\sqrt{1+\sinh^2 x}\ (+C)$ (or equiv.) A1	
	Notes **(a)1M1** Complete attempt to use parts **1A1** One term correct. **2A1** All correct. **2M1** All integration completed. Need a ln term. **3A1** c.a.o. (in x) o.e, any correct form, simplified or not **(b)1M1** Use of limits 0 and 2 and 1/10. **1A1** c.s.o.	

Question Number	Scheme	Notes	Marks
17	$x = 4\cosh\theta \Rightarrow \frac{dx}{d\theta} = 4\sinh\theta$ $\Rightarrow \int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,dx = \int \frac{4\sinh\theta}{(16\cosh^2\theta-16)^{\frac{3}{2}}}\,d\theta$ Full attempt to use the given substitution. Award for $\int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,dx = k\int \frac{\sinh\theta}{((4\cosh\theta)^2-16)^{\frac{3}{2}}}\,d\theta$ Condone $4\cosh^2\theta$ for $(4\cosh\theta)^2$		M1
	$= \int \frac{4\sinh\theta}{(16\sinh^2\theta)^{\frac{3}{2}}}\,d\theta = \int \frac{4\sinh\theta}{64\sinh^3\theta}\,d\theta$ Simplifies $(16\cosh^2\theta-16)^{\frac{3}{2}}$ to the form $k\sinh^3\theta$ which may be implied by: $\int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,dx = k\int \frac{1}{\sinh^2\theta}\,d\theta$ **Note that this is not dependent on the first M**		M1
	$= \int \frac{1}{16\sinh^2\theta}\,d\theta$ Fully correct simplified integral. Allow equivalents e.g. $\frac{1}{16}\int \operatorname{cosech}^2\theta\,d\theta, \int \frac{1}{(4\sinh\theta)^2}\,d\theta, \int (4\sinh\theta)^{-2}\,d\theta$ etc. May be implied by subsequent work.		A1
	$= \int \frac{1}{16\sinh^2\theta}\,d\theta = \frac{1}{16}\int \operatorname{cosech}^2\theta\,d\theta = -\frac{1}{16}\coth\theta(+c)$ Integrates to obtain $k\coth\theta$. **Depends on both previous method marks.**		**d**M1
	$= -\frac{1}{16}\frac{\cosh\theta}{\sinh\theta} + c = -\frac{1}{16}\frac{\frac{x}{4}}{\sqrt{\frac{x^2}{16}-1}} + c$ or e.g. $-\frac{1}{4}\frac{\frac{x}{4}}{\sqrt{x^2-16}} + c$ Substitutes back **correctly** for x by replacing $\cosh\theta$ with $\frac{x}{4}$ or equivalent e.g. $4\cosh\theta$ with x **and** $\sinh\theta$ with $\sqrt{\left(\frac{x}{4}\right)^2-1}$ or equivalent e.g. $4\sinh\theta$ with $\sqrt{x^2-16}$ **Depends on all previous method marks and must be fully correct work for their "$-\frac{1}{16}$"**		**d**M1
	$\frac{-x}{16\sqrt{x^2-16}}(+c)$ oe e.g. $\frac{-\frac{1}{16}x}{\sqrt{x^2-16}}(+c)$	Correct answer. Award once the correct answer is seen and apply isw if necessary. Condone the omission of "+ c"	A1
	Note that you can condone the omission of the "dθ" throughout		
			(6)
			Total 6

Question Number	Scheme	Marks
18.	(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{arsinh} x^{\frac{1}{2}}\right) = \frac{1}{\sqrt{(1+x)}} \times \frac{1}{2}x^{-\frac{1}{2}} \quad \left(= \frac{1}{2\sqrt{x}\sqrt{(1+x)}} \right)$	M1 A1
	At $x = 4$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4\sqrt{5}}$ accept equivalents	A1 **(3)**
	(b) $x = \sinh^2\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sinh\theta\cosh\theta$	
	$\int \operatorname{arsinh}\sqrt{x}\,\mathrm{d}x = \int \theta \times 2\sinh\theta\cosh\theta\,\mathrm{d}\theta$	M1 A1
	$= \int \theta\sinh 2\theta\,\mathrm{d}\theta = \frac{\theta\cosh 2\theta}{2} - \int \frac{\cosh 2\theta}{2}\,\mathrm{d}\theta$	M1 A1 + A1
	$= \ \ldots \ - \frac{\sinh 2\theta}{4}$	M1
	$\left[\frac{\theta\cosh 2\theta}{2} - \frac{\sinh 2\theta}{4}\right]_0^{\operatorname{arsinh} 2} = \ \ldots$ attempt at substitution	M1
	$= \left[\frac{\theta\left(1+2\sinh^2\theta\right)}{2} - \frac{2\sinh\theta\cosh\theta}{4}\right] = \frac{1}{2}\operatorname{arsinh} 2 \times (1+8) - \frac{4\sqrt{5}}{4}$	M1 A1
	$= \frac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$	A1 **(10)**
		[13]
	Alternative for (a)	
	$x = \sinh^2 y$, $2\sinh y\cosh y\frac{\mathrm{d}y}{\mathrm{d}x} = 1$	M1
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sinh y\cosh y} = \frac{1}{2\sinh y\sqrt{\left(\sinh^2 y + 1\right)}} \left(= \frac{1}{2\sqrt{x}\sqrt{(1+x)}} \right)$	A1
	At $x = 4$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4\sqrt{5}}$ accept equivalents	A1 **(3)**
	An alternative for (b) *is given on the next page*	

Question Number	Scheme	Marks
18.	*Alternative for* (b)	
	$\int 1\times \operatorname{arsinh}\sqrt{x}\,\mathrm{d}x = x\operatorname{arsinh}\sqrt{x} - \int x\times\frac{1}{2\sqrt{x}\sqrt{(1+x)}}\mathrm{d}x$	M1 A1 + A1
	$= x\operatorname{arsinh}\sqrt{x} - \int \frac{\sqrt{x}}{2\sqrt{(1+x)}}\mathrm{d}x$	
	Let $x=\sinh^2\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sinh\theta\cosh\theta$	
	$\int \frac{\sqrt{x}}{\sqrt{(1+x)}}\mathrm{d}x = \int \frac{\sinh\theta}{\cosh\theta}\times 2\sinh\theta\cosh\theta\,\mathrm{d}\theta$	M1 A1
	$= 2\int \sinh^2\theta\,\mathrm{d}\theta = 2\int \frac{\cosh 2\theta - 1}{2}\mathrm{d}\theta, \; = \frac{\sinh 2\theta}{2} - \theta$	M1, M1
	$\left[\frac{\sinh 2\theta}{2} - \theta\right]_0^{\operatorname{arsinh} 2} = \left[\frac{2\sinh\theta\cosh\theta}{2} - \theta\right]_0^{\operatorname{arsinh} 2} = \frac{2\times 2\times\sqrt{5}}{2} - \operatorname{arsinh} 2$	M1 A1
	$\int_0^4 \operatorname{arsinh}\sqrt{x}\,\mathrm{d}x = 4\operatorname{arsinh} 2 - \frac{1}{2}\left(\frac{2\times 2\times\sqrt{5}}{2} - \operatorname{arsinh} 2\right) = \frac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$	A1 **(10)**
	The last 7 marks of the alternative solution can be gained as follows	
	Let $x=\tan^2\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\tan\theta\sec^2\theta$	
	$\int \frac{\sqrt{x}}{\sqrt{(1+x)}}\mathrm{d}x = \int \frac{\tan\theta}{\sec\theta}\times 2\tan\theta\sec^2\theta\,\mathrm{d}\theta$ dependent on first M1	M1 A1
	$= \int 2\sec\theta\tan^2\theta\,\mathrm{d}\theta$	
	$\int(\sec\theta\tan\theta)\tan\theta\,\mathrm{d}\theta = \sec\theta\tan\theta - \int \sec^3\theta\,\mathrm{d}\theta$	M1
	$= \sec\theta\tan\theta - \int \sec\theta\left(1+\tan^2\theta\right)\mathrm{d}\theta$	
	Hence $\int \sec\theta\tan^2\theta\,\mathrm{d}\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int\sec\theta\,\mathrm{d}\theta$	
	$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln\left(\sec\theta + \tan\theta\right)$	M1
	$[\ \ldots\ ]_0^{\arctan 2} = \frac{1}{2}\times\sqrt{5}\times 2 - \frac{1}{2}\ln\left(\sqrt{5}+2\right)$	M1 A1
	$\int_0^4 \operatorname{arsinh}\sqrt{x}\,\mathrm{d}x = 4\operatorname{arsinh} 2 - \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right) = \frac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$	A1