London Academy of Maths

Integration Techniques 1

Mark Scheme

Question			Answer	Mks	AO	Guidance	
1			In all three parts: Allow unsimplified answers. ISW for incorrect simplification. Ignore incorrect $\mathrm{d}x$ and/or $\int$				
1	**(a)**		$3 + 15x^{-4}$ or $3+\frac{15}{x^4}$ oe	**B1** **M1** **A1** **[3]**	**1.1** **1.1a** **1.1**	$3x$ correctly differentiated to 3 kx^{-4} oe $15x^{-4}$ or $\frac{15}{x^4}$ oe	ISW eg $5\times3x^{-4} = \frac{5}{3x^4}$ M1A1
1	**(b)**		$-60x^{-5}$ or $-\frac{60}{x^5}$ oe	**B1f** **B1f** **[2]**	**1.1a** **1.1**	zero term stated or implied by absence other term, ft their 2nd term in (a) if includes negative index	or ft their 1st term in (a) ISW
1	**(c)**		$\frac{3x^2}{2}+\frac{5}{2}x^{-2}+c$ or $\frac{3x^2}{2}+\frac{5}{2x^2}+c$ oe	**B1** **M1** **A1** **[3]**	**1.1** **1.1a** **1.1**	$\frac{3x^2}{2}$ kx^{-2} oe All correct and + c. ISW	

Question			Answer	Marks	AO	Guidance
2	**(a)**		$3x^2 - 3 - \frac{10}{x^3}$ oe	**M1**	**1.1**	Allow M1 for $\pm\frac{k}{x^3}$,
				B1	**1.1**	B1 for either $3x^2$ or -3
				A1	**1.1**	A1 for all correct
				[3]		
2	**(b)**		$2x^3 + \frac{1}{x^2}$ or $2x^3 + x^{-2}$ oe	**M1**	**1.1**	Allow M1 for ax^3 or $\pm\frac{b}{x^2}$ $(a, b \neq 0)$
				A1	**1.1**	A1 for both terms correct. Allow unsimplified form, eg $-\frac{2}{-2x^2}$
			$+ c$	**B1**	**1.1**	
				[3]		

Question			Answer	Mark	AO	Guidance
3						**DR**
			$x(x^2-4)=0$	**B1**	**3.1a**	Evidence of factorising or otherwise attempting to solve (=0 not required for this mark)
			$x=0,-2,2$	**B1**	**1.1**	This mark may be implied by correct limits
			$A_1=\int_0^2\left(x^3-4x\right)\mathrm{d}x$	**M1***	**1.1**	Ignore limits for this mark
			$=\left[\frac{x^4}{4}-2x^2\right]_0^2=-4$	**A1**	**1.1**	Must be seen for this mark (or clear indication of taking modulus) Condone area from [0,-2] as -4 or from [2,0] as +4 but must be consistent with their limits
			$A_2=\lvert A_1\rvert=4$ or $A_2=-A_1=4$	**dM1**	**2.1**	By symmetry: Total area = 2 × (their A_1) or adding together two areas of the same sign from their two integrals (or just 4 + 4)
			Total area = 8	**A1**	**1.1**	www, Area must be positive
			Alternative method for final M1*A1dM1A1			
			$\int_{-2}^0\left(x^3-4x\right)\mathrm{d}x-\int_0^2\left(x^3-4x\right)\mathrm{d}x$	**M1***		Ignore limits for this mark
			$=4-(-4)$	**A1** **dM1**		Correct area of -4 seen Attempt combine the two areas, with correct signs
			$=8$	**A1**		www, Area must be positive
				[6]		NB $\int_{-2}^2\left(x^3-4x\right)\mathrm{d}x=0$ scores B1B1M1A0M0A0 if working seen SC, no working or inadequate working: One area = 4: **SCB3** or Total area = 8: **SCB4**

Question			Answer	Marks	AO	Guidance
4			$(y =)\ -x^2 + c$	M1	3.1a	Allow omission of "$y =$". Must include "+ c"
			$-13 = -4^2 + c$ ($c = 3$)	M1	1.1	FT their integral of $-2x$. Must include "+ c"
			$-x^2 + 3 = 2x$	A1FT	1.1	FT their integral of $-2x$ and their c
			$x^2 + 2x - 3\ (= 0)$	M1	2.1	Rearrange their quadratic equation to solvable form. Must see this step
			$(x + 3)(x - 1) = 0$ or $(x + 1)^2 - 4 = 0$ or $x = \frac{-2\pm\sqrt{2^2-4\times1\times(-3)}}{2}$	M1	1.1	FT their c. May be implied by correct values for x
			$x = 1$ or -3	A1	1.1	
			$(1, 2)\ (-3, -6)$	A1	1.1	Condone y=2, y=–6 as long as clearly identified and paired. SC **B1B1** for correct solutions without working in second half (i.e. max M1M1A1M0M0B1B1 5/7)
				[7]		

Question			Answer	Marks	AO	Guidance
5	(a)		$\int (x^2 - 3x)\mathrm{d}x$	M1	3.1a	$\geq$ one term or both powers correct. May be implied by result
			$= \frac{x^3}{3} - \frac{3x^2}{2} + c$	A1	1.1	Allow without "+ c"
			$20 = \frac{6^3}{3} - \frac{3\times6^2}{2} + c$ $(\Rightarrow c = 2)$	M1	2.1	Substitute $x = 6$ into their integral, dep M1, & = 20
			$y = \frac{x^3}{3} - \frac{3x^2}{2} + 2$	A1	1.1	Correct answer, including "$y =$ ". Allow f(x) = …
				[4]		NB, if no working seen for finding c, but fully correct answer given: SC3
5	(b)		$\int_1^p (\frac{x^3}{3} - \frac{3x^2}{2} + 2)\mathrm{d}x$	M1	2.1	ft their equation, dep cubic. $\geq$ two terms or all three powers correct. May be implied by result
			$= \left[\frac{x^4}{12} - \frac{x^3}{2} + 2x\right]_1^p$	A1ft	1.1	Correct integral of their curve, dep quartic
				M1	1.1	Substitute limits 1 and p, dep integration attempted
			$= \frac{p^4}{12} - \frac{p^3}{2} + 2p - \frac{19}{12}$ oe	A1ft	1.1	ft their integral, dep their integral is a quartic.
				[4]		

Question			Answer	Marks	AO	Guidance
6	(a)		Cubic curve, correct orientation cuts x-axis twice to left of O and once to right	B1 B1 [2]	1.2 1.1	 Allow cubic of incorrect orientation
6	(b)		$(-3a, 0)$, $(-a, 0)$, $(b, 0)$ all shown correctly $(0, -3a^2b)$ shown correctly	B1 B1 [2]	1.1 1.1	allow $-3a$, $-a$, b marked on x-axis allow $-3a^2b$ marked on y-axis
6	(c)		$\int_{-3}^{-1}(x+3)(x+1)(x-4)\,dx$ $\quad$ $\int_{-1}^{4}(x^3-13x-12)\,dx$) $\left[\frac{x^4}{4}-\frac{13x^2}{2}-12x\right]_{-3}^{-1}$ $\quad$ $\left[\frac{x^4}{4}-\frac{13x^2}{2}-12x\right]_{-1}^{4}$	M1	1.1	State or imply integrating f(x) with any limits or none Allow incorrect expansion
			$= 8$ $\quad$ $= -\frac{375}{4}$ or -93.75	A1	1.1	A1 for either answer or for $\frac{375}{4}$. **BC** Allow A1 for $\pm\frac{343}{4}$
			$8 - (-93.75)$ or $8 + 93.75$	M1	2.1	Attempt subtract their integrals with correct limits, one +ve, one –ve ie $I_1 - I_2$ or $I_1 + (-I_2)$ or $I_1 + \lvert I_2\rvert$ dep I_2 being –ve
			$\frac{407}{4}$ or 101.75 or 102 (3 sf)	A1	2.2a	
				[4]		NB $\int_{-3}^{-1} + \int_{-1}^{4} = 8 + 93.75$ M1A1M0A0 (Two errors)

Question			Answer	Marks	AO	Guidance	
7	(a)		$y=1-x+\frac{6}{\sqrt{x}}$ leading to $y'=\ldots$	M1	2.1	Derivative of the form $-1+kx^{-\frac{3}{2}}$	
			$y'=-1-3x^{-\frac{3}{2}}$	A1	1.1		
			At $x=1, m_T=-4 \Rightarrow m_N=\frac{1}{4}$	M1*	1.2	Substitutes $x = 1$ into their derivative and correct use of $mm'=-1$	
			$y-6=\frac{1}{4}(x-1)$	M1dep*	1.1	Use of $y-6=m_N(x-1)$	
			$-x+4y=23$	A1	1.1	oe	
				[5]			
7	(b)		$x=4, y=1-4+\frac{6}{\sqrt{4}}=0$	B1	1.1	**AG –** must show sufficient working and must see = 0	
				[1]			
7	(c)		**DR**				
			$\int\left(1-x+\frac{6}{\sqrt{x}}\right)\mathrm{d}x=$	M1*	2.1	Attempt to integrate with at least two terms correct	
			$=x-\frac{1}{2}x^2+12\sqrt{x}$	A1	1.1		
			$\left(4-\frac{1}{2}(4^2)+12\sqrt{4}\right)-\left(1-\frac{1}{2}+12\right)=\ldots$	M1dep*	1.1	Use of correct limits (1 and 4)	If correct, then expect to see 7.5
			$\frac{1}{2}\left(\frac{23}{4}+6\right)(1)$	B1ft	3.1a	Any correct numerical expression for the area of the trapezium between $x = 0$ and $x = 1$ using their result from **(a)**	
			$\frac{107}{8}$	A1	2.2a	Or exact equivalent (e.g. 13.375)	
				[5]			

Question			Answer	Marks	AO	Guidance	
8			**DR**				
				M1*	**2.1**	**M1** for either term integrated correctly	
			$8x^{\frac{1}{2}}+3x$	**A1**	**1.1**		
			$\left(8a^{\frac{1}{2}}+3a\right)-(16+12)=7$	**M1dep***	**1.1**	Correct use of correct limits and equating to 7 – allow one substitution error	
			$3a+8a^{\frac{1}{2}}-35=0$	**M1**	**1.1**	Forming a 3TQ in $a^{\frac{1}{2}}$	Any three-term form (so terms do not need to be on the same side)
			$\left(3a^{\frac{1}{2}}-7\right)\left(a^{\frac{1}{2}}+5\right)=0$	**M1**	**3.1a**	Dependent on all previous **M** marks – correct method for solving for $a^{\frac{1}{2}}$	Or $8a^{\frac{1}{2}}=35-3a$ $9a^2-274a+1225=0$ $(9a-49)(a-25)=0$
			$a^{\frac{1}{2}}\neq-5$ as $a^{\frac{1}{2}}$ can't be negative	**A1**	**2.3**	Explicit rejection of -5 No specific justification required	Explicit rejection of a = 25 No specific justification required
			$a^{\frac{1}{2}}=\frac{7}{3}\Rightarrow a=\frac{49}{9}$	**A1**	**2.2a**	Correct value only	
				[7]			

Question			Answer	Marks	AO	Guidance	
9			$\frac{32}{3}$	B1	1.1	Seen or implied by later working	
				M1*	3.1a	Attempt integration on a 3 term quadratic in x	(increase in power by 1 for at least 1 term but not just multiplying each term by x)
			$\int(-x^2+6x-5)\,dx = -\frac{x^3}{3}+3x^2-5x$	A1	1.1	Ignore lack of $+c$	
			$-\frac{a^3}{3}+3a^2-5a-\left(-\frac{5^3}{3}+75-25\right)$	Dep*M1	1.1	$\pm\left(F(a)-F(5)\right)$	
			$\frac{32}{3}+\frac{a^3}{3}-3a^2+5a+\frac{25}{3}=19$	A1	1.1	oe	
			$a^3-9a^2+15a=0 \Rightarrow a^2-9a+15=0 \because a \neq 0$	M1	3.1a	solve their cubic (which comes from attempt at both areas and 19) leading to an exact value for a	**Dependent on both previous M marks**
			$a \neq \frac{9-\sqrt{21}}{2} \because a > 5$	B1	3.2a	**BC –** must give a reason for rejection of this value of a	Allow rejection of 2.21
			$a = \frac{9+\sqrt{21}}{2}$ only	A1	2.2a	**BC**	
				[8]			

Question			Answer	Marks	AOs	Guidance	
10			**DR**			If a = 27 with no working then 0/9	
			$\int_8^a 2x^{\frac{1}{3}} - 7x^{-\frac{1}{3}} \mathrm{d}x = 45$				
			$\left[\frac{2x^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{7x^{\frac{2}{3}}}{\left(\frac{2}{3}\right)}\right]_8^a (= 45)$	**M1***	**3.1a**	M1 – attempt integration (increase in power by 1 for at least 1 term)	
				A1	**1.1**	A1 – 1 term correct (accept unsimplified)	
				A1	**1.1**	A1 – both correct (accept unsimplified)	
			$\frac{3}{2}a^{\frac{4}{3}} - \frac{21}{2}a^{\frac{2}{3}} - \left(\frac{3}{2}(8)^{\frac{4}{3}} - \frac{21}{2}(8)^{\frac{2}{3}}\right)(= 45)$	**Dep*M1**	**1.1**	$F(a) - F(8)$	
			$\frac{3}{2}a^{\frac{4}{3}} - \frac{21}{2}a^{\frac{2}{3}} - (24 - 42)(= 45)$	**A1**	**1.1**	oe	
				M1	**1.1**	Equate integrated expression to 45 – dependent on both previous M marks	
			$a^{\frac{4}{3}} - 7a^{\frac{2}{3}} - 18 = 0$				
			$\left(a^{\frac{2}{3}} - 9\right)\left(a^{\frac{2}{3}} + 2\right) = 0$	**M1**	**3.1a**	Attempt to solve quadratic in $a^{\frac{2}{3}}$	**SC** if M0 for fourth M mark then award
			$a^{\frac{2}{3}} = 9$ $\left(\text{and } a^{\frac{2}{3}} = -2\right)$	**A1**	**1.1**		B1 for $a^{\frac{2}{3}} = 9$
			$a = 27$ only	**A1**	**2.2a**		B1 $a = 27$ only
				[9]			

Question			Answer	Marks	AO	Guidance	
11			$\frac{\mathrm{d}y}{\mathrm{d}x} = 20x^3 + 3ax^2 + b$	**M1***	**2.1**	Attempt to differentiate (at least two terms correct)	
			$12a + b = -160$	**M1dep***	**1.1**	Substitutes $x = 2$ into their derivative and set derivative equal to zero (need not be simplified)	
				M1*	**1.1**	Attempt to integrate (all terms with powers increased by 1 and at least one term correct)	
			$\int (5x^4 + ax^3 + bx)\,\mathrm{d}x = \frac{5x^5}{5} + \frac{ax^4}{4} + \frac{bx^2}{2} (+c)$	**A1**	**1.1**	cao (need not be simplified)	
			$\int_0^2 (5x^4 + ax^3 + bx)\,\mathrm{d}x$ $= (2)^5 + \frac{a}{4}(2)^4 + \frac{b}{2}(2)^2 = -48$	**M1dep***	**3.1a**	Correct use of limits $x = 0$ and $x = 2$ in their integrated expression (need not be simplified) and set equal to ± 48 (oe). Needs to be in terms of a and b.	For reference (if simplified): $2a + b = -40$
			$a = -12,\ b = -16$	**A1**	**1.1**	**BC** (For reference if correct: $y = 5x^4 - 12x^3 - 16x$)	
			y-coordinate of P is -48	**A1**	**2.2a**	cao	www
				[7]			

Question			Answer	Marks	AO	Guidance	
12	(a)		$3x^2+2>0$ for all values of x therefore stationary point is a minimum	B1 [1]	2.4		
12	(b)		$y'=\int(3x^2+2)\mathrm{d}x=x^3+2x+k$	M1*	2.1	Attempt to integrate (at least one of the terms in x correct)	Condone with no $+k$
			$y'=0$ at $x=-1$ $\Rightarrow(-1)^3+2(-1)+k=0$ leading to $k=\ldots$	M1dep*	1.1	Uses correct conditions to find the value of k (candidates may use the fact that when $x=0$, $y'=3$)	If correct $k=3$
			$y=\int(x^3+2x+'3')\mathrm{d}x=\frac{1}{4}x^4+x^2+'3'x+c$	M1	1.1	Integrates their y' correctly (allow with $k=0$)	Condone with no $+c$ (allow use of same letter for second constant)
			$\left(-1,\frac{1}{4}\right)\Rightarrow\frac{1}{4}(-1)^4+(-1)^2+'3'(-1)+c=\frac{1}{4}$ leading to $c=\ldots$	M1	1.1	Uses correct conditions to find the value of c	
			$y=\frac{1}{4}x^4+x^2+3x+2$	A1 [5]	2.5	cao (must include $y=$)	

Question			Answer	Marks	AO	Guidance	
13	(a)		$f(x+h)=a(x+h)^2+b(x+h)$ $=a(x^2+2xh+h^2)+b(x+h)$	M1	2.1	Considers $f(x+h)$ and attempts to expand bracket squared	
			$f(x+h)-f(x)$ $=(ax^2+2ahx+ah^2+bx+bh)-(ax^2+bx)$ $=2xah+ah^2+bh$	A1	1.1	Correct simplified expression for $f(x+h)-f(x)$	
			$\frac{f(x+h)-f(x)}{h}=2ax+ah+b$	A1	1.1	Correct simplified expression	
			$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=2ax+b$	A1	2.2a	cao – must be explicit that the limit (and not simply $h=0$) is considered	
				[4]			
13	(b)		$\int(ax^2+bx)\,dx=\frac{1}{3}ax^3+\frac{1}{2}bx^2(+c)$	M1*	2.1	Attempt to integrate (with at least one term correct)	
			$\int_1^4(ax^2+bx)\,dx=\left(\frac{64}{3}a+8b\right)-\left(\frac{1}{3}a+\frac{1}{2}b\right)\left(=21a+\frac{15}{2}b\right)$	M1dep*	1.1	Correct use of limits $x=1$ and $x=4$ in their integrated expression (need not be simplified)	
			$21a+\frac{15}{2}b=9$	M1	3.1a	Dependent on both previous M marks – setting up an equation in a and b using the area of shaded region	
			$(f'(4)=)8a+b=-0.75$	B1	1.1	Correct equation in a and b	
			$a=-0.375, b=2.25$	A1	1.1	**BC** (oe)	
			$y=-0.375x^2+2.25x$ with $x=4$ gives $y=3$ Equation of tangent: $y-3=-0.75(x-4)$	M1	1.1	Sets up the equation of the tangent at $x=4$ using 4, –0.75 and their y value at $x=4$ (dependent on all previous M marks) or for $-\frac{\text{their } y}{k-4}=-0.75$	Equation of tangent may have y set to 0 and x equal to k
			$0-3=-0.75(k-4)\Rightarrow k=8$	A1	2.2a		
				[7]			