London Academy of Maths

Number Theory

Question Paper

1 **In this question you must show detailed reasoning.**

Express the number 41723_{10} in hexadecimal (base 16). **[3]**

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Answer ALL questions. Write your answers in the spaces provided.

2. (i) Use the Euclidean algorithm to find the highest common factor of 602 and 161.

Show each step of the algorithm.

(3)

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3 **(a)** Express 205 in the form $7q + r$ for positive integers q and r, with $0 \leqslant r < 7$. **[1]**

(b) Given that $7 \mid (205 \times 8666)$, use the result of part **(a)** to justify that $7 \mid 8666$. **[2]**

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Answer ALL questions. Write your answers in the spaces provided.

4. (i) Using a suitable algorithm and without performing any division, determine whether 23 738 is divisible by 11

(2)

(ii) Use the Euclidean algorithm to find the highest common factor of 2322 and 654

(3)

5. The highest common factor of 963 and 657 is c.

(a) Use the Euclidean algorithm to find the value of c.

(3)

(b) Hence find integers a and b such that

$$963a + 657b = c$$

(3)

6. **In this question you must show detailed reasoning.**

Without performing any division, explain why $n = 20\,210\,520$ is divisible by 66

(4)

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7 In this question you must show detailed reasoning.

The number N is written as 28A3B in base-12 form.

Express N in decimal (base-10) form. **[2]**

8. (a) Use the Euclidean Algorithm to find integers a and b such that

$$125a + 87b = 1$$

(5)

(b) Hence write down a multiplicative inverse of 87 modulo 125

(1)

(c) Solve the linear congruence

$$87x \equiv 16 \pmod{125}$$

(2)

9. (i) Without performing any division, explain why 8184 is divisible by 6

(2)

(ii) Use the Euclidean algorithm to find integers a and b such that

$$27a + 31b = 1$$

(4)

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10 **(a)** Evaluate 13×19 modulo 31. **[1]**

(b) Solve the linear congruence $13x \equiv 9 \pmod{31}$. **[3]**

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11. (a) Use the Euclidean algorithm to show that 124 and 17 are relatively prime (coprime).

(2)

(b) Hence solve the equation

$$124x + 17y = 10$$

(3)

(c) Solve the congruence equation

$$124x \equiv 6 \bmod 17$$

(2)

12 In decimal (base 10) form, the number N is 15 260.

(a) Express N in binary (base 2) form. **[1]**

(b) Using the binary form of N, show that N is divisible by 7. **[2]**

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13. (i) Making your reasoning clear and using modulo arithmetic, show that

$$214^6 \text{ is divisible by } 8$$

(3)

(ii) The following 7-digit number has four unknown digits

$$\boxed{a}\,5\,\boxed{b}\,8\,\boxed{a}\,\boxed{b}\,0$$

Given that the number is divisible by 11

(a) determine the value of the digit a.

(2)

Given that the number is also divisible by 3

(b) determine the possible values of the digit b.

(3)

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14. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(i) (a) Use the Euclidean algorithm to find the highest common factor h of 416 and 72

(3)

(b) Hence determine integers a and b such that

$$416a + 72b = h$$

(3)

(c) Determine the value c in the set $\{0,1,2...,415\}$ such that

$$23 \times 72 \equiv c \pmod{416}$$

(2)

(ii) Evaluate $5^{10} \pmod{13}$ giving your answer as the smallest positive integer solution.

(3)

15 Given that n is a positive integer, show that the numbers $(4n + 1)$ and $(6n + 1)$ are co-prime. **[3]**

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16. (a) Use the Euclidean algorithm to show that the highest common factor of 168 and 66 is 6

(2)

(b) Use back substitution to determine integers a and b such that

$$168a + 66b = 6$$

(3)

(c) Explain why there are no integer solutions to the equation

$$168x + 66y = 10$$

(1)

(d) Solve the congruence equation

$$11v \equiv 8 \pmod{28}$$

(3)

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17 **(a)** Express as a decimal (base-10) number the base-23 number 7119_{23}. **[2]**

(b) Solve the linear congruence $7n + 11 \equiv 9 \pmod{23}$. **[3]**

(c) Let $N = 10a + b$ and $M = a + 7b$, where a and b are integers and $0 \leqslant b \leqslant 9$.

(i) By considering $3N - 7M$, prove that $23 | N$ if and only if $23 | M$. **[4]**

(ii) Use a procedure based on this result to show that $N = 711\,965$ is a multiple of 23. **[2]**

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18. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

(a) Use the Euclidean Algorithm to determine the highest common factor h of 234 and 96

(3)

(b) Hence determine integers a and b such that

$$234a + 96b = h$$

(3)

(c) Solve the congruence equation

$$96x \equiv 36 \pmod{234}$$

(5)

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19 In this question, N is the number 26 132 652.

(a) Without dividing N by 13, explain why 13 is a factor of N. **[1]**

(b) Use standard divisibility tests to show that 36 is a factor of N. **[3]**

It is given that $N = 36 \times 725\,907$.

(c) Use the results of parts **(a)** and **(b)** to deduce that 13 is a factor of 725 907. **[2]**

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20 For integers a and b, with $a \geqslant 0$ and $0 \leqslant b \leqslant 99$, the numbers M and N are such that

$$M = 100a + b \quad \text{and} \quad N = a - 9b.$$

(i) By considering the number $M + 2N$, show that $17 \mid M$ if and only if $17 \mid N$. **[4]**

(ii) Demonstrate step-by-step how an algorithm based on the result of part **(i)** can be used to show that $2\,058\,376\,813\,901$ is a multiple of 17. **[4]**

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21 Consider the integers a and b, where, for each integer n, $a = 7n + 4$ and $b = 8n + 5$.

Let $h = \text{hcf}(a, b)$.

(a) Determine all possible values of h. **[3]**

(b) Find all values of n for which a and b are **not** co-prime. **[2]**

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22 **(a)** Determine all values of x for which $16x \equiv 5 \pmod{101}$. **[4]**

(b) Solve

(i) $95x \equiv 6 \pmod{101}$, **[2]**

(ii) $95x \equiv 5 \pmod{101}$. **[2]**

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23 Let $N = 10a + b$ and $M = a + 3b$, where a and b are integers such that $a \geqslant 1$ and $0 \leqslant b \leqslant 9$.

(a) Prove that $29 \mid N$ if and only if $29 \mid M$. **[5]**

(b) Use an iterative method based on the result of part **(a)** to show that 899 364 472 is a multiple of 29. **[3]**

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24 **(a)** Let $a = 1071$ and $b = 67$.

(i) Find the unique integers q and r such that $a = bq + r$, where $q > 0$ and $0 \leqslant r < b$. **[1]**

(ii) Hence express the answer to **(a)(i)** in the form of a linear congruence modulo b. **[1]**

(b) Use the fact that $358 \times 715 - 239 \times 1071 = 1$ to prove that 715 and 1071 are co-prime. **[4]**

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25 For positive integers n, let $f(n) = 1 + 2^n + 4^n$.

(a) **(i)** Given that n is a multiple of 3, but **not** of 9, use the division algorithm to write down the two possible forms that n can take. **[1]**

(ii) Show that when n is a multiple of 3, but **not** of 9, $f(n)$ is a multiple of 73. **[6]**

(b) Determine the value of $f(n)$, modulo 73, in the case when n is a multiple of 9. **[2]**

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26. (i) Determine all the possible integers a, where $a > 3$, such that

$$15 \equiv 3 \bmod a$$

(2)

(ii) Show that if p is prime, x is an integer and $x^2 \equiv 1 \bmod p$ then either

$$x \equiv 1 \bmod p \qquad \text{or} \qquad x \equiv -1 \bmod p$$

(3)

(iii) A company has £13 940 220 to share between 11 charities.

Without performing any division and showing all your working, decide if it is possible to share this money equally between the 11 charities.

(2)

27 **(i)** Let $N = 10a + b$ and $M = a - 5b$ where a and b are integers such that $a \geq 1$ and $0 \leq b \leq 9$. N is to be tested for divisibility by 17.

(a) Prove that $17 \mid N$ if and only if $17 \mid M$. **[5]**

(b) Demonstrate step-by-step how an algorithm based on these forms can be used to show that $17 \mid 4097$. **[2]**

(ii) **(a)** Show that, for $n \geq 2$, any number of the form 1001_n is composite. **[3]**

(b) Given that n is a positive even number, provide a counter-example to show that the statement "any number of the form 10001_n is prime" is false. **[3]**

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28 **(a)** The number N has the base-10 form $N = abba\ abba\ \ldots\ abba$, consisting of blocks of four digits, as shown, where a and b are integers such that $1 \leqslant a < 10$ and $0 \leqslant b < 10$.

Use a standard divisibility test to show that N is always divisible by 11. **[3]**

(b) The number M has the base-n form $M = cddc\ cddc\ \ldots\ cddc$, where $n > 11$ and c and d are integers such that $1 \leqslant c < n$ and $0 \leqslant d < n$.

Show that M is always divisible by a number of the form $k_1 n + k_2$, where k_1 and k_2 are integers to be determined. **[3]**

29 **(a)** Let $f(n) = 2^{4n+3} + 3^{3n+1}$.

Use arithmetic modulo 11 to prove that $f(n) \equiv 0 \pmod{11}$ for all integers $n \geqslant 0$. **[4]**

(b) Use the standard test for divisibility by 11 to prove the following statements.

(i) $10^{33} + 1$ is divisible by 11 **[2]**

(ii) $10^{33} + 1$ is divisible by 121 **[4]**

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30 Let $\mathrm{f}(n)$ denote the base-n number 2121_n where $n \geqslant 3$.

(a) **(i)** For each $n \geqslant 3$, show that $\mathrm{f}(n)$ can be written as the product of two positive integers greater than 1, $\mathrm{a}(n)$ and $\mathrm{b}(n)$, each of which is a function of n. **[2]**

(ii) Deduce that $\mathrm{f}(n)$ is always composite. **[1]**

(b) Let h be the highest common factor of $\mathrm{a}(n)$ and $\mathrm{b}(n)$.

(i) Prove that h is either 1 or 5. **[4]**

(ii) Find a value of n for which $h = 5$. **[2]**