London Academy of Maths

Invariant Points and Lines

Question Paper

1 Find the equation of the line of invariant points under the transformation given by the matrix $\mathbf{M} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$. [3]

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2 Find the invariant line of the transformation of the *x*-*y* plane represented by the matrix $\begin{pmatrix} 2 & 0 \\ 4 & -1 \end{pmatrix}$. **[4]**

3 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

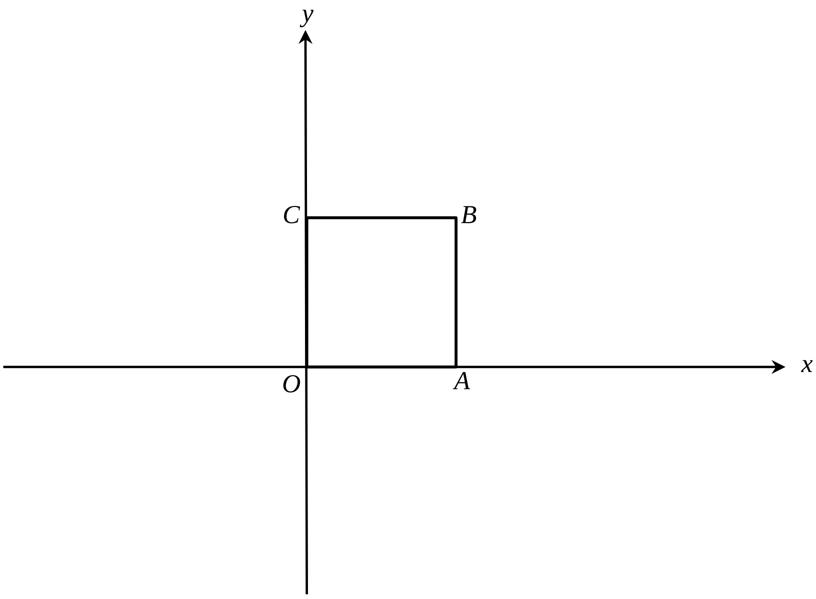
(i) The diagram in the Printed Answer Booklet shows the unit square *OABC*. The image of the unit square under the transformation represented by **M** is $OA'B'C'$. Draw and clearly label $OA'B'C'$. **[3]**

(ii) Find the equation of the line of invariant points of this transformation. **[3]**

(iii) **(a)** Find the determinant of **M**. **[1]**

(b) Describe briefly how this value relates to the transformation represented by **M**. **[2]**

3(i)



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4 **(i)** The matrix $\mathbf{S} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$ represents a transformation.

(*A*) Show that the point $(1, 1)$ is invariant under this transformation. [1]

(*B*) Calculate $\mathbf{S}^{-1}$. [2]

(*C*) Verify that $(1, 1)$ is also invariant under the transformation represented by $\mathbf{S}^{-1}$. [1]

(ii) Part **(i)** may be generalised as follows.

If (x, y) is an invariant point under a transformation represented by the non-singular matrix $\mathbf{T}$, it is also invariant under the transformation represented by $\mathbf{T}^{-1}$.

Starting with $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, or otherwise, prove this result. [2]

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5.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix $\mathbf{A}$.
(3)

The transformation V, represented by the 2×2 matrix $\mathbf{B}$, is a reflection in the line $y = -x$

(b) Write down the matrix $\mathbf{B}$.
(1)

Given that U followed by V is the transformation T, which is represented by the matrix $\mathbf{C}$,

(c) find the matrix $\mathbf{C}$.
(2)

(d) Show that there is a real number k for which the point $(1, k)$ is invariant under T.
(4)

6

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix **M** is non-singular.

(2)

The transformation T of the plane is represented by the matrix **M**.

The triangle R is transformed to the triangle S by the transformation T.

Given that the area of S is 63 square units,

(b) find the area of R.

(2)

(c) Show that the line $y = 2x$ is invariant under the transformation T.

(2)

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7 You are given that the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ represents a transformation T.

(a) You are given that the line with equation $y = kx$ is invariant under T.

Determine the value of k. **[4]**

(b) Determine whether the line with equation $y = kx$ in part **(a)** is a line of invariant points under T. **[1]**

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8.

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix **A** represents the linear transformation M.

Prove that, for the linear transformation M, there are no invariant lines.

(5)

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9 P, Q and T are three transformations in 2-D.

P is a reflection in the x-axis. **A** is the matrix that represents P.

(a) Write down the matrix **A**. **[1]**

Q is a shear in which the y-axis is invariant and the point $\begin{pmatrix}1\\0\end{pmatrix}$ is transformed to the point $\begin{pmatrix}1\\2\end{pmatrix}$. **B** is the matrix that represents Q.

(b) Find the matrix **B**. **[2]**

T is P followed by Q. **C** is the matrix that represents T.

(c) Determine the matrix **C**. **[2]**

L is the line whose equation is $y = x$.

(d) Explain whether or not L is a line of invariant points under T. **[2]**

An object parallelogram, M, is transformed under T to an image parallelogram, N.

(e) Explain what the value of the determinant of **C** means about

- the area of N compared to the area of M,
- the orientation of N compared to the orientation of M. **[3]**

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10. (i)

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}$$

where a and b are non-zero constants.

Given that the matrix **A** is self-inverse,

(a) determine the value of b and the possible values for a.

(5)

The matrix **A** represents a linear transformation M.

Using the smaller value of a from part (a),

(b) show that the invariant points of the linear transformation M form a line, stating the equation of this line.

(3)

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11 You are given that matrix $\mathbf{M} = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$.

(i) Prove that, for all positive integers n, $\mathbf{M}^n = \begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}$. **[6]**

(ii) Determine the equation of the line of invariant points of the transformation represented by the matrix $\mathbf{M}$. **[3]**

It is claimed that the answer to part **(ii)** is also a line of invariant points of the transformation represented by the matrix $\mathbf{M}^n$, for any positive integer n.

(iii) Explain *geometrically* why this claim is true. **[2]**

(iv) Verify *algebraically* that this claim is true. **[3]**

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12 A linear transformation T of the x-y plane has an associated matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} \lambda & k \\ 1 & \lambda - k \end{pmatrix}$, and λ and k are real constants.

(a) You are given that $\det \mathbf{M} > 0$ for all values of λ.

(i) Find the range of possible values of k. **[3]**

(ii) What is the significance of the condition $\det \mathbf{M} > 0$ for the transformation T? **[1]**

For the remainder of this question, take $k = -2$.

(b) Determine whether there are any lines through the origin that are invariant lines for the transformation T. **[4]**

(c) The transformation T is applied to a triangle with area 3 units^2. The area of the resulting image triangle is 15 units^2.
Find the possible values of λ. **[3]**

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13.

$$\mathbf{M} = \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}$$

where k is a constant.

Given that

$$\mathbf{M}^2 + 11\mathbf{M} = a\mathbf{I}$$

where a is a constant and $\mathbf{I}$ is the 2×2 identity matrix,

(a) (i) determine the value of a

(ii) show that $k = -9$

(3)

(b) Determine the equations of the invariant lines of the transformation represented by $\mathbf{M}$.

(6)

(c) State which, if any, of the lines identified in (b) consist of fixed points, giving a reason for your answer.

(1)

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14 A transformation T of the plane is represented by the matrix $\mathbf{M} = \begin{pmatrix} k+1 & -1 \\ 1 & k \end{pmatrix}$, where k is a constant.

Show that, for all values of k, T has no invariant lines through the origin. **[6]**

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15 The matrix **A** is given by $\mathbf{A} = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$.

You are given that **A** represents the transformation T which is a reflection in a certain straight line. You are also given that this straight line, the mirror line, passes through the origin, O.

(a) Explain why there must be a line of invariant points for T. State the geometric significance of this line. **[2]**

(b) By considering the line of invariant points for T, determine the equation of the mirror line. Give your answer in the form $y = mx + c$. **[4]**

The coordinates of the point P are (1, 5).

(c) By considering the image of P under the transformation T, or otherwise, determine the coordinates of the point on the mirror line which is closest to P. **[3]**

(d) The line with equation $y = ax + 2$ is an invariant line for T.

Determine the value of a. **[2]**

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16 The 2×2 matrix **A** represents a transformation T which has the following properties.

- The image of the point $(0, 1)$ is the point $(3, 4)$.
- An object shape whose area is 7 is transformed to an image shape whose area is 35.
- T has a line of invariant points.

(i) Find a possible matrix for **A**. **[8]**

The transformation S is represented by the matrix **B** where $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

(ii) Find the equation of the line of invariant points of S. **[2]**

(iii) Show that any line of the form $y = x + c$ is an invariant line of S. **[3]**

17 A transformation T of the plane has matrix **M**, where $\mathbf{M} = \begin{pmatrix} \cos\theta & 2\cos\theta - \sin\theta \\ \sin\theta & 2\sin\theta + \cos\theta \end{pmatrix}$.

(a) Show that T leaves areas unchanged for all values of θ. **[2]**

(b) Find the value of θ, where $0 < \theta < \frac{1}{2}\pi$, for which the y-axis is an invariant line of T. **[4]**

The matrix **N** is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(c) **(i)** Find $\mathbf{MN}^{-1}$. **[2]**

(ii) Hence describe fully a sequence of two transformations of the plane that is equivalent to T. **[4]**