

Invariant Points and Lines

Question Paper

- 1** Find the equation of the line of invariant points under the transformation given by the matrix $\mathbf{M} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$. [3]

- 2 Find the invariant line of the transformation of the x - y plane represented by the matrix $\begin{pmatrix} 2 & 0 \\ 4 & -1 \end{pmatrix}$. [4]

3 The matrix \mathbf{M} is given by $\mathbf{M} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

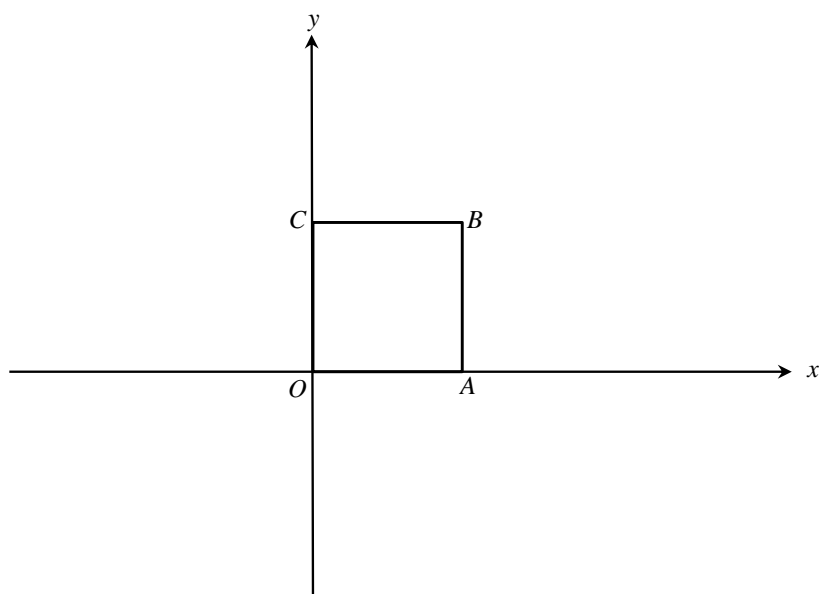
(i) The diagram in the Printed Answer Booklet shows the unit square $OABC$. The image of the unit square under the transformation represented by \mathbf{M} is $OA'B'C'$. Draw and clearly label $OA'B'C'$. [3]

(ii) Find the equation of the line of invariant points of this transformation. [3]

(iii) (a) Find the determinant of \mathbf{M} . [1]

(b) Describe briefly how this value relates to the transformation represented by \mathbf{M} . [2]

3(i)



4 (i) The matrix $\mathbf{S} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$ represents a transformation.

(A) Show that the point $(1, 1)$ is invariant under this transformation. [1]

(B) Calculate \mathbf{S}^{-1} . [2]

(C) Verify that $(1, 1)$ is also invariant under the transformation represented by \mathbf{S}^{-1} . [1]

(ii) Part (i) may be generalised as follows.

If (x, y) is an invariant point under a transformation represented by the non-singular matrix \mathbf{T} , it is also invariant under the transformation represented by \mathbf{T}^{-1} .

Starting with $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, or otherwise, prove this result. [2]

5.

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix \mathbf{A} . (3)

The transformation V , represented by the 2×2 matrix \mathbf{B} , is a reflection in the line $y = -x$

(b) Write down the matrix \mathbf{B} . (1)

Given that U followed by V is the transformation T , which is represented by the matrix \mathbf{C} ,

(c) find the matrix \mathbf{C} . (2)

(d) Show that there is a real number k for which the point $(1, k)$ is invariant under T . (4)

6

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

- (a) Show that the matrix \mathbf{M} is non-singular. (2)

The transformation T of the plane is represented by the matrix \mathbf{M} .

The triangle R is transformed to the triangle S by the transformation T .

Given that the area of S is 63 square units,

- (b) find the area of R . (2)

- (c) Show that the line $y = 2x$ is invariant under the transformation T . (2)

7 You are given that the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ represents a transformation T.

(a) You are given that the line with equation $y = kx$ is invariant under T.

Determine the value of k .

[4]

(b) Determine whether the line with equation $y = kx$ in part (a) is a line of invariant points under T.

[1]

8.
$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix \mathbf{A} represents the linear transformation M .

Prove that, for the linear transformation M , there are no invariant lines.

(5)

9 ~~P, Q and T are three transformations in 2-D.~~

~~P is a reflection in the x axis. \mathbf{A} is the matrix that represents P.~~

~~(a) Write down the matrix \mathbf{A} .~~ [1]

~~Q is a shear in which the y axis is invariant and the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is transformed to the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. \mathbf{B} is the matrix that represents Q.~~

~~(b) Find the matrix \mathbf{B} .~~ [2]

~~T is P followed by Q. \mathbf{C} is the matrix that represents T.~~

~~(c) Determine the matrix \mathbf{C} .~~ [2]

~~L is the line whose equation is $y = x$.~~

~~(d) Explain whether or not L is a line of invariant points under T .~~ [2]

~~An object parallelogram, M , is transformed under T to an image parallelogram, N .~~

~~(e) Explain what the value of the determinant of \mathbf{C} means about~~
~~• the area of N compared to the area of M ,~~
~~• the orientation of N compared to the orientation of M .~~ [3]

10. (i)

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ a - 4 & b \end{pmatrix}$$

where a and b are non-zero constants.

Given that the matrix \mathbf{A} is self-inverse,

(a) determine the value of b and the possible values for a . (5)

The matrix \mathbf{A} represents a linear transformation M .

Using the smaller value of a from part (a),

(b) show that the invariant points of the linear transformation M form a line, stating the equation of this line. (3)

11 You are given that matrix $\mathbf{M} = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$.

(i) Prove that, for all positive integers n , $\mathbf{M}^n = \begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}$. [6]

(ii) Determine the equation of the line of invariant points of the transformation represented by the matrix \mathbf{M} . [3]

It is claimed that the answer to part (ii) is also a line of invariant points of the transformation represented by the matrix \mathbf{M}^n , for any positive integer n .

(iii) Explain *geometrically* why this claim is true. [2]

(iv) Verify *algebraically* that this claim is true. [3]

12 A linear transformation T of the x - y plane has an associated matrix \mathbf{M} , where $\mathbf{M} = \begin{pmatrix} \lambda & k \\ 1 & \lambda - k \end{pmatrix}$, and λ and k are real constants.

(a) You are given that $\det \mathbf{M} > 0$ for all values of λ .

(i) Find the range of possible values of k . [3]

(ii) What is the significance of the condition $\det \mathbf{M} > 0$ for the transformation T ? [1]

For the remainder of this question, take $k = -2$.

(b) Determine whether there are any lines through the origin that are invariant lines for the transformation T . [4]

(c) The transformation T is applied to a triangle with area 3 units². The area of the resulting image triangle is 15 units².
Find the possible values of λ . [3]

13.

$$\mathbf{M} = \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}$$

where k is a constant.

Given that

$$\mathbf{M}^2 + 11\mathbf{M} = a\mathbf{I}$$

where a is a constant and \mathbf{I} is the 2×2 identity matrix,

(a) (i) determine the value of a

(ii) show that $k = -9$

(3)

(b) Determine the equations of the invariant lines of the transformation represented by \mathbf{M} .

(6)

(c) State which, if any, of the lines identified in (b) consist of fixed points, giving a reason for your answer.

(1)

- 14 A transformation T of the plane is represented by the matrix $\mathbf{M} = \begin{pmatrix} k+1 & -1 \\ 1 & k \end{pmatrix}$, where k is a constant.

Show that, for all values of k , T has no invariant lines through the origin.

[6]

15 The matrix \mathbf{A} is given by $\mathbf{A} = \frac{1}{13} \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$.

You are given that \mathbf{A} represents the transformation T which is a reflection in a certain straight line. You are also given that this straight line, the mirror line, passes through the origin, O .

- (a) Explain why there must be a line of invariant points for T . State the geometric significance of this line. [2]
- (b) By considering the line of invariant points for T , determine the equation of the mirror line. Give your answer in the form $y = mx + c$. [4]

The coordinates of the point P are $(1, 5)$.

- (c) By considering the image of P under the transformation T , or otherwise, determine the coordinates of the point on the mirror line which is closest to P . [3]
- (d) The line with equation $y = ax + 2$ is an invariant line for T . Determine the value of a . [2]

16 The 2×2 matrix **A** represents a transformation T which has the following properties.

- The image of the point $(0, 1)$ is the point $(3, 4)$.
- An object shape whose area is 7 is transformed to an image shape whose area is 35.
- T has a line of invariant points.

(i) Find a possible matrix for **A**. **[8]**

The transformation S is represented by the matrix **B** where $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

(ii) Find the equation of the line of invariant points of S. **[2]**

(iii) Show that any line of the form $y = x + c$ is an invariant line of S. **[3]**

17 A transformation T of the plane has matrix \mathbf{M} , where $\mathbf{M} = \begin{pmatrix} \cos \theta & 2 \cos \theta - \sin \theta \\ \sin \theta & 2 \sin \theta + \cos \theta \end{pmatrix}$.

(a) Show that T leaves areas unchanged for all values of θ . [2]

(b) Find the value of θ , where $0 < \theta < \frac{1}{2}\pi$, for which the y -axis is an invariant line of T . [4]

The matrix \mathbf{N} is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(c) (i) Find \mathbf{MN}^{-1} . [2]

~~(ii) Hence describe fully a sequence of two transformations of the plane that is equivalent to T . [4]~~