London Academy of Maths

Invariant Points and Lines

Worked Solutions

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1 Find the equation of the line of invariant points under the transformation given by the matrix $\mathbf{M} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$. [3]

$$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$-x - y = x \Rightarrow y = -2x$

$2x + 2y = y \Rightarrow y = -2x$

So $y = -2x$ is line of invariant points

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2 Find the invariant line of the transformation of the x-y plane represented by the matrix $\begin{pmatrix} 2 & 0 \\ 4 & -1 \end{pmatrix}$. **[4]**

$$\begin{pmatrix} 2 & 0 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix}$$

(A) $2x = X$

(B) $4x - (mx+c) = mX + c$

sub (A) into (B):

$$4x - (mx+c) = m(2x) + c$$

compare coefficients:

$[x]$: $4 - m = 2m \Rightarrow 3m = 4$
$\Rightarrow m = 4/3$

$[h]$ $-c = c \Rightarrow c = 0$

I use h to refer to constant terms.

So $\boxed{y = 4/3\, x}$ is an invariant line

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3 The matrix **M** is given by $\mathbf{M}=\begin{pmatrix}-\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{pmatrix}$.

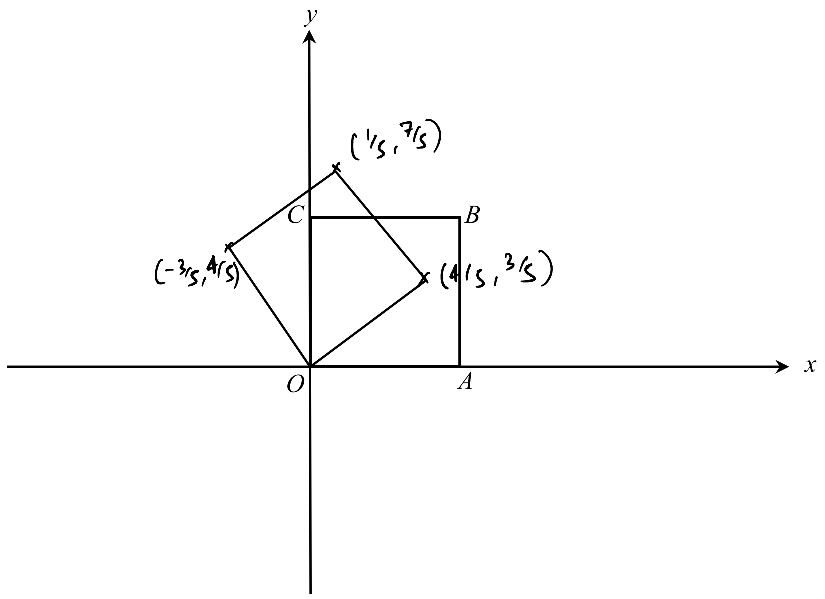
(i) The diagram in the Printed Answer Booklet shows the unit square $OABC$. The image of the unit square under the transformation represented by **M** is $OA'B'C'$. Draw and clearly label $OA'B'C'$. **[3]**

(ii) Find the equation of the line of invariant points of this transformation. **[3]**

(iii) (a) Find the determinant of **M**. **[1]**

(b) Describe briefly how this value relates to the transformation represented by **M**. **[2]**

3(i)



$$\begin{pmatrix}-3/5 & 4/5\\ 4/5 & 3/5\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}=\begin{pmatrix}0 & -3/5 & 1/5 & 4/5\\ 0 & 4/5 & 7/5 & 3/5\end{pmatrix}$$

ii) $\begin{pmatrix}-3/5 & 4/5\\ 4/5 & 3/5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$

$-3/5\,x + 4/5\,y = x \Rightarrow 4/5\,y = 8/5\,x \Rightarrow y = 2x$

$4/5\,x + 3/5\,y = y \Rightarrow 2/5\,y = 4/5\,x \Rightarrow y = 2x$

So $\boxed{y = 2x}$

iii) a) $\det M = -3/5 \times 3/5 - 4/5 \times 4/5$

$= -1$

b) so the area remains the same

but the orientation of the image has changed.

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4 **(i)** The matrix $\mathbf{S} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$ represents a transformation.

(*A*) Show that the point $(1, 1)$ is invariant under this transformation. [1]

(*B*) Calculate $\mathbf{S}^{-1}$. [2]

(*C*) Verify that $(1, 1)$ is also invariant under the transformation represented by $\mathbf{S}^{-1}$. [1]

(ii) Part **(i)** may be generalised as follows.

If (x, y) is an invariant point under a transformation represented by the non-singular matrix $\mathbf{T}$, it is also invariant under the transformation represented by $\mathbf{T}^{-1}$.

Starting with $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, or otherwise, prove this result. [2]

i) A) $\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+2 \\ -3+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

B) $S^{-1} = \frac{1}{\det(S)}\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$

c) $\frac{1}{2}\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4-2 \\ 3-1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ii) suppose (x, y) is an invariant point under T where T is non-singular

then $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow T^{-1}T\begin{pmatrix} x \\ y \end{pmatrix} = T^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ since T is non-singular T^{-1} exists

$I\begin{pmatrix} x \\ y \end{pmatrix} = T^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ since $I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

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5.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **A**.
(3)

The transformation V, represented by the 2×2 matrix **B**, is a reflection in the line $y = -x$

(b) Write down the matrix **B**.
(1)

Given that U followed by V is the transformation T, which is represented by the matrix **C**,

(c) find the matrix **C**.
(2)

(d) Show that there is a real number k for which the point $(1, k)$ is invariant under T.
(4)

a) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad \cos\theta = -1/2$

$\Rightarrow \theta = 120^\circ$ or 240°

Since $\cos(360-\theta) = \cos\theta$

$\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60^\circ$ or 120°

Since $\sin(180-\theta) = \sin\theta$

So A represents a rotation 120° anticlockwise about the origin.

b) From formula booklet:

reflection in line $y = \pm x$: $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$

So $B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

c) order matters.

since its U followed by V, we do B × A

$$C = BA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

d)
$$\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

$$-\sqrt{3}/2 + \tfrac{1}{2}k = 1 \quad \Rightarrow \quad k = 2+\sqrt{3}$$

$$\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}k = k \quad \Rightarrow \quad (2-\sqrt{3})k = 1$$

$$k = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$

So $\boxed{k = 2+\sqrt{3}}$

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6

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix **M** is non-singular.

(2)

The transformation T of the plane is represented by the matrix **M**.

The triangle R is transformed to the triangle S by the transformation T.

Given that the area of S is 63 square units,

(b) find the area of R.

(2)

(c) Show that the line $y = 2x$ is invariant under the transformation T.

(2)

a) $\det M = 4(-7) - (-5)(2)$

$= -28 + 10 = -18$

$-18 \neq 0$ so Matrix M is non-singular

b) Area of $R \times |\det M| =$ Area of S

Area of $R \times 18 = 63$

Area of $R = \frac{63}{18} = \boxed{\frac{7}{2}}$ or $\boxed{3.5}$ (square units)

c) $\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 4x - 10x \\ 2x - 14x \end{pmatrix}$

$= \begin{pmatrix} -6x \\ -12x \end{pmatrix} = -6\begin{pmatrix} x \\ 2x \end{pmatrix}$

so $y = 2x$ is invariant.

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7 You are given that the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ represents a transformation T.

(a) You are given that the line with equation $y = kx$ is invariant under T.

Determine the value of k. **[4]**

(b) Determine whether the line with equation $y = kx$ in part **(a)** is a line of invariant points under T. **[1]**

a) $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} X \\ kX \end{pmatrix}$

A $\quad 2x + kx = X$

B $\quad -x = kX$

Sub A into B

$-x = k(2x + kx)$

$-x = 2kx + k^2x$

Take coefficients of x:

$-1 = 2k + k^2$

$k^2 + 2k + 1 = 0$

$(k+1)^2 = 0$

$k = -1$

b) $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} 2x - x \\ -x \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix}$

So yes, it is a line of invariant point since each point maps to itself.

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8. $$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix **A** represents the linear transformation M.

Prove that, for the linear transformation M, there are no invariant lines.

(5)

Suppose there are invariant lines of the form $y = mx + c$,

then $$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix} \quad (*)$$

(A) $4x - 2(mx+c) = X$

(B) $5x + 3(mx+c) = mX + c$

sub (A) into (B):

$$5x + 3(mx+c) = m(4x - 2(mx+c)) + c$$

Take coefficients of x:

$$5 + 3m = 4m - 2m^2$$

$$\Rightarrow 2m^2 - m + 5 = 0$$

$$b^2 - 4ac = (-1)^2 - 4(2)(5)$$
$$= -39 < 0$$

So there are no real values of m that satisfy $(*)$

Contradiction, thus there are no invariant lines.

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9 P, Q and T are three transformations in 2-D.

P is a reflection in the x-axis. **A** is the matrix that represents P.

(a) Write down the matrix **A**. **[1]**

Q is a shear in which the y-axis is invariant and the point $\begin{pmatrix}1\\0\end{pmatrix}$ is transformed to the point $\begin{pmatrix}1\\2\end{pmatrix}$. **B** is the matrix that represents Q.

(b) Find the matrix **B**. **[2]**

T is P followed by Q. **C** is the matrix that represents T.

(c) Determine the matrix **C**. **[2]**

L is the line whose equation is $y = x$.

(d) Explain whether or not L is a line of invariant points under T. **[2]**

An object parallelogram, M, is transformed under T to an image parallelogram, N.

(e) Explain what the value of the determinant of **C** means about
- the area of N compared to the area of M,
- the orientation of N compared to the orientation of M. **[3]**

a) $A = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

b) $\begin{pmatrix}1 & 0\\k & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$\Rightarrow k = 2$ so $B = \begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}$

c) $C = BA$

$= \begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\2 & -1\end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} x \\ 2x-x \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$

So yes, it's a line of invariant poits.

e) $\det(C) = 1(-1) - 0(2)$

$= -1$

- So area of N = area of M
- orientation of N is the everse of the orientation of M.

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10. (i)

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}$$

where a and b are non-zero constants.

Given that the matrix **A** is self-inverse,

(a) determine the value of b and the possible values for a.

(5)

The matrix **A** represents a linear transformation M.

Using the smaller value of a from part (a),

(b) show that the invariant points of the linear transformation M form a line, stating the equation of this line.

(3)

i a) A is self inverse

$\Rightarrow A = A^{-1}$

$\times A(\quad) \times A$

$A^2 = A^{-1}A = I$

So $\begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}\begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 4 + a(a-4) & 2a + ab \\ (a-4)2 + b(a-4) & (a-4)a + b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} a^2 - 4a + 4 & 2a + ab \\ 2a - 8 + ab - 4b & a^2 - 4a + b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$a^2 - 4a + 4 = 1$

$a^2 - 4a + 3 = 0$

$(a-3)(a-1) = 0$

$a = 3$ or $a = 1$

Sub into $2a + ab = 0$

$2(3) + 3b = 0$

$6 + 3b = 0$

$\Rightarrow b = -2$

or

$2(1) + b = 0$

$2 + b = 0$

$b = -2$

So $\boxed{a = 3 \text{ or } 1 \quad \text{and } b = -2}$

b) $\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$2x + y = x \quad \Rightarrow y = x$

$-3x - 2y = y \quad 3y = 3x$

$\Rightarrow y = x$

So $y = x$ is a line of invariant points

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11 You are given that matrix $\mathbf{M} = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$.

(i) Prove that, for all positive integers n, $\mathbf{M}^n = \begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}$. **[6]**

(ii) Determine the equation of the line of invariant points of the transformation represented by the matrix $\mathbf{M}$. **[3]**

It is claimed that the answer to part **(ii)** is also a line of invariant points of the transformation represented by the matrix $\mathbf{M}^n$, for any positive integer n.

(iii) Explain *geometrically* why this claim is true. **[2]**

(iv) Verify *algebraically* that this claim is true. **[3]**

i) Proof by induction

Base case:

if $n=1$, $M^n = \begin{pmatrix} 1-4(1) & 8(1) \\ -2(1) & 1+4(1) \end{pmatrix} = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$

So the result is true for $n=1$

Inductive Hypothesis

Assume the result is true for $n=k$,

i.e. $M^k = \begin{pmatrix} 1-4k & 8k \\ -2k & 1+4k \end{pmatrix}$

Inductive Step

or $M \times M^k$

$M^{k+1} = M^k \times M$

$= \begin{pmatrix} 1-4k & 8k \\ -2k & 1+4k \end{pmatrix} \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} -3+12k-16k & 8-32k+40k \\ 6k-2-8k & -16k+5+20k \end{pmatrix}$$

$$= \begin{pmatrix} -3-4k & 8+8k \\ -2k-2 & 4k+5 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4-4k & 8+8k \\ -2k-2 & 1+4+4k \end{pmatrix}$$

$$= \begin{pmatrix} 1-4(k+1) & 8(k+1) \\ -2(k+1) & 1+4(k+1) \end{pmatrix}$$

So result is also true for $n=k+1$.

Since, the result is true for $n=1$ and if it's true for $n=k$ then it's also true for $n=k+1$, we have shown by induction it's true for all positive integers n.

ii) $\begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$-3x+8y=x \Rightarrow y=\frac{1}{2}x$

$-2x+5y=y \Rightarrow y=\frac{1}{2}x$

So $y=\frac{1}{2}$ is the line of invariant points

iii) M^n is just the transformation repeated n times and each repeat leaves points on $y=\frac{1}{2}x$ invariant. So $y=\frac{1}{2}x$ must also be a line of invariant points for M^n.

iv) Recall that the long way to do this (when you have the line) is to apply the same method as usual, i.e.

$$\begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and find that $y = \frac{1}{2}x$

Instead,

$$\begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{2}x \end{pmatrix} = \begin{pmatrix} (1-4n)x + (8n)(\frac{1}{2}x) \\ -2nx + \frac{1}{2}x(1+4n) \end{pmatrix}$$

$$= \begin{pmatrix} x - 4nx + 4nx \\ -2nx + \frac{1}{2}x + 2nx \end{pmatrix}$$

$$= \begin{pmatrix} x \\ \frac{1}{2}x \end{pmatrix}$$

So it's the same invariant line for all values of n.

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12 A linear transformation T of the x-y plane has an associated matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} \lambda & k \\ 1 & \lambda - k \end{pmatrix}$, and λ and k are real constants.

(a) You are given that $\det \mathbf{M} > 0$ for all values of λ.

(i) Find the range of possible values of k. **[3]**

(ii) What is the significance of the condition $\det \mathbf{M} > 0$ for the transformation T? **[1]**

For the remainder of this question, take $k = -2$.

(b) Determine whether there are any lines through the origin that are invariant lines for the transformation T. **[4]**

(c) The transformation T is applied to a triangle with area 3 units^2. The area of the resulting image triangle is 15 units^2.
Find the possible values of λ. **[3]**

a) i) $\det M = \lambda(\lambda - k) - k = \lambda^2 - \lambda k - k > 0$ for all $\lambda \in \mathbb{R}$

$\Rightarrow b^2 - 4ac < 0$ i.e. $k^2 - 4(-k)(1) > 0$

$k^2 + 4k < 0$

$k(k+4) < 0$

$-4 < k < 0$

ii) the Transformation T preserves orientation of shapes.

b) $\begin{pmatrix} \lambda & -2 \\ 1 & \lambda+2 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$

(A) $\lambda x - 2(mx) = X$

(B) $x + (\lambda+2)mx = mX$

sub (A) into (B) : $x + (\lambda+2)\,mx = m(\lambda x - 2mx)$

$[x]$: $1 + (\lambda+2)m = \lambda m - 2m^2$

$\Rightarrow 1 + \lambda m + 2m = \lambda m - 2m^2 \Rightarrow 2m^2 + 2m + 1 = 0$

$b^2-4ac = 2^2-2\times 4$
$= -4 < 0$

So no real roots for m

$\therefore$ no invariant lines exist.

c) $3\times|\det M| = 15$

$3\times|\lambda^2+2\lambda+2| = 15$

$3\times(\lambda^2+2\lambda+2) = 15$

Since $\det M > 0$ for all values of λ if $-4 < k < 0$

$\lambda^2+2\lambda+2 = 5$

$\lambda^2+2\lambda-3 = 0$

$(\lambda+3)(\lambda-1) = 0$

$\lambda = -3$ or $\lambda = 1$

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13. $$\mathbf{M} = \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}$$

where k is a constant.

Given that

$$\mathbf{M}^2 + 11\mathbf{M} = a\mathbf{I}$$

where a is a constant and $\mathbf{I}$ is the 2×2 identity matrix,

(a) (i) determine the value of a

(ii) show that $k = -9$

(3)

(b) Determine the equations of the invariant lines of the transformation represented by $\mathbf{M}$.

(6)

(c) State which, if any, of the lines identified in (b) consist of fixed points, giving a reason for your answer.

(1)

a) i) $\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix} + 11\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 34 & -10+5k \\ -12+6k & 30+k^2 \end{pmatrix} + \begin{pmatrix} -22 & 55 \\ 66 & 11k \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$\begin{pmatrix} 12 & 45+5k \\ 54+6k & 30+k^2+11k \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$12 = a$

ii) $54 + 6k = 0$

$\Rightarrow 6k = -54$

$k = -9$

b) $\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix}$

Ⓐ $-2x + 5(mx+c) = X$

Ⓑ $6x - 9(mx+c) = mX + c$

sub Ⓐ into Ⓑ :

$6x - 9(mx+c) = m(-2x + 5(mx+c)) + c$

$[x]$: $6 - 9m = -2m + 5m^2$ $\qquad$ $[n]$: $-9c = 5mc + c$

$5m^2 + 7m - 6 = 0$ $\qquad$ $\Rightarrow -10c = 5mc$

$(5m - 3)(m+2) = 0$ $\qquad$ $-2c = mc$

$m = 3/5$ or $m = -2$

if $m = -2$, $-2c = -2c$

$\Rightarrow c$ is any real number

if $m = 3/5$ $\quad -2c = 3/5\, c$

$\Rightarrow c = 0$

so $\boxed{y = 3/5\,x \;\&\; y = -2x + c, \text{ where } c \in \mathbb{R}}$ are the invariant lines.

c) $\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $\begin{array}{ll} -2x + 5y = x & \Rightarrow y = 3/5\,x \\ 6x - 9y = y & \Rightarrow y = 3/5\,x \end{array}$

So $y = 3/5$ is also a line of invariant points

So it must contain fixed points.

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14 A transformation T of the plane is represented by the matrix $\mathbf{M} = \begin{pmatrix} k+1 & -1 \\ 1 & k \end{pmatrix}$, where k is a constant.

Show that, for all values of k, T has no invariant lines through the origin. **[6]**

Suppose there are invariant lines through the origin, i.e.

$$\begin{pmatrix} k+1 & -1 \\ 1 & k \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$ for some $m \in \mathbb{R}$

(A) $(k+1)x - mx = X$

(B) $x + kmx = mX$

Sub (A) into (B):

$$x + kmx = m\left((k+1)x - mx\right)$$

$[x]$: $1 + km = m(k+1) - m^2$

$1 + \cancel{km} = \cancel{km} + m - m^2$

$m^2 - m + 1 = 0$

$b^2 - 4ac = (-1)^2 - 4(1) = -3 < 0$

So no real roots of m.

Contradiction.

Thus there are no invariant lines.

15 The matrix **A** is given by $\mathbf{A} = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$.

You are given that **A** represents the transformation T which is a reflection in a certain straight line. You are also given that this straight line, the mirror line, passes through the origin, O.

(a) Explain why there must be a line of invariant points for T. State the geometric significance of this line. **[2]**

(b) By considering the line of invariant points for T, determine the equation of the mirror line. Give your answer in the form $y = mx + c$. **[4]**

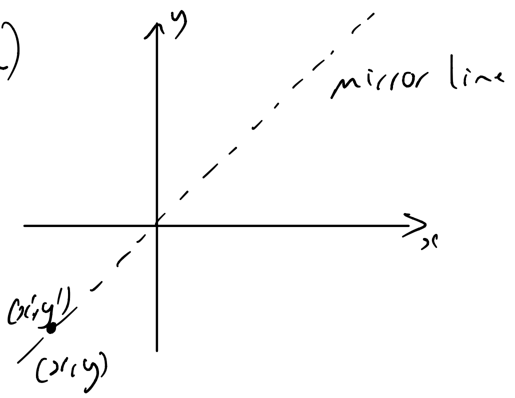
The coordinates of the point P are (1, 5).

(c) By considering the image of P under the transformation T, or otherwise, determine the coordinates of the point on the mirror line which is closest to P. **[3]**

(d) The line with equation $y = ax + 2$ is an invariant line for T.

Determine the value of a. **[2]**

a)



There must be a line of invariant points because every point on the mirror line, following a reflection remains invariant.

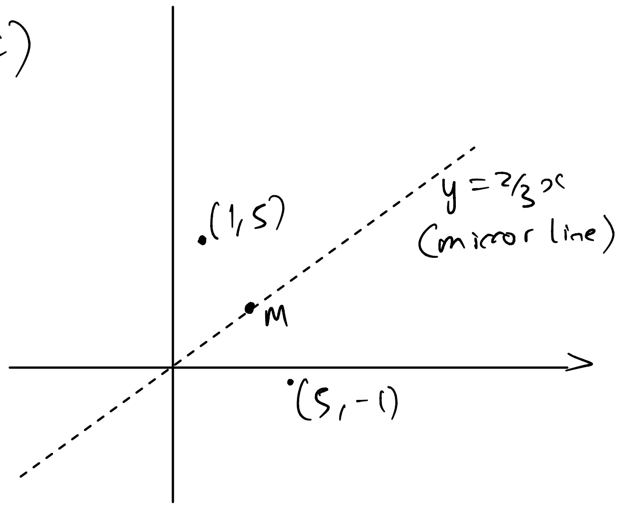
The significance of the mirror line is that it must itself be a line of invariant points.

b) $\frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\frac{1}{13}(5x + 12y) = x \quad \rightarrow \quad 5x + 12y = 13x \quad \Rightarrow \quad y = \frac{2}{3}x$

$\frac{1}{13}(12x - 5y) = y \quad \Rightarrow \quad 12x - 5y = 13y \quad \Rightarrow \quad y = \frac{2}{3}x$

so $y = 2/3$ is the line of invariant points c)

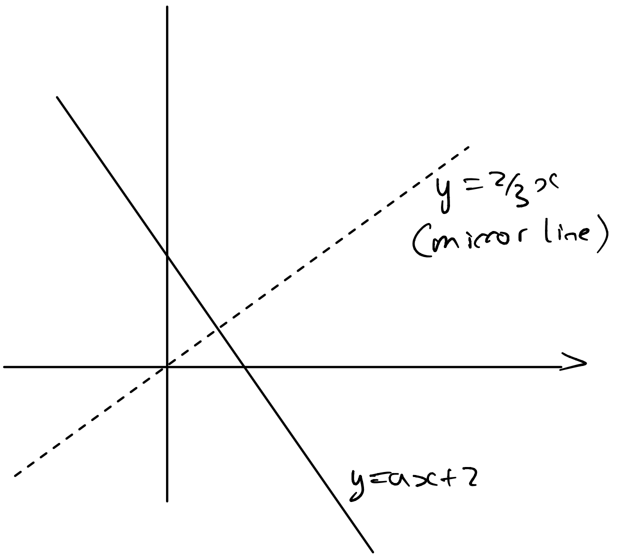


$$\frac{1}{13}\begin{pmatrix}5 & 12\\12 & -5\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix}=\frac{1}{13}\begin{pmatrix}65\\-13\end{pmatrix}$$

$$=\begin{pmatrix}5\\-1\end{pmatrix}$$

$$M=\left(\frac{1+5}{2},\ \frac{5-1}{2}\right)=\boxed{(3,2)}$$

d) Method 1: (Fastest)



Invariant lines must be perpendicular to the mirror line because they need to reflect back onto themselves.

So by simple coordinate geometry rules. $a=-3/2$

Method 2: (Almost identical to method 3)

$$\frac{1}{13}\begin{pmatrix}5 & 12\\12 & -5\end{pmatrix}\begin{pmatrix}x\\ax+2\end{pmatrix}=\frac{1}{13}\begin{pmatrix}5x+12(ax+2)\\12x-5(ax+2)\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{13}(5+12a)x+\frac{24}{13}\\ \frac{1}{13}x(12-5a)-\frac{10}{13}\end{pmatrix}$$

So if this is X

then this must be $aX+2$

i.e. $a\left(\frac{1}{13}(5+12a)x+\frac{24}{13}\right)+2=\frac{1}{13}x(12-5a)-\frac{10}{13}$

We either check coefficients of x, or the constants. Since we know the invariant line exists, we can just pick the easier option (the constants):

$$\frac{24}{13}a + 2 = -\frac{10}{13}$$

$$24a + 26 = -10$$

$$24a = -36 \quad \Rightarrow a = -3/2$$

Method 3: (Classic method)

$$\frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}\begin{pmatrix} x \\ ax+2 \end{pmatrix} = \begin{pmatrix} X \\ aX+2 \end{pmatrix}$$

(A): $\frac{1}{13}(5x + 12ax + 24) = X$

(B) $\frac{1}{13}(12x - 5(ax+2)) = aX + 2$

Sub (A) into (B): and compare coefficients

. . . . .

. . . .

$a = -3/2$

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16 The 2×2 matrix **A** represents a transformation T which has the following properties.

- The image of the point $(0, 1)$ is the point $(3, 4)$.
- An object shape whose area is 7 is transformed to an image shape whose area is 35.
- T has a line of invariant points.

(i) Find a possible matrix for **A**. **[8]**

The transformation S is represented by the matrix **B** where $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

(ii) Find the equation of the line of invariant points of S. **[2]**

(iii) Show that any line of the form $y = x + c$ is an invariant line of S. **[3]**

i) • $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$b = 3$

$d = 4$

so $A = \begin{pmatrix} a & 3 \\ c & 4 \end{pmatrix}$

• Now, $|\det A| = \frac{35}{7}$

$\Rightarrow 4a - 3c = \pm 5$

• $\begin{pmatrix} a & 3 \\ c & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$ax + 3y = x \Rightarrow (a-1)x + 3y = 0$

$cx + 4y = y \Rightarrow cx + 3y = 0$

so $a - 1 = c$

since questions asks to find a possible matrix of A, we can take $4a - 3c = 5$

Subbing $c = a - 1$ into $4a - 3c = 5$ gives:

$4a - 3(a-1) = 5$

$a + 3 = 5$

$a = 2 \Rightarrow c = 1$

so $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

ii) $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$3x + y = x \quad \Rightarrow \quad y = -2x$

$2x + 2y = y \quad \rightarrow \quad y = -2x$

So $y = -2x$ is a line of invariant points.

iii) when the line is given you don't want to do the classic method of $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix}$ because this is what you use to find m & c, and takes longer.

You do the following:

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ x+c \end{pmatrix} = \begin{pmatrix} 3x + x+c \\ 2x+2(x+c) \end{pmatrix}$$

$$= \begin{pmatrix} 4x+c \\ 4x+2c \end{pmatrix}$$

$$= \begin{pmatrix} 4x+c \\ 4x+c \ + c \end{pmatrix}$$

So $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4x+c \\ 4x+c \ +c \end{pmatrix}$ and $Y = X + c$

See Question 6c, 7b, 9d for similar examples

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17 A transformation T of the plane has matrix **M**, where $\mathbf{M} = \begin{pmatrix} \cos\theta & 2\cos\theta - \sin\theta \\ \sin\theta & 2\sin\theta + \cos\theta \end{pmatrix}$.

(a) Show that T leaves areas unchanged for all values of θ. **[2]**

(b) Find the value of θ, where $0 < \theta < \frac{1}{2}\pi$, for which the y-axis is an invariant line of T. **[4]**

The matrix **N** is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(c) (i) Find $\mathbf{MN}^{-1}$. **[2]**

(ii) Hence describe fully a sequence of two transformations of the plane that is equivalent to T. **[4]**

a) $\det M = \cos\theta(2\sin\theta + \cos\theta) - (2\cos\theta - \sin\theta)\sin\theta$

$= 2\cos\theta\sin\theta + \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta$

$= \cos^2\theta + \sin^2\theta$

$\equiv 1$

det M represents the area scale factor. so the areas must be unchanged.

b) $\begin{pmatrix} \cos\theta & 2\cos\theta - \sin\theta \\ \sin\theta & 2\sin\theta + \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ Y \end{pmatrix}$

$(2\cos\theta - \sin\theta)y = 0$ for all y

So $2\cos\theta - \sin\theta = 0$

$\Rightarrow$ $2\cos\theta = \sin\theta$

$\div\cos\theta$ $\quad$ $\div\cos\theta$

$\tan\theta = 2$ $\quad$ $\boxed{\theta = \arctan(2) = 1.11 \text{ radians}}$

c i) $N^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ by calculation of determinant and so on.

$$MN^{-1} = \begin{pmatrix} \cos\theta & 2\cos\theta - \sin\theta \\ \sin\theta & 2\sin\theta + \cos\theta \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -2\cos\theta + 2\cos\theta - \sin\theta \\ \sin\theta & -2\sin\theta + 2\sin\theta + \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

ii) let $A = MN^{-1}$

$MN^{-1} = A$

$MN^{-1}N = AN$

$M = AN$

so T is the transformation represented by N followed by the transformation represented by A

i.e. a shear with x axis invariant mapping $(0,1)$ to $(2,1)$ followed by a rotation anticlockwise through θ, about the origin.