London Academy of Maths

Linear Transformations

Worked Solutions

1. (ii) The matrix **B** is defined by

$$\mathbf{B} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

where p and q are integers.

State the value of p and the value of q when **B** represents

(a) an enlargement about the origin with scale factor –2

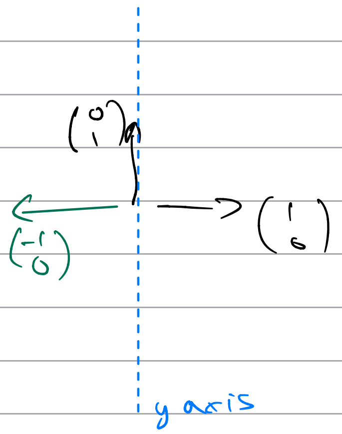
(b) a reflection in the y–axis.

(2)

a) $p = -2$, $q = -2$.

b) $p = -1$, $q = 1$

b)



$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

2.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

(a) Describe the single geometrical transformation represented by the matrix **A**.
(2)

The matrix **B** represents a rotation of 210° anticlockwise about centre (0, 0).

(b) Write down the matrix **B**, giving each element in exact form.
(1)

The transformation represented by matrix **A** followed by the transformation represented by matrix **B** is represented by the matrix **C**.

(c) Find **C**.
(2)

The hexagon H is transformed onto the hexagon H' by the matrix **C**.

(d) Given that the area of hexagon H is 5 square units, determine the area of hexagon H'
(2)

a) stretch parallel to y-axis, x-axis invariant, with scale factor 3

b) $B = \begin{pmatrix} \cos(210) & -\sin(210) \\ \sin(210) & \cos(210) \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

c) $C = BA = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix}$

d) $\det C = 3$ (either by calculation or by recognising that determinant of a rotation matrix is always 1 and $\det(A) = 3$). area of $H' = 5 \times 3 = \boxed{15}$

3. $\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad\quad$ $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

(a) (i) Describe fully the single geometrical transformation P represented by the matrix **P**.

(ii) Describe fully the single geometrical transformation Q represented by the matrix **Q**.
(4)

The transformation P followed by the transformation Q is the transformation R, which is represented by the matrix **R**.

(b) Determine **R**.
(1)

(c) (i) Evaluate the determinant of **R**.

(ii) Explain how the value obtained in (c)(i) relates to the transformation R.
(2)

ai) $\cos\theta = 0 \Rightarrow \theta = 90$ or $270°$

$\sin\theta = 1 \Rightarrow \theta = 90°$

so P is a rotation anticlockwise by 90° (about O)

ii) stretch parallel to the y-axis with scale factor 3 with x-axis invariant.

b) $R = QP = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$

ci) $\det R = 3$

ii) it means that transformation R will increase the area of a shape by scale factor 3.

Leave blank

4. (i)
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

(a) Describe fully the single transformation represented by the matrix **A**.
(2)

The matrix **B** represents a rotation of 45° clockwise about the origin.

(b) Write down the matrix **B**, giving each element of the matrix in exact form.
(1)

The transformation represented by matrix **A** followed by the transformation represented by matrix **B** is represented by the matrix **C**.

(c) Determine **C**.
(2)

(ii) The trapezium T has vertices at the points $(-2, 0)$, $(-2, k)$, $(5, 8)$ and $(5, 0)$, where k is a positive constant. Trapezium T is transformed onto the trapezium T' by the matrix

$$\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$$

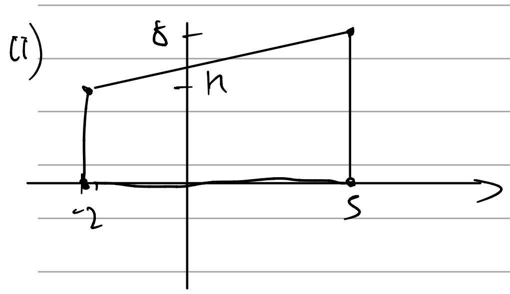
Given that the area of trapezium T' is 510 square units, calculate the exact value of k.
(5)

i) a) stretch parallel to y axis, x axis invariant, scale factor 3.

b) $B = \begin{pmatrix} \cos(315) & -\sin(315) \\ \sin(315) & \cos(315) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

c) $C = BA = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \end{pmatrix}$

(ii)



Area of trapezium T =

$\frac{7(8+h)}{2}$

Question 4 continued

$$\det\left(\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}\right) = 17$$

so $$\frac{7(8+h)}{2} \times 17 = 510$$

$$\frac{7(8+h)}{2} = 30$$

$$7(8+h) = 60$$

$$h + 8 = \frac{60}{7}$$

$$\boxed{h = \frac{4}{7}}$$

5. $\mathbf{M} = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}^4$

The transformation represented by matrix **M** followed by the transformation represented by matrix **N** is represented by the matrix **B**

(b) (i) Determine **N** in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c and d are integers.

(ii) Determine **B**

(3)

Hexagon S is transformed onto hexagon S' by matrix **B**

(c) Given that the area of S' is 720 square units, determine the area of S

(2)

bi) $N = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}^4 = \left(\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}\right)^2$

$= \begin{pmatrix} 1 & -6 \\ 0 & 4 \end{pmatrix}^2$

$= \begin{pmatrix} 1 & -6 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -6 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -30 \\ 0 & 16 \end{pmatrix}$

ii) $B = NM$

$= \begin{pmatrix} 1 & -30 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 4 & -150 \\ 0 & 80 \end{pmatrix}$

c) $\det B = 320$

so area of $S \times 320 = 720$

$\Rightarrow \boxed{\text{area of } S = \frac{9}{4} = 2.25}$

Leave blank

6. The triangle T has vertices $A(2,1)$, $B(2,3)$ and $C(0,1)$.

The triangle T' is the image of T under the transformation represented by the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(a) Find the coordinates of the vertices of T'

(2)

(b) Describe fully the transformation represented by **P**

(2)

The 2×2 matrix **Q** represents a reflection in the x-axis and the 2×2 matrix **R** represents a rotation through 90° anticlockwise about the origin.

(c) Write down the matrix **Q** and the matrix **R**

(2)

(d) Find the matrix **RQ**

(2)

(e) Give a full geometrical description of the single transformation represented by the answer to part (d).

(2)

a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ -2 & -2 & 0 \end{pmatrix}$

T': $(1,-2)$, $(3,-2)$, $(1,0)$

b) $\cos\theta = 0 \Rightarrow \theta = 90°$ or $270°$
$\sin\theta = -1 \Rightarrow \theta = 270°$

So a rotation by 270° anticlockwise about the origin.

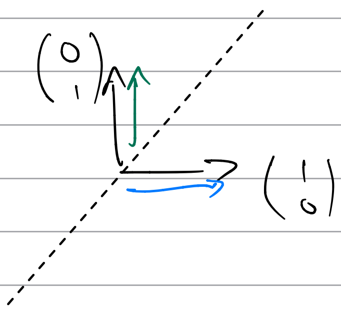
c) $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

d) $RQ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 6 continued

e) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\boxed{\text{is a reflection in the line } y = x}$



7.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe fully the single geometric transformation A represented by the matrix **A**.

(2)

$$\mathbf{B} = \begin{pmatrix} 1 & 3 & 0 \\ \sqrt{3} & 0 & 5\sqrt{3} \\ 1 & 2 & 0 \end{pmatrix}$$

The transformation B is represented by the matrix **B**.
The transformation A followed by the transformation B is the transformation C, which is represented by the matrix **C**.

To determine matrix **C**, a student attempts the following matrix multiplication.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ \sqrt{3} & 0 & 5\sqrt{3} \\ 1 & 2 & 0 \end{pmatrix}$$

(b) State the error made by the student.

(1)

(c) Determine the correct matrix **C**.

(1)

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ $\quad \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ \text{ or } 330^\circ$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = 30^\circ \text{ or } 150^\circ$

so it's a rotation 30° anticlockwise about the x-axis (with respect to the right hand rule)

Question 7 continued

b) The student did $A \times B$ but it should be $B \times A$

c) $C = B \times A$

$$= \begin{pmatrix} 1 & 3 & 0 \\ \sqrt{3} & 0 & 5\sqrt{3} \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ \sqrt{3} & \frac{5\sqrt{3}}{2} & \frac{15}{2} \\ 1 & \sqrt{3} & -1 \end{pmatrix}$$

8. (ii)

$$\mathbf{P} = \begin{pmatrix} p & 2p \\ -1 & 3p \end{pmatrix}$$

where p is a positive constant.

The matrix **P** represents a linear transformation U.
The triangle T has vertices at the points with coordinates (1, 2), (3, 2) and (2, 5).
The area of the image of T under the linear transformation U is 15

(a) Determine the value of p.

(4)

The transformation V consists of a stretch scale factor 3 parallel to the x-axis with the y-axis invariant followed by a stretch scale factor –2 parallel to the y-axis with the x-axis invariant. The transformation V is represented by the matrix **Q**.

(b) Write down the matrix **Q**.

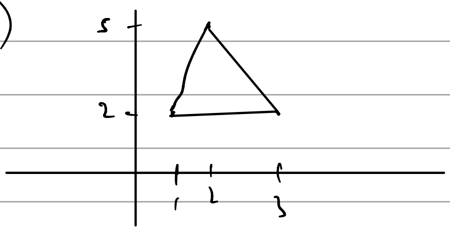
(2)

Given that U followed by V is the transformation W, which is represented by the matrix **R**,

(c) find the matrix **R**.

(2)

a)



Area of triangle $T = \frac{2 \times 3}{2} = 3$

$\Rightarrow$ area scalefactor $= 5$

i.e. $\det P = \pm 5$

$\Rightarrow 3p^2 + 2p = \pm 5$

$3p^2 + 2p \pm 5 = 0$

$3p^2 + 2p + 5 = 0$ has no solutions since $b^2 - 4ac = 4 - 4(3)(5) = -56 < 0$

$3p^2 + 2p - 5 = 0$

$\Rightarrow p = 1$ or $p = -\frac{5}{3}$ but $p > 0$ so $\boxed{p = 1}$

***

Question 8 continued

b) $Q = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$

c) $R = Q \times P$ not $P \times Q$

$= \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 3 & 6 \\ 2 & -6 \end{pmatrix}$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

9. The transformation P is an enlargement, centre the origin, with scale factor k, where $k > 0$

The transformation Q is a rotation through angle θ degrees anticlockwise about the origin.

The transformation P followed by the transformation Q is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

(a) Determine

(i) the value of k,

(ii) the smallest value of θ

(4)

A square S has vertices at the points with coordinates $(0, 0)$, $(a, -a)$, $(2a, 0)$ and (a, a) where a is a constant.

The square S is transformed to the square S' by the transformation represented by $\mathbf{M}$.

(b) Determine, in terms of a, the area of S'

(2)

a) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$

Q P M

$\det(Q)\det(P) = \det(M)$

$1 \times k^2 = 16 + 16 \times 3$

$\Rightarrow k^2 = 64$ so $\boxed{k = 8}$ since $k > 0$

$\cos(\theta)k = -4$ & $\sin\theta(k) = 4\sqrt{3}$

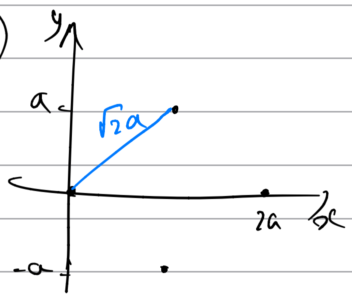
$\Rightarrow \cos\theta = -\frac{1}{2}$ & $\sin\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = 120, 240^\circ$ $\theta = 60^\circ, 120^\circ$

$\boxed{\theta = 120^\circ}$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Question 9 continued

b) 

Area of $S = (\sqrt{2}a)^2 = 2a^2$

so Area of $S' = 64 \times 2a^2$

$= \boxed{128a^2}$

Leave blank

10. The matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 4 & -5 \\ -3 & 2 \end{pmatrix}$$

The transformation represented by **A** maps triangle T onto triangle T'

Given that the area of triangle T is 23 cm^2

(a) determine the area of triangle T'

(2)

The point P has coordinates $(3p + 2, 2p - 1)$ where p is a constant. The transformation represented by **A** maps P onto the point P' with coordinates $(17, -18)$

(b) Determine the value of p.

(2)

Given that

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(c) describe fully the single geometrical transformation represented by matrix **B**

(2)

The transformation represented by matrix **A** followed by the transformation represented by matrix **C** is equivalent to the transformation represented by matrix **B**

(d) Determine **C**

(3)

a) $\det A = 8 - 15 = -7$

So Area of $T' = 23 \times 7 = 161$

b) $\begin{pmatrix} 4 & -5 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 3p+2 \\ 2p-1 \end{pmatrix} = \begin{pmatrix} 17 \\ -18 \end{pmatrix}$

$4(3p+2) - 5(2p-1) = 17$

$\Rightarrow 2p + 13 = 17 \quad \Rightarrow p = 2$

DO NOT WRITE IN THIS AREA

Question 10 continued

c) $\cos\theta = 0 \Rightarrow \theta = 90°, 270°$

$\sin\theta = -1 \Rightarrow \theta = 270°$

so it's a rotation 270° anticlockwise (or 90° clockwise) about the origin.

d) $CA = B$

$CAA^{-1} = BA^{-1}$

$C = BA^{-1}$

$A^{-1} = \frac{1}{-7}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$

$C = \frac{-1}{7}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$

$= \frac{-1}{7}\begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix}$

11.

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

The matrix **P** represents the transformation U

(a) Give a full description of U as a single geometrical transformation.

(2)

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line $y = -x$

(b) Write down the matrix **Q**

(1)

The transformation U followed by the transformation V is represented by the matrix **R**

(c) Determine the matrix **R**

(2)

The transformation W is represented by the matrix $3\mathbf{R}$

The transformation W maps a triangle T to a triangle T'

The transformation W' maps the triangle T' back to the original triangle T

(d) Determine the matrix that represents W'

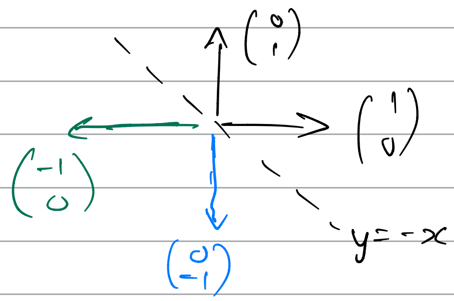
(3)

a) $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°, 300°$

$\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60°, 120°$

so it's a rotation $60°$ anticlockwise about the origin.

b) $\boxed{Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$



DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 11 continued

c) $R = QP = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

d) $W: \quad 3R = \begin{pmatrix} -\frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3\sqrt{3}}{2} \end{pmatrix}$

$W^{-1}: \quad (3R)^{-1} = \frac{1}{\det(3R)}\begin{pmatrix} \frac{3\sqrt{3}}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix}$

$\det(3R) = -9$ so $(3R)^{-1} = \frac{1}{-9}\begin{pmatrix} \frac{3\sqrt{3}}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix}$

$= \boxed{\begin{pmatrix} -\frac{\sqrt{3}}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{\sqrt{3}}{6} \end{pmatrix}}$

12. (i)

$$\mathbf{P} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

The matrix **P** represents a geometrical transformation U

(a) Describe U fully as a single geometrical transformation.

(2)

The transformation V, represented by the 2×2 matrix **Q**, is a rotation through 240° anticlockwise about the origin followed by an enlargement about (0, 0) with scale factor 6

(b) Determine the matrix **Q**, giving each entry in exact numerical form.

(2)

Given that U followed by V is the transformation T, which is represented by the matrix **R**

(c) determine the matrix **R**

(2)

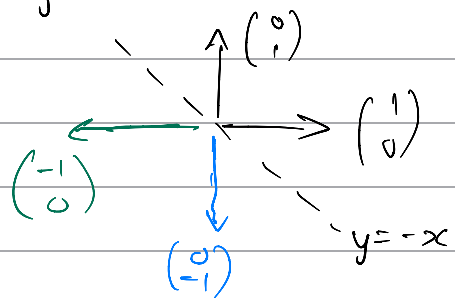
(ii) The transformation W is represented by the matrix

$$\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}$$

Show that there is a real number λ for which W maps the point $(\lambda, 1)$ onto the point $(4\lambda, 4)$, giving the exact value of λ

(5)

i) a) reflection in the line $y = -x$



b) $Q = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} \cos(240) & -\sin(240) \\ \sin(240) & \cos(240) \end{pmatrix}$

$= \begin{pmatrix} 6\cos(240) & -6\sin(240) \\ 6\sin(240) & 6\cos(240) \end{pmatrix} = \boxed{\begin{pmatrix} -3 & 3\sqrt{3} \\ -3\sqrt{3} & -3 \end{pmatrix}}$

Question 12 continued

c) $R = QP$

$$= \begin{pmatrix} -3 & 3\sqrt{3} \\ -3\sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3\sqrt{3} & 3 \\ 3 & 3\sqrt{3} \end{pmatrix}$$

ii) $\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} 4\lambda \\ 4 \end{pmatrix}$

$-2\lambda + 2\sqrt{3} = 4\lambda \implies 6\lambda = 2\sqrt{3}$ so $\lambda = \frac{\sqrt{3}}{3}$

$2\sqrt{3}\lambda + 2 = 4 \implies 2\sqrt{3}\lambda = 2$ so $\lambda = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$
$= \frac{\sqrt{3}}{3}$

$\boxed{\lambda = \frac{\sqrt{3}}{3}}$

here you must solve both equations because solving one only shows that either the x or y coordinate gets mapped to the desired point, but not both.

13. (i)
$$\mathbf{A} = \begin{pmatrix} -3 & 8 \\ -3 & k \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

The transformation represented by **A** transforms triangle T to triangle T'

The area of triangle T' is three times the area of triangle T

Determine the possible values of k

(4)

(ii)
$$\mathbf{B} = \begin{pmatrix} a & -4 \\ 2 & 3 \end{pmatrix} \text{ and } \mathbf{BC} = \begin{pmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \end{pmatrix} \text{ where } a \text{ is a constant}$$

Determine, in terms of a, the matrix **C**

(4)

i) $\det A = -3k + 24$

so $|-3k+24| = 3$ since $|\det A|$ is the asf.

$-3k+24 = 3$ or $-3k+24=-3$

$\Rightarrow -3k = -21$ $\quad -3k = -27$

so $k = 7$ $\quad k = 9$

$\boxed{k = 7 \text{ or } 9}$

ii)
$$\begin{pmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} a & -4 \\ 2 & 3 \end{pmatrix} C$$

$$B^{-1}\begin{pmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \end{pmatrix} = C$$

$$C = \frac{1}{3a+8}\begin{pmatrix} 3 & 4 \\ -2 & a \end{pmatrix}\begin{pmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \end{pmatrix}$$

$$= \boxed{\frac{1}{3a+8}\begin{pmatrix} 10 & 31 & 11 \\ a-4 & 4a-10 & 2a-2 \end{pmatrix}}$$

14.
$$\mathbf{A} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Determine the matrix $\mathbf{A}^2$ **(1)**

(b) Describe fully the single geometrical transformation represented by the matrix $\mathbf{A}^2$ **(2)**

(c) Hence determine the smallest positive integer value of n for which $\mathbf{A}^n = \mathbf{I}$ **(1)**

The matrix **B** represents a stretch scale factor 4 parallel to the x-axis.

(d) Write down the matrix **B** **(1)**

The transformation represented by matrix **A** followed by the transformation represented by matrix **B** is represented by the matrix **C**

(e) Determine the matrix **C** **(2)**

The parallelogram P is transformed onto the parallelogram P' by the matrix **C**

(f) Given that the area of parallelogram P' is 20 square units, determine the area of parallelogram P **(2)**

a) $A^2 = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

b) $\cos\theta = 1/2 \qquad \theta = 60° \text{ or } 300°$

$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \theta = -60°, 240 \text{ or } 300°$

So a rotation by 300° anticlockwise (or 60° clockwise) about the origin.

DO NOT WRITE IN THIS AREA

Question 14 continued

c) A is a rotation $30°$ clockwise about O.

so A^{12} is a rotation $360°$ clockwise, i.e. the identity.

so $\boxed{n = 12}$

d) $B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

e) $C = BA = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -2\sqrt{3} & -2 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

f) $\det C = 4$ (because it's a rotation (det always will be 1) followed by a stretch).

area of $P = \frac{20}{4} = \boxed{5}$ square units

15. *With respect to the **right-hand rule**, a rotation through $\theta°$ anticlockwise about the y-axis is represented by the matrix*

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

The point P has coordinates (8, 3, 2)

The point Q is the image of P under the transformation reflection in the plane $y = 0$

(a) Write down the coordinates of Q

(1)

The point R is the image of P under the transformation rotation through 120° anticlockwise about the y-axis, with respect to the **right-hand rule**.

(b) Determine the exact coordinates of R

(2)

(c) Hence find $|\overrightarrow{PR}|$ giving your answer as a simplified surd.

(2)

(d) Show that $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ are perpendicular.

(1)

(e) Hence determine the exact area of triangle PQR, giving your answer as a surd in simplest form.

(2)

a) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ 2 \end{pmatrix}$$

$$\boxed{(8, -3, 2)}$$

coordinates

b) $$\begin{pmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \sqrt{3} - 4 \\ 3 \\ -4\sqrt{3} - 1 \end{pmatrix}$$

Question 15 continued

$$\boxed{\left(\sqrt{3}-4,\ 3,\ -4\sqrt{3}-1\right)}$$

c) $\overrightarrow{PR} = \begin{pmatrix} \sqrt{3}-4 \\ 3 \\ -4\sqrt{3}-1 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}-12 \\ 0 \\ -4\sqrt{3}-3 \end{pmatrix}$

$$|\overrightarrow{PR}| = \sqrt{(\sqrt{3}-12)^2 + (-4\sqrt{3}-3)^2}$$

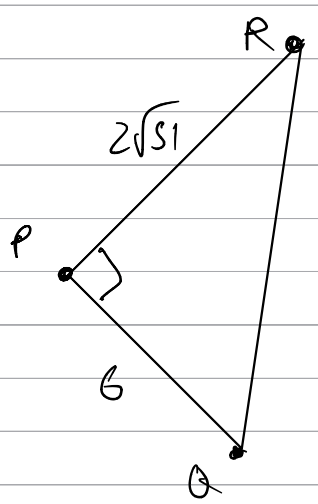
$$= \sqrt{204} = 2\sqrt{51}$$

d) $\overrightarrow{PQ} = \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix}$

$$\overrightarrow{PR} \cdot \overrightarrow{PQ} = \begin{pmatrix} \sqrt{3}-12 \\ 0 \\ -4\sqrt{3}-3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix} = 0$$

so $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ are perpendicular.

e)



$|\overrightarrow{PQ}| = 6$

Area of PRQ $= 6 \times 2\sqrt{51} \times \frac{1}{2}$

$= 6\sqrt{51}$

16. $\mathbf{M} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ where a is a constant

(a) Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$$

(6)

Triangle T has vertices A, B and C.

Triangle T is transformed to triangle T' by the transformation represented by $\mathbf{M}^n$ where $n \in \mathbb{N}$

Given that

- triangle T has an area of 5 cm^2
- triangle T' has an area of 1215 cm^2
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$

(b) determine

(i) the value of n

(ii) the value of a

(5)

a) Base case

let $n=1$: $M^1 = \begin{pmatrix} 3^1 & \frac{a}{2}(3^1-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} = M$

So the result is true for $n=1$.

Inductive Hypothesis

Assume the result is true for $n=k$, $(k \in \mathbb{Z}^+)$

i.e. $M^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$

P.T.O.

Question 1 continued

Inductive Step

$$M^{k+1} = M^k M$$

$$= \begin{pmatrix} 3^k & \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & 3^k a + \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & \frac{a}{2}(3^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$$

since $3^k a + \frac{a}{2}(3^k - 1)$

$= \frac{a}{2}(2 \times 3^k + 3^k - 1)$

$= \frac{a}{2}(3 \times 3^k - 1)$

$= \frac{a}{2}(3^{k+1} - 1)$

we have shown the result is true for $n=1$ and we have also shown it is true for $n=k+1$, if we assume it's true for $n=k$, so by the principle of induction the result is true for all $n \in \mathbb{Z}^+$.

b) i) $\det M^n = 3^n$

so $5 \times 3^n = 1215 \Rightarrow 3^n = 243 \Rightarrow \boxed{n=5}$

ii) $\begin{pmatrix} 243 & 121a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix}$

by subbing in $n=5$.

$\Rightarrow 486 - 242a = 123$

$\Rightarrow \boxed{a = \frac{3}{2} = 1.5}$

17. (i)

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single transformation represented by the matrix **A**.
(2)

The matrix **B** represents a stretch scale factor 2 parallel to the x-axis.

(b) Write down the matrix **B**.
(1)

The transformation represented by matrix **B** followed by the transformation represented by matrix **A** is the transformation represented by the matrix **C**.

(c) Determine **C**.
(2)

(ii)
$$\mathbf{M} = \begin{pmatrix} k & -2 \\ -1 & 2k \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

Given that the transformation represented by matrix **M** maps the point (k, k) onto the point $(35, 91)$

(a) determine the value of k.
(4)

A quadrilateral Q is transformed to another quadrilateral Q' by the matrix **M**.

Given that Q' has area 336

(b) use the value of k found in part (ii)(a) to determine the area of Q.
(2)

i)a) $\cos\theta = -\frac{1}{2} \Rightarrow \theta = 120°, 240°$

$\sin\theta = -\frac{\sqrt{3}}{2} \quad \theta = -60, 240, 300°$

rotation anticlockwise by 240° (or clockwise 120°)

about O.

b) $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

c) $C = AB$

Question 17 continued

$$= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \frac{\sqrt{3}}{2} \\ -\sqrt{3} & -\frac{1}{2} \end{pmatrix}$$

ii) a $\begin{pmatrix} k & -2 \\ -1 & 2k \end{pmatrix} \begin{pmatrix} k \\ k \end{pmatrix} = \begin{pmatrix} 35 \\ 91 \end{pmatrix}$

$k^2 - 2k = 35 \Rightarrow k^2 - 2k - 35 = 0$

$(k-7)(k+5) = 0$

$k = 7$ or $k = -5$

$-k + 2k^2 = 91$

$2k^2 - k - 91 = 0 \qquad k = \frac{1 \pm \sqrt{1+4(2)(91)}}{4} = 7$ or $-\frac{13}{2}$

So $\boxed{k = 7}$ not $\underline{k = -5}$

b) $\det M = 7 \times 14 - 2 = 96$

area of $Q \times 96 = 336$

$\boxed{\text{area of } Q = 3.5}$

****

18. With respect to the **right-hand rule**, a rotation through $\theta°$ anticlockwise about the z-axis is represented by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Given that the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents a rotation through $\alpha°$ anticlockwise about the z-axis with respect to the **right-hand rule**,

(a) determine the value of α.

(1)

(b) Hence determine the smallest possible positive integer value of k for which $\mathbf{M}^k = \mathbf{I}$

(2)

The 3×3 matrix **N** represents a reflection in the plane with equation $y = 0$

(c) Write down the matrix **N**.

(1)

The point A has coordinates $(-2, 4, 3)$

The point B is the image of the point A under the transformation represented by matrix **M** followed by the transformation represented by matrix **N**.

(d) Show that the coordinates of B are $(2 + \sqrt{3}, 2\sqrt{3} - 1, 3)$

(2)

Given that O is the origin,

(e) show that, to 3 significant figures, the size of angle AOB is 66.9°

(2)

(f) Hence determine the area of triangle AOB, giving your answer to 3 significant figures.

(2)

a) $\cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = 150°$ or $210°$

$\sin\theta = -\frac{1}{2} \Rightarrow \theta = -30, 210°$ or $330°$

so $\boxed{\alpha = 210°}$

Question 18 continued

b) we want to find k, $k \in \mathbb{Z}$ such that

$210k$ is a multiple of 360.

Either by trial and error or prime factorisation

$210 = 2 \times 3 \times 5 \times 7$

$360 = 2^3 \times 3^2 \times 5$

so $LCM = 2^3 \times 3^2 \times 5 \times 7$

$= 2520$

$\frac{2520}{210} = 12$

so $\boxed{k = 12}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

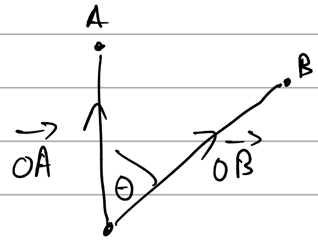
d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{3} \\ 2\sqrt{3}-1 \\ 3 \end{pmatrix}$

so $\boxed{B = (2+\sqrt{3}, 2\sqrt{3}-1, 3)}$

Question 1 continued

e)



$$\cos\theta = \frac{\vec{OA} \cdot \vec{OB}}{|\vec{OA}|\,|\vec{OB}|}$$

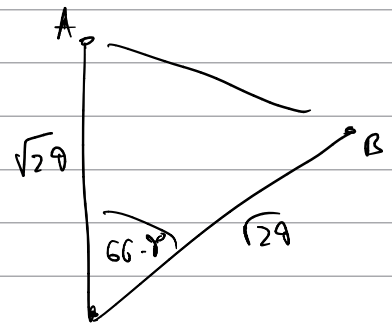
$$= \frac{\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} \begin{pmatrix} 2+\sqrt{3} \\ 2\sqrt{3}-1 \\ 3 \end{pmatrix}}{\sqrt{2^2+4^2+3^2}\ \sqrt{(2+\sqrt{3})^2+(2\sqrt{3}-1)^2+3^2}}$$

$$= \frac{1+6\sqrt{3}}{\sqrt{29}\ \sqrt{29}}$$

$$\theta = \arccos\left(\frac{1+6\sqrt{3}}{29}\right) = 66.86\ldots$$

$$= 66.9^\circ \text{ to } 3sf$$

f)



$$\text{Area} = \frac{1}{2} \times \sqrt{29} \times \sqrt{29} \times \sin(66.86\ldots)$$

$$= 13.334\ldots$$

$$= \boxed{13.3} \text{ to } 3sf$$

*****

Leave blank

DO NOT WRITE IN THIS AREA

19. In part (i), the elements of each matrix should be expressed in exact numerical form.

(i) (a) Write down the 2×2 matrix that represents a rotation of 210° anticlockwise about the origin.

(1)

(b) Write down the 2×2 matrix that represents a stretch parallel to the y-axis with scale factor 5

(1)

The transformation T is a rotation of 210° anticlockwise about the origin followed by a stretch parallel to the y-axis with scale factor 5

(c) Determine the 2×2 matrix that represents T

(2)

(ii)

$$\mathbf{M} = \begin{pmatrix} k & k+3 \\ -5 & 1-k \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Find det $\mathbf{M}$, giving your answer in simplest form in terms of k.

(2)

A closed shape R is transformed to a closed shape R' by the transformation represented by the matrix $\mathbf{M}$.

Given that the area of R is 2 square units and that the area of R' is $16k$ square units,

(b) determine the possible values of k.

(3)

i) a) $\begin{pmatrix} \cos(210) & -\sin(210) \\ \sin(210) & \cos(210) \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{5}{2} & -\frac{5\sqrt{3}}{2} \end{pmatrix}$

*****

Question 19 continued

ii) $\det M = h(1-h) + 5(h+3)$

$= h - h^2 + 5h + 15$

$= -h^2 + 6h + 15$

$2 \times |-h^2+6h+15| = 16h$

$-2h^2 + 12h + 30 = 16h$ or $2h^2 - 12h - 30 = 16h$

$-2h^2 - 4h + 30 = 0$ $\quad$ $2h^2 - 28h - 30 = 0$

$h^2 + 2h - 15 = 0$ $\quad$ $h^2 - 14h - 15 = 0$

$(h+5)(h-3) = 0$ $\quad$ $(h-15)(h+1) = 0$

$h = -5$ or $h = 3$ $\quad$ $h = 15$ or $h = -1$

but $16h$ is the area of R' so $h > 0$

because the area of R' can't be negative.

i.e. $\boxed{h = 3 \quad \text{or} \quad h = 15}$

Note that there is a mistake in the official mark scheme for this question. The solution above is correct. $h \neq -5$ or -1

Leave blank

20.

$$\mathbf{P} = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **P**.
(3)

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$

(b) Write down the matrix **Q**.
(1)

Given that the transformation V followed by the transformation U is the transformation T, which is represented by the matrix **R**,

(c) find the matrix **R**.
(2)

(d) Show that there is a value of k for which the transformation T maps each point on the straight line $y = kx$ onto itself, and state the value of k.
(4)

a) $\cos\theta = \frac{5}{13}$ $\quad \theta = 67.4°, 292.6°$

$\sin\theta = \frac{12}{13}$ $\quad \theta = 67.4°, 112.6°$

so U is a rotation 67.4° anticlockwise about O.

b) $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

c) $R = PQ = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$

d) P-T-O

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 20 continued

d) $\begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix} \begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} x \\ kx \end{pmatrix}$

This is essentially an invariant point question. See that pack for more questions like this.

$$-\frac{12}{13}x + \frac{5}{13}kx = x$$

$$\frac{5}{13}kx = \frac{25}{13}x$$

$5kx = 25x$ but this is true for all x

so $5k = 25$

$\Rightarrow k = 5$

Check: $\frac{5}{13}x + \frac{12}{13}(5)x = \frac{5}{13}x + \frac{60}{13}x$

$= \frac{65}{13}x = 5x = kx$

So $\boxed{k=5}$

the difficulty rating of 5 here, entirely depends on if you've seen invariant points questions before. If you have, you should find it easier than 5 *.

21. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2\text{i}$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \text{i}y$, where x and y are real and exact.

(6)

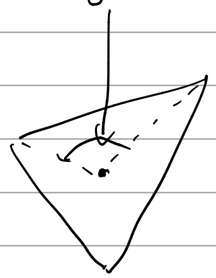
The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)

a) This is difficult because there's no nice geometric method to solve it and you have to use linear transformations.

The centre is at the origin, and I know this angle is 120° because we're given that the triangle is equitateateral. so I just rotate the point (6,2) by 120° anticlockwise:



$$\begin{pmatrix} \cos(120) & -\sin(120) \\ \sin(120) & \cos(120) \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -3-\sqrt{3} \\ 3\sqrt{3}-1 \end{pmatrix}$$

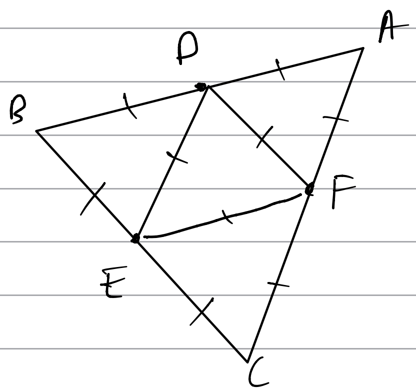
to find the other point.

$$\begin{pmatrix} \cos(240) & -\sin(240) \\ \sin(240) & \cos(240) \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -3+\sqrt{3} \\ -1-3\sqrt{3} \end{pmatrix}$$

so $(-3-\sqrt{3}) + (3\sqrt{3}-1)i$ and $(-3+\sqrt{3}) + (-1-3\sqrt{3})i$ are the other vertices.

Question 21 continued

b)



Area of DEF $= \frac{1}{4}$ Area of ABC

$|AB| = \sqrt{(6+3+\sqrt{3})^2 + (2+1-3\sqrt{3})^2}$

$= \sqrt{(9+\sqrt{3})^2 + (3-3\sqrt{3})^2}$

$= \sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$

so Area of ABC $= \frac{1}{2} \times 6\sqrt{2} \times 6\sqrt{2} \times \sin(60)$

$= 18\sqrt{3}$

$\boxed{\text{so Area of DEF} = \frac{9\sqrt{3}}{2}}$

Since ABC is equilateral so all lengths are same and angles are all 60°.