London Academy of Maths

Linear Transformations

Worked Solutions

1 **(i)** Describe fully the transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. **[2]**

(ii) A triangle of area 5 square units undergoes the transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Explaining your reasoning, find the area of the image of the triangle following this transformation. **[2]**

i) shear, x-axis invariant, $(0, 1)$ goes to $(2, 1)$.

ii) $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$\det M = 1$

So area of image $= 5 \times 1 = 5$

2 Matrix is given by $\mathbf{M} = \begin{pmatrix} k & 1 & -5 \\ 2 & 3 & -3 \\ -1 & 2 & 2 \end{pmatrix}$, where k is a constant.

(i) Show that $\det \mathbf{M} = 12(k-3)$. **[2]**

(iv) Find the values of k for which the transformation represented by $\mathbf{M}$ has a volume scale factor of 6. **[3]**

i) $\det M = k(3\times 2 - -3\times 2) - 1(2\times 2 - -3\times -1) - 5(2\times 2 - 3\times -1)$

$= 12k - 1 - 35$

$= 12k - 36 = 12(k-3)$

iv) $|\det M| = 6$

i.e. $|12(k-3)| = 6$

$\Rightarrow 12(k-3) = 6$ or $12(k-3) = -6$

$k - 3 = \frac{1}{2}$ or $k - 3 = -\frac{1}{2}$

$k = \frac{7}{2}$ So $k = 5/2$

3 The transformation R of the plane is reflection in the line $x = 0$.

(a) Write down the matrix **M** associated with R. **[1]**

(b) Find $\mathbf{M}^2$. **[1]**

(c) Interpret the result of part **(b)** in terms of the transformation R. **[1]**

a) $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ it's just a reflection in the y-axis.

b) $M^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c) M^2 is the identity matrix, so it represents the combination of two of the same reflections which will always be the identity ~~matix~~ matrix.

4 The matrix **M** is $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) **(i)** Calculate det **M**. **[1]**

(ii) State two geometrical consequences of this value for the transformation associated with **M**. **[2]**

(b) Describe fully the transformation associated with **M**. **[1]**

a(i) $\det M = 0(\ldots) + 1(1\times 1 - 0) + 0(\ldots)$

$= 1$

ii) orientation and volume are both preserved.

b) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ compare with formula booklet.

looks like $R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So $\cos\theta = 0$ & $\sin\theta = 1$

$\Rightarrow \theta = 90, 270$ $\Rightarrow \theta = 90$

So it's a rotation $90°$ anti-clockwise about the z-axis

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5 A 2-D transformation T is a shear which leaves the y-axis invariant and which transforms the object point (2, 1) to the image point (2, 9). **A** is the matrix which represents the transformation T.

(a) Find **A**. **[3]**

(b) By considering the determinant of **A**, explain why the area of a shape is invariant under T. **[2]**

a) $\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

recall shear x axis invariant is of form $\begin{pmatrix} 1 & H \\ 0 & 1 \end{pmatrix}$

and y axis invariant is of form $\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 2h+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

$\Rightarrow 2h+1 = 9$

$\Rightarrow \quad h = 4$

So $A = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

b) $\det A = 1 - 0 = 1$

The determinant is the area scale factor so since $\det A = 1$ the area must be unchanged

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6 **(a)** Find the volume scale factor of the transformation with associated matrix $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ -1 & 0 & 2 \end{pmatrix}$. **[2]**

(b) The transformations S and T of the plane have associated 2×2 matrices **P** and **Q** respectively.

(i) Write down an expression for the associated matrix of the combined transformation S followed by T. **[1]**

The determinant of **P** is 3 and $\mathbf{Q} = \begin{pmatrix} k & 3 \\ -1 & 2 \end{pmatrix}$, where k is a constant.

(ii) Given that this combined transformation preserves both orientation and area, determine the value of k. **[3]**

a) $\det M = 1(3\times 2) - 2(0-1) + 0(0+3)$

$= 6+2 = 8$

So vsf $= 8$

saying $\det M = 8$ is not enough. Need to link it to vsf.

b) i) QP

ii) $\det(QP) = 1$

$\Rightarrow \det(Q)\det(P) = 1$

$(2k+3) \times 3 = 1$

$2k+3 = \frac{1}{3}$

$2k = -\frac{8}{3}$

$k = -\frac{4}{3}$

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7 You are given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

(a) Find $\mathbf{A}^4$. **[1]**

(b) Describe the transformation that **A** represents. **[2]**

The matrix **B** represents a reflection in the plane $x = 0$.

(c) Write down the matrix **B**. **[1]**

The point P has coordinates (2, 3, 4). The point P' is the image of P under the transformation represented by **B**.

(d) Find the coordinates of P'. **[1]**

a) $A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

purely a calculator question. Can also recognise the type of transformation it is, however this is asked in (b).

b) looks like $R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

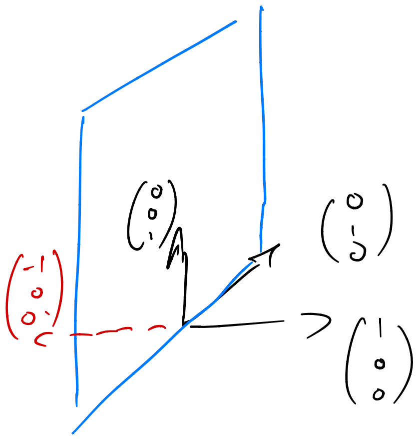
$\cos\theta = 0 \Rightarrow \theta = 90°$ or $270°$

$\sin\theta = -1 \Rightarrow \theta = -90°$ or $270°$

so it's a rotation $270°$ anticlockwise about the x-axis

or $90°$ clockwise.

c) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



d) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$

So $\underline{\underline{(-2, 3, 4)}}$ because Q asks for coordinates.

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8 (a) Specify fully the transformation T of the plane associated with the matrix **M**, where $\mathbf{M} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and λ is a non-zero constant. **[2]**

(b) (i) Find $\det \mathbf{M}$. **[1]**

(ii) Deduce **two** properties of the transformation T from the value of $\det \mathbf{M}$. **[2]**

(c) Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix}$, where n is a positive integer. **[4]**

(d) Hence specify fully a **single** transformation which is equivalent to n applications of the transformation T. **[1]**

a) Shear, x axis invariant, $(0,1)$ goes to $(\lambda, 1)$

bi) $\det M = 1$

ii) orientation and area are preserved.

c) Base case:

let $n=1$, $M^1 = \begin{pmatrix} 1 & 1\times\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = M$

So the result is true for $n=1$

Inductive Hypothesis

assume result is true for $n=k$.

i.e. $M^k = \begin{pmatrix} 1 & k\lambda \\ 0 & 1 \end{pmatrix}$

Inductive step

$M^{k+1} = M^k \times M = \begin{pmatrix} 1 & k\lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda + k\lambda \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & \lambda(k+1) \\ 0 & 1 \end{pmatrix}$

So if the result is true for $n=k$, it's also true for $n=k+1$

Since we have also proved the result true for $n=1$,

by induction the result is true for all positive integers

n.

d) a shear, x-axis invariant, mapping

$(0,1)$ to $(n\lambda, 1)$

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9 A transformation T of the plane has associated matrix $\mathbf{M} = \begin{pmatrix} 1 & \lambda+1 \\ \lambda-1 & -1 \end{pmatrix}$, where λ is a non-zero constant.

(a) **(i)** Show that T reverses orientation. **[3]**

(ii) State, in terms of λ, the area scale factor of T. **[1]**

(b) **(i)** Show that $\mathbf{M}^2 - \lambda^2\mathbf{I} = \mathbf{0}$. **[2]**

(ii) Hence specify the transformation equivalent to two applications of T. **[1]**

(c) In the case where $\lambda = 1$, T is equivalent to a transformation S followed by a reflection in the x-axis.

(i) Determine the matrix associated with S. **[3]**

(ii) Hence describe the transformation S. **[2]**

ai) $\det M = 1(-1) - (\lambda+1)(\lambda-1)$

$= -1 - (\lambda^2 - 1)$

$= -\lambda^2 < 0$ for all $\lambda \in \mathbb{R}, \lambda \neq 0$.

So $\det M < 0 \Rightarrow$ orientation is reversed.

ii) area scale factor $= |\det M| = \lambda^2$

bi) $M^2 = \begin{pmatrix} 1 & \lambda+1 \\ \lambda-1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \lambda+1 \\ \lambda-1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1+\lambda^2-1 & \lambda+1-\lambda-1 \\ \lambda-1-\lambda+1 & \lambda^2-1+1 \end{pmatrix}$

$= \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}$

$$M^2 - \lambda^2 I = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} - \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} - \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underline{0}$$

ii) $\Rightarrow M^2 = \lambda^2 I$

So two application of T is equivalent to an enlargement about O of scale factor λ^2.

c) i) let matrix N represent transformation S.

$$M_{\lambda=1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} N$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=A} N$$

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

since $\det A = -1$

$$\Rightarrow N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

because we multiply by A^{-1} on the left of both sides.

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

ii) shear, x axis invariant, mapping $(0,1)$ to $(2,1)$

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10 A transformation T is represented by the matrix $\mathbf{N} = \begin{pmatrix} a & 4 & 2 \\ 5 & 1 & 0 \\ 3 & 6 & 3 \end{pmatrix}$, where a is a constant.

(a) Find $\mathbf{N}^2$ in terms of a. **[3]**

(b) Find det $\mathbf{N}$ in terms of a. **[2]**

The value of a is 13 to the **nearest integer**.

A shape S_1 has volume 11.6 to 1 decimal place. Shape S_1 is mapped to shape S_2 by the transformation T.

A student claims that the volume of S_2 is less than 400.

(c) Comment on the student's claim. **[3]**

a) $N^2 = \begin{pmatrix} a & 4 & 2 \\ 5 & 1 & 0 \\ 3 & 6 & 3 \end{pmatrix}\begin{pmatrix} a & 4 & 2 \\ 5 & 1 & 0 \\ 3 & 6 & 3 \end{pmatrix}$

$= \begin{pmatrix} a^2+26 & 4a+16 & 2a+6 \\ 5a+5 & 21 & 10 \\ 3a+39 & 36 & 15 \end{pmatrix}$

b) $\det N = a(1\times3) - 4(5\times3) + 2(5\times6 - 3\times1)$

$= 3a - 60 + 54 = 3a - 6$

c) $12.5 \le a < 13.5$

$\Rightarrow 31.5 \le \det N < 34.5$ since $\det N = 3a-6$

Now $11.55 \le V_{S_1} < 11.65$

$\Rightarrow 31.5 \times 11.55 \le V_{S_2} < 34.5 \times 11.65$

$363.825 \le V_{S_2} < 401.925$

So the student's claim isn't necessarily true since the upper bound is greater than 400.

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11 A transformation T is represented by the matrix $\mathbf{T}$ where $\mathbf{T} = \begin{pmatrix} x^2+1 & -4 \\ 3-2x^2 & x^2+5 \end{pmatrix}$.

A quadrilateral Q, whose area is 12 units, is transformed by T to Q'.

Find the smallest possible value of the area of Q'. **[5]**

$$\det T = (x^2+1)(x^2+5) - (-4(3-2x^2))$$

$$= x^4 + 6x^2 + 5 + 12 - 8x^2$$

$$= x^4 - 2x^2 + 17$$

we want to make this as small as possible

Method 1: Complete the square

$$x^4 - 2x^2 + 17 = (x^2-1)^2 - 1 + 17$$

$$= (x^2-1)^2 + 16$$

So $\det T \geqslant 16$

Method 2: differentiate

let $\det T = y$

$$\frac{dy}{dx} = 4x^3 - 4x = 0$$

$$4x(x^2-1) = 0$$

$$\Rightarrow x = 0, 1 \text{ or } -1$$

Sub back into $y = x^4 - 2x^2 + 17$

$\Rightarrow x = 0, y = 17$

$x = \pm 1, y = 16$

So $\det T \geqslant 16$.

since $(\pm 1, 16)$ are the minima

Therefore smallest possible area

$= 16 \times 12 = \boxed{192}$

Note that $x^4 - 2x^2 + 17$ is a quartic. You should have a rough idea of what this looks like on a graph but also that it doesn't necessarily have roots.

So if your method was to try set the quartic equal to zero, you need to be taking more time to think about the problem, before trying to find the roots straight away.

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12 You are given that a is a parameter which can take only real values.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 2 & 4 & -6 \\ -3 & 10-4a & 9 \\ 7 & 4 & 4 \end{pmatrix}$.

(a) Find an expression for the determinant of **A** in terms of a. **[2]**

The transformation represented by **A** is denoted by T.
A 3-D object of volume $|5a-20|$ is transformed by T to a 3-D image.

(c) (i) Determine the range of values of a for which the orientation of the image is the reverse of the orientation of the object. **[1]**

(ii) Determine the range of values of a for which the volume of the image is less than the volume of the object. **[2]**

a)

$\det A = 2\big((10-4a)4 - 9(4)\big) - 4\big(-3(4) - 9(7)\big) - 6\big(-3(4) - (10-4a)7\big)$

$= 2(40-16a-36) - 4(-12-63) - 6(-12-70+28a)$

$= 2(4-16a) - 4(-75) - 6(-82+28a)$

$= 800 - 200a$

ci) want $\det A < 0$ i.e. $800-200a < 0$

$\Rightarrow a > 4$

ii) when $|\det A| < 1$

i.e. $-1 < 800-200a < 1$

$-801 < -200a < -799$

$799 < 200a < 801$

$\boxed{\frac{799}{200} < a < \frac{801}{200}}$

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13 The transformation T of the plane has associated matrix **M**, where $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$.

(a) On the grid in the Printed Answer Booklet, plot the image OA′B′C′ of the unit square OABC under the transformation T. **[2]**

(b) (i) Calculate the value of det **M**. **[1]**

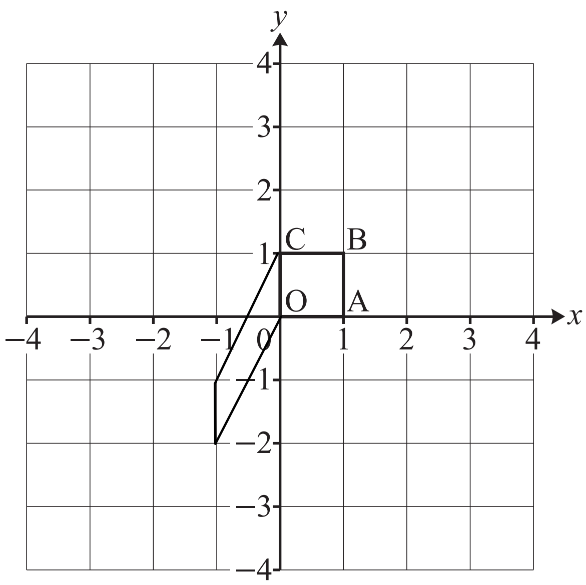
(ii) Explain the significance of the value of det **M** in relation to the image OA′B′C′. **[2]**

(c) T is equivalent to a sequence of two transformations of the plane.

(i) Specify fully **two** transformations equivalent to T. **[3]**

(ii) Use matrices to verify your answer. **[3]**

13(a)



$$\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \end{pmatrix}$$

reflection

bi) $\det M = -1(1) = -1$

ii) Area is preserved i.e. area of OA′B′C′ = 1 and orientation clearly has been reversed.

c) i) a reflection in the y axis followed by a shear mapping points from $(-1, 0)$ to $(-1, -2)$

or $(1, 0)$ to $(1, 2)$

both can be seen as correct.

(ii) The classic mistake here is to think that the shear is $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ but this isn't the case because the x coordinates of $(-1,0)$ and $(-1,1)$ are negative. So $k=-2$, would make them positive.

So $\underline{\underline{k=2}}$.

reflection: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

shear: $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} = M$$

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14 Matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}$.

(a) Use **A** and **B** to disprove the proposition: "Matrix multiplication is commutative". **[2]**

Matrix **B** represents the transformation T_B.

(b) Describe the transformation T_B. **[2]**

(c) By considering the inverse transformation of T_B, determine $\mathbf{B}^{-1}$. **[2]**

Matrix **C** is given by $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ and represents the transformation T_C.

The transformation T_{BC} is transformation T_C followed by transformation T_B.

An object shape of area 5 is transformed by T_{BC} to an image shape *N*.

(d) Determine the area of *N*. **[2]**

a) $AB = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix} = \begin{pmatrix} \frac{-5}{13} & \frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}$

$BA = \begin{pmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{-5}{13} & \frac{-12}{13} \\ \frac{-12}{13} & \frac{5}{13} \end{pmatrix}$

$AB \neq BA$ so matrix multiplication isn't commutative

b) $\cos^{-1}\left(\frac{5}{13}\right) = 67.4^\circ, 292.6^\circ$

$\sin^{-1}\left(\frac{5}{13}\right) = 67.4^\circ, 112.6^\circ$

So a rotation anticlockwise by 67.4° about the origin.

c) B^{-1} would be a rotation clockwise of $67.4°$, i.e. a rotation anticlockwise of $-67.4°$

we know $\cos(-67.4°) = \cos(67.4°) = \frac{5}{13}$

and $\sin(-67.4°) = -\sin(67.4)°$

$= -\frac{12}{13}$

Since $\cos(-x) = \cos(x)$
& $\sin(-x) = -\sin(x)$

so $B^{-1} = \begin{pmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{-12}{13} & \frac{5}{13} \end{pmatrix}$

d) $\det BC = \det B \det C$

$= 1 \times -3 = -3$

Area of $N = |-3| \times 5 = 15$.

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15 The transformations T_A and T_B are represented by the matrices **A** and **B** respectively, where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Describe geometrically the **single** transformation consisting of T_A followed by T_B. **[2]**

(b) By considering the transformation T_A, determine the matrix $\mathbf{A}^{423}$. **[3]**

The transformation T_C is represented by the matrix **C**, where

$$\mathbf{C} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

The region R is defined by the set of points (x, y) satisfying the inequality $x^2 + y^2 \leqslant 36$.

The region R' is defined as the image of R under T_C.

(c) **(i)** Find the exact area of the region R'. **[2]**

(ii) Sketch the region R', specifying all the points where the boundary of R' intersects the coordinate axes. **[4]**

a) T_A followed by T_B:

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is a reflection in the x axis.

b) $\cos(\theta) = 0$

$\theta = 90°, 270°$

$\sin(\theta) = 1$

$\theta = 90°$

so T_A is a rotation $90°$ anticlockwise about the origin.

$A^{423} = A^{420} \times A^3$

$= I \times A^3$

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$A^{420} = I$ Since $A^4 = I$ because it's a rotation $360°$ which does nothing.

because A^3 is just a rotation $270°$ anticlockwise about O.

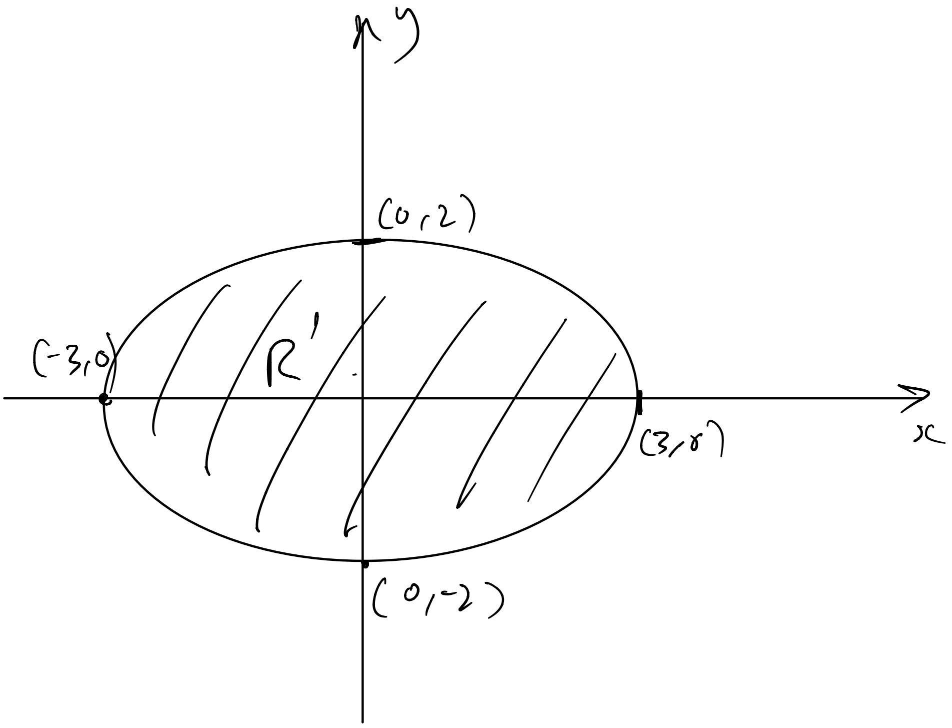
ci) Area of $R = 36\pi$

since $x^2+y^2 \leq 36$ is a circle with radius 6.

$\det C = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

so Area of $R' = \frac{1}{6} \times 36\pi = 6\pi$

ii) C represents the transformation of a stretch parallel to the x axis by scale factor $1/2$ and stretch parallel to the y axis by scale factor $1/3$



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16 **(a)** A transformation with associated matrix $\begin{pmatrix} m & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 3 \end{pmatrix}$, where m is a constant, maps the vertices of a cube to points that all lie in a plane.

Find m. **[3]**

(b) The transformations S and T of the plane have associated matrices **M** and **N** respectively, where $\mathbf{M} = \begin{pmatrix} k & 1 \\ -3 & 4 \end{pmatrix}$ and the determinant of **N** is $3k+1$. The transformation U is equivalent to the combined transformation consisting of S followed by T.

Given that U preserves orientation and has an area scale factor 2, find the possible values of k. **[4]**

a) volume scale factor = 0

i.e. det M = 0

$\Rightarrow m(1\times3) - 2(0\times3 - -2\times2) + 1(-1\times2) = 0$

$3m - 8 - 2 = 0$

$3m = 10 \Rightarrow \boxed{m = 10/3}$

b) let U be represented by the matrix P.

$P = NM$ (not MN!)

det $P = 2$ (not -2, since orientation preserved)

So $2 = \det(N)\det(M)$

$= (3k+1)(4k+3)$

$0 = 12k^2 + 13k + 1$

$= (12k+1)(k+1) \Rightarrow \boxed{k = \frac{-1}{12} \text{ or } k = -1}$

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17 A transformation of the x-y plane is represented by the matrix $\begin{pmatrix} \cos\theta & 2\sin\theta \\ 2\sin\theta & -\cos\theta \end{pmatrix}$, where θ is a positive acute angle.

(i) Write down the image of the point (2, 3) under this transformation. **[2]**

(ii) You are given that this image is the point $(a, 0)$. Find the value of a. **[5]**

i) $\begin{pmatrix} \cos\theta & 2\sin\theta \\ 2\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\cos\theta + 6\sin\theta \\ 4\sin\theta - 3\cos\theta \end{pmatrix}$

So image $= \underline{\underline{(2\cos\theta + 6\sin\theta,\ 4\sin\theta - 3\cos\theta)}}$

ii) $\begin{pmatrix} 2\cos\theta + 6\sin\theta \\ 4\sin\theta - 3\cos\theta \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$

$\Rightarrow 4\sin\theta - 3\cos\theta = 0$

$\div \cos\theta$ $\quad$ $\div \cos\theta$

$4\tan\theta - 3 = 0$

$\tan\theta = 3/4$

$\theta = 36.869...$

sub into $2\cos\theta + 6\sin\theta = a$:

$a = 2\cos(36.869...) + 6\sin(36.869...)$

$= \boxed{5.2}$

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18 **(a)** The matrix **M** is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(i) Find $\mathbf{M}^2$. **[1]**

(ii) Write down the transformation represented by **M**. **[1]**

(iii) Hence state the geometrical significance of the result of part **(i)**. **[1]**

(b) The matrix **N** is $\begin{pmatrix} k+1 & 0 \\ k & k+2 \end{pmatrix}$, where k is a constant.

Using determinants, investigate whether **N** can represent a reflection. **[4]**

a) i) $M^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

ii) reflection in the line $y = -x$

iii) two reflections in $y = -x$ are equivalent to the identity transformation (i.e. all points would remain invariant.

b) reflections preserve area but not orientation.

so if N is a reflection, $\det N = -1$

$\Rightarrow (k+1)(k+2) = -1$

$k^2 + 3k + 2 = -1$

$k^2 + 3k + 3 = 0$

$b^2 - 4ac = 3^2 - 4(3) = -3 < 0$

so no real roots

$\Rightarrow$ N can never be a reflection.

19 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 0.6 & 2.4 \\ -0.8 & 1.8 \end{pmatrix}$.

(a) Find det **A**. **[1]**

The matrix **A** represents a stretch parallel to one of the coordinate axes followed by a rotation about the origin.

(b) By considering the determinants of these transformations, determine the scale factor of the stretch. **[2]**

(c) Explain whether the stretch is parallel to the x-axis or the y-axis, justifying your answer. **[1]**

(d) Find the angle of rotation. **[2]**

a) $\det A = 0.6 \times 1.8 + 2.4 \times 0.8 = 3$

b) $A = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}}_{Q}$ where one of a or $c = 1$.

$\det P = 1$ $\qquad \det Q = ac$

$\det A = 1 \times ac = ac$

So $ac = 3$

$\Rightarrow$ scale factor of stretch is 3 since either a or c is 1.

c) By multiplication of P with Q

$A = \begin{pmatrix} a\cos\theta & -c\sin\theta \\ a\sin\theta & c\cos\theta \end{pmatrix} = \begin{pmatrix} 0.6 & 2.4 \\ -0.8 & 1.8 \end{pmatrix}$

if $c = 1$, then $\sin\theta = -2.4$ & $\cos\theta = 1.8$ which is impossible. so $c = 3$ $\Rightarrow$ stretch parallel to y-axis

$a = 1 \Rightarrow \quad \cos\theta = 0.6 \quad \& \quad \sin\theta = -0.8$

$\theta = 53.1^\circ, 306.9^\circ \qquad \theta = -53.1^\circ, 233.1^\circ, 306.9$

so $\theta = 306.9^\circ$ (anticlockwise)

or 53.1° clockwise.

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20 Matrix **R** is given by $\mathbf{R} = \begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}$ where a and b are constants.

(a) Find $\mathbf{R}^2$ in terms of a and b. **[2]**

The constants a and b are given by $a = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$ and $b = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$.

(b) By determining exact expressions for ab and $a^2 - b^2$ and using the result from part **(a)**,

show that $\mathbf{R}^2 = k\begin{pmatrix} \sqrt{3} & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & \sqrt{3} \end{pmatrix}$ where k is a real number whose value is to be determined.

[2]

(c) Find $\mathbf{R}^6$, $\mathbf{R}^{12}$ and $\mathbf{R}^{24}$. **[3]**

(d) Describe fully the transformation represented by **R**. **[3]**

a) $R^2 = \begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}\begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix} = \begin{pmatrix} a^2-b^2 & 0 & -2ab \\ 0 & 1 & 0 \\ 2ab & 0 & a^2-b^2 \end{pmatrix}$

b) $ab = \frac{\sqrt{2}}{4}(\sqrt{3}+1)\frac{\sqrt{2}}{4}(\sqrt{3}-1)$

$= \frac{2}{16}(\sqrt{3}+1)(\sqrt{3}-1)$

$= \frac{1}{8}(3-1) = \frac{1}{4}$

$a^2 - b^2 = (a-b)(a+b) = \left(\frac{\sqrt{2}}{4}(\sqrt{3}+1) - \frac{\sqrt{2}}{4}(\sqrt{3}-1)\right)\left(\frac{\sqrt{2}}{4}(\sqrt{3}+1) + \frac{\sqrt{2}}{4}(\sqrt{3}-1)\right)$

$= \left(\frac{2\sqrt{2}}{4}\right)\left(\frac{2\sqrt{2}\sqrt{3}}{4}\right)$

$= \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2}$

$$R^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sqrt{3} & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & \sqrt{3} \end{pmatrix}$$

c) $$R^6 = (R^2)^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

perfectly acceptable to do with calculator here. The question doesn't ask for detailed reasoning.

$$R^{12} = (R^6)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$R^{24} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d) $R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$ from formula booklet

$$R = \begin{pmatrix} \frac{\sqrt{2}}{4}(\sqrt{3}+1) & 0 & -\frac{\sqrt{2}}{4}(\sqrt{3}-1) \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{4}(\sqrt{3}-1) & 0 & \frac{\sqrt{2}}{4}(\sqrt{3}+1) \end{pmatrix}$$

Since $R^{24} = I$, the angle of rotation must be to do with $\frac{360}{24} = 15°$.

Now, $\cos(15) = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$ which means the top left entries are the same.

but $\sin(15) = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$ which means the bottom left entries are not the same.

So instead of $15°$ we do $345°$ since

$\cos(15) = \cos(360-15) = \cos(345)$

So R is a rotation $345°$ anticlockwise

or $15°$ clockwise

about the y-axis.

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21 Three transformations, T_A, T_B and T_C, are represented by the matrices **A**, **B** and **C** respectively.

You are given that $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Find the matrix which represents the inverse transformation of T_A. **[1]**

(b) By considering matrix multiplication, determine whether T_A followed by T_B is the same transformation as T_B followed by T_A. **[2]**

Transformations R and S are each defined as being the result of successive transformations, as specified in the table.

Transformation	First transformation	followed by
R	T_A followed by T_B	T_C
S	T_A	T_B followed by T_C

(c) Explain, using a property of matrix multiplication, why R and S are the same transformations. **[2]**

D E F G

A quadrilateral, Q, has vertices D, E, F and G in anticlockwise order from D. Under transformation R, Q's image, Q', has vertices D', E', F' and G' (where D' is the image of D, etc). The area of Q, in suitable units, is 5.

You are given that $\det \mathbf{C} = a^2 + 1$ where a is a real constant.

(d) **(i)** Determine the order of the vertices of Q', starting anticlockwise from D'. **[2]**

(ii) Find, in terms of a, the area of Q'. **[1]**

(iii) Explain whether the inverse transformation for R exists. Justify your answer. **[2]**

a) $A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$

b) $BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$

so T_A followed by T_B is not the same as T_B followed by T_A

$R = C(BA)$

$S = (CB)A$

$R = S$ since matrix multiplication is associative

d) i) $R = C(BA)$

$\det R = \det(C)\det(B)\det(A)$

$= (a^2+1)(-1)(3) = -3(a^2+1) < 0$ for all $a \in \mathbb{R}$

so orientation is reversed :

D', G', F', E'

ii) Area of $Q' = 5 \times 3(a^2+1) = 15(a^2+1)$

(ii) The inverse transformation exists

since $\det R \neq 0$ for all $a \in \mathbb{R}$.

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22 A transformation A is represented by the matrix **A** where $\mathbf{A}=\begin{pmatrix} -1 & x & 2 \\ 7-x & -6 & 1 \\ 5 & -5x & 2x \end{pmatrix}$.

The tetrahedron H has vertices at O, P, Q and R. The volume of H is 6 units.

P', Q', R' and H' are the images of P, Q, R and H under A.

(a) In the case where $x = 5$

- find the volume of H',
- determine whether A preserves the orientation of H. **[3]**

(b) Find the values of x for which O, P', Q' and R' are coplanar (i.e. the four points lie in the same plane). **[4]**

a) $A_{x=5} = \begin{pmatrix} -1 & 5 & 2 \\ 2 & -6 & 1 \\ 5 & -25 & 10 \end{pmatrix}$

$\det A = -1(-60+25) - 5(20-5) + 2(-50+30)$

$= 35 - 75 - 40$

$= -80$

so Volume of $H' = 6 \times 80 = 480$

$\det A < 0$, so A does not preserve the orientation.

b) V. similar to Q16a. $V_{sf} = 0$ since we're going from 3D to 2D. i.e. $\det A = 0$

$$\det A = -1(-6(2x) + 5x) - x((7-x)(2x) - 5) + 2((7-x)(-5x) + 30)$$

$$= -1(-7x) - x(14x - 2x^2 - 5) + 2(-35x + 5x^2 + 30)$$

$$= 7x - 14x^2 + 2x^3 + 5x - 70x + 10x^2 + 60$$

$$= 2x^3 - 4x^2 - 58x + 60 \quad = 0$$

$$x^3 - 2x^2 - 29x + 30 = 0$$

Although you could find the roots here by guessing a root and then factorising, there's nothing about showing detailed working. So I use my calculator (although it's good practice to factorise anyway!)

$x = 1, 6$ or -5