London Academy of Maths

Number Theory

Worked Solutions

1 In this question you must show detailed reasoning.

Express the number 41723_{10} in hexadecimal (base 16). **[3]**

$41723 \div 16 = 2607 \text{ r } 11$

$2607 \div 16 = 162 \text{ r } 15$

$162 \div 16 = 10 \text{ r } 2$

$10 \div 16 = 0 \text{ r } 10$

$41723_{10} = A2FB_{16}$

because you just take the remainders in reverse order

Answer ALL questions. Write your answers in the spaces provided.

2. (i) Use the Euclidean algorithm to find the highest common factor of 602 and 161.

Show each step of the algorithm.

(3)

$602 = 161(3) + 119$

$161 = 119(1) + 42$

$119 = 42(2) + 35$

$42 = 35(1) + 7$

$35 = 7(5) + 0$

so hcf $(602, 161) = 7$

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3 **(a)** Express 205 in the form $7q+r$ for positive integers q and r, with $0 \leqslant r < 7$. **[1]**

(b) Given that $7 \mid (205 \times 8666)$, use the result of part **(a)** to justify that $7 \mid 8666$. **[2]**

1 a) $205 = 7(29) + 2$

b) $7 \mid (205 \times 8666) \Rightarrow 7 \mid 205$ or $7 \mid 8666$

by Euclid's lemm

$\Rightarrow 7 \mid 8666$ since $7 \nmid 205$ by (a)

Answer ALL questions. Write your answers in the spaces provided.

4. (i) Using a suitable algorithm and without performing any division, determine whether 23 738 is divisible by 11

(2)

(ii) Use the Euclidean algorithm to find the highest common factor of 2322 and 654

(3)

i) $2-3+7-3+8 = 11 = 11(1)$

So $11 \mid 23738$

ii)

$2322 = 654(3) + 360$

$654 = 360(1) + 294$

$360 = 294(1) + 66$

$294 = 66(4) + 30$

$66 = 30(2) + 6$

$30 = 6(5) + 0$

$\text{hcf}(2322, 654) = 6$

*

5. The highest common factor of 963 and 657 is c.

(a) Use the Euclidean algorithm to find the value of c.

(3)

(b) Hence find integers a and b such that

$$963a + 657b = c$$

(3)

a) $963 = 657(1) + 306$

$657 = 306(2) + 45$

$306 = 45(6) + 36$

$45 = 36(1) + 9$ $\Rightarrow c = 9$

$36 = 9(4) + 0$

b) $9 = 45 - 36$

$= 45 - (306 - 45(6))$

$= 7(45) - 306$

$= 7(657 - 306(2)) - 306$

$= 7(657) - 15(306)$

$= 7(657) - 15(963 - 657)$

$= 22(657) - 15(963)$

$a = -15$, $b = 22$.

*

6. **In this question you must show detailed reasoning.**

Without performing any division, explain why $n = 20\,210\,520$ is divisible by 66

(4)

$n = 20210520$

need to check $6 \mid n$ and $11 \mid n$ (works because $hcf(6, 11) = 1$)

$2+0+2+1+0+5+2+0 = 12 = 3(4)$

and n is even

so $6 \mid n$

$2-0+2-1+0-5+2-0 = 0 = 11(0)$

so $11 \mid n$

$\Rightarrow 66 \mid n$ ▨

7 **In this question you must show detailed reasoning.**

The number N is written as 28A3B in base-12 form.

Express N in decimal (base-10) form. **[2]**

$$28A3B_{12} = 12^4 \times 2 + 12^3 \times 8 + 12^2 \times 10 + 12 \times 3 + 11$$

$$= 56783_{10}$$

8. (a) Use the Euclidean Algorithm to find integers a and b such that

$$125a + 87b = 1$$

(5)

(b) Hence write down a multiplicative inverse of 87 modulo 125

(1)

(c) Solve the linear congruence

$$87x \equiv 16 \pmod{125}$$

(2)

a) $125 = 87(1) + 38$

$87 = 38(2) + 11$

$38 = 11(3) + 5$

$11 = 5(2) + 1$

$\Rightarrow 1 = 11 - 5(2)$

$= 11 - (38 - 11(3))2$

$= 7(11) - 2(38)$

$= 7(87 - 38(2)) - 2(38)$

$= 7(87) - 16(38)$

$= 7(87) - 16(125 - 87(1))$

$= 23(87) - 16(125)$

$a = -16, b = 23$

b) $23(87) - 16(125) \equiv 1 \bmod 125$

$23(87) \equiv 1 \bmod 125$

so, $87^{-1} \equiv 23 \bmod 125$

*

c) $87x \equiv 16 \bmod 125$

$\times 23$ $\qquad\qquad\qquad$ $\times 23$

$$x \equiv 16 \times 23 \bmod 125$$

$$\equiv 368 \bmod 125$$

$$\equiv 118 \bmod 125$$

9. (i) Without performing any division, explain why 8184 is divisible by 6

(2)

(ii) Use the Euclidean algorithm to find integers a and b such that

$$27a + 31b = 1$$

(4)

i) $8+1+8+4 = 21 = 3(7)$

so $3 \mid 8184$

8184 is even so $2 \mid 8184$

$\therefore$ $6 \mid 8184$ since hcf$(3, 2) = 1$

ii) $31 = 27(1) + 4$

$27 = 4(6) + 3$

$4 = 3(1) + 1$

$\Rightarrow$ $1 = 4 - 3$

$= 4 - (27 - 4(6))$

$= 7(4) - 27$

$= 7(31 - 27) - 27$

$= 7(31) - 8(27)$

$\underline{a = -8 \quad b = 7}$

**

10 **(a)** Evaluate 13×19 modulo 31. **[1]**

(b) Solve the linear congruence $13x \equiv 9 \pmod{31}$. **[3]**

a) $13 \times 19 \equiv 13 \times -12$

$\equiv -156 \mod 31$

$\equiv 30 \mod 31$

b) $13x \equiv 9 \mod 31$

$\times 19$ $\quad$ $\times 19$

$30x \equiv 16 \mod 31$

$-x \equiv 16 \mod 31$

$x \equiv -16 \mod 31$

$x \equiv 15 \mod 31$

**

11. (a) Use the Euclidean algorithm to show that 124 and 17 are relatively prime (coprime).
(2)

(b) Hence solve the equation

$$124x + 17y = 10$$

(3)

(c) Solve the congruence equation

$$124x \equiv 6 \bmod 17$$

(2)

a) $124 = 17(7) + 5$

$17 = 5(3) + 2$

$5 = 2(2) + 1$

$2 = 1(2) + 0$

so $hcf(124, 17) = \underline{1}$

$\Leftrightarrow$ 124 & 17 are coprime.

b) $1 = 5 - 2(2)$

$= 5 - 2(17 - 5(3))$

$= 7(5) - 2(17)$

$= 7(124 - 17(7)) - 2(17)$

$1 = 7(124) - 51(17)$

$\times 10 \quad \times 10$

$10 = 70(124) - 510(17)$

$x = 70, \ y = -510$

c) $124x \equiv 6 \bmod 17$

$7(124) - 51(17) \equiv 1 \bmod 17$

$7(124) \equiv 1 \bmod (17)$

$\times 7 \quad \times 7$

$x \equiv 42 \bmod 17$

$x \equiv 8 \bmod 17$

**

12 In decimal (base 10) form, the number N is 15 260.

(a) Express N in binary (base 2) form. **[1]**

(b) Using the binary form of N, show that N is divisible by 7. **[2]**

a) 1 mark so use calculator

1. On Casio fx-991EX go to menu setup and find Base-N
2. Type in the number: 15260 and press enter
3. Then Shift $\log_{\square}\square$ or wherever you calculator says BIN
4. Answer is read from left to right from the first 1.

$N = 11101110011100_2$

b) $7_{10} = 111_2$

$$\begin{array}{r} 100010000100 \\ 111 \overline{)\,11101110011100_2} \end{array}$$

So $7 \mid N$ since $N = 111_2 \times 100010000100_2$

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13. (i) Making your reasoning clear and using modulo arithmetic, show that

$$214^6 \text{ is divisible by } 8$$

(3)

(ii) The following 7-digit number has four unknown digits

$$\boxed{a}\,5\,\boxed{b}\,8\,\boxed{a}\,\boxed{b}\,0$$

Given that the number is divisible by 11

(a) determine the value of the digit a.

(2)

Given that the number is also divisible by 3

(b) determine the possible values of the digit b.

(3)

$214^6 \equiv 6^6 \mod 8$

$\equiv (-2)^6 \mod 8$

$\equiv 64 \mod 8$

$\equiv 0 \mod 8$ so $8 \mid 214^6$

ii) a) $a-5+b-8+a-b+0 = 11k$ for some $k \in \mathbb{Z}$

$2a - 13 = 11k$

$0 \leq a \leq 9$ so $\underline{a = 1}$ since $2-13 = -11 = 11(-1)$

b) $a+5+b+8+a+b = 3m$ for some $m \in \mathbb{Z}$.

$a=1$, so $2b+15 = 3m$

$0 \leq b \leq 9$ by trial & error $2b = 0, 6, 12$ or 18

$\underline{b = 0, 3, 6 \text{ or } 9}$

 **

14. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(i) (a) Use the Euclidean algorithm to find the highest common factor h of 416 and 72 **(3)**

(b) Hence determine integers a and b such that

$$416a + 72b = h$$

(3)

(c) Determine the value c in the set $\{0,1,2...,415\}$ such that

$$23 \times 72 \equiv c \pmod{416}$$

(2)

(ii) Evaluate $5^{10} \pmod{13}$ giving your answer as the smallest positive integer solution. **(3)**

i) a)

$416 = 72(5) + 56$

$72 = 56(1) + 16$

$56 = 16(3) + 8$ $\quad h = 8$

$16 = 8(2) + 0$

b)

$8 = 56 - 16(3)$

$= 56 - (72 - 56)(3)$

$= 4(56) - 3(72)$

$= 4(416 - 72(5)) - 3(72)$

$8 = 4(416) - 23(72) \quad (*)$

$a = 4, \; b = -23$

c)

$23 \times 72 \equiv c \bmod 416$

if we reduce $*$ mod 416 we get

$-23(72) \equiv 8 \bmod 416$

$\Rightarrow 23(72) \equiv -8 \bmod 416$

$\equiv 408 \bmod 416$ $\quad$ so $\underline{c = 408}$

(i) $5^{10} \bmod 13$ **

$\equiv (5^2)^5 \bmod 13$

$\equiv (25)^5 \bmod 13$

$\equiv 12^5 \bmod 13$

$\equiv (-1)^5 \bmod 13$

$\equiv -1 \bmod 13$

$\equiv 12 \bmod 13$

 **

15 Given that n is a positive integer, show that the numbers $(4n + 1)$ and $(6n + 1)$ are co-prime. **[3]**

let $h = hcf(4n+1, 6n+1)$

$h \mid 4n+1$ & $h \mid 6n+1$

$h \mid 3(4n+1) - 2(6n+1)$

$h \mid 1$

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16. (a) Use the Euclidean algorithm to show that the highest common factor of 168 and 66 is 6

(2)

(b) Use back substitution to determine integers a and b such that

$$168a + 66b = 6$$

(3)

(c) Explain why there are no integer solutions to the equation

$$168x + 66y = 10$$

(1)

(d) Solve the congruence equation

$$11v \equiv 8 \pmod{28}$$

(3)

a) $168 = 66(2) + 36$
$66 = 36(1) + 30$
$36 = 30(1) + 6$
$30 = 6(5) + 0$

so $hcf(168, 36) = 6$

b) $6 = 36 - 30$
$= 36 - (66 - 36)$
$= 2(36) - 66$
$= 2(168 - 66(2)) - 66$
$= 2(168) - 5(66)$

$a = 2 \quad b = -5$

c) no integer solutions since 10 is not a multiple of the $hcf(66, 168) = 6$.

P.T.O

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d) $11v \equiv 8 \bmod 28$

want to find inverse of 11 mod 28.

Now from (6) $\quad 2(168) - 5(66) = 6$

$$\div 6 \qquad\qquad \div 6$$

$$2(28) - 5(11) = 1$$

$$\Rightarrow \quad -5(11) \equiv 1 \bmod 28$$

$$11v \equiv 8 \bmod 28$$

$$-5 \qquad\qquad \times -5$$

$$v \equiv -40 \bmod 28$$

$$\equiv 16 \bmod 28$$

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17 **(a)** Express as a decimal (base-10) number the base-23 number 7119_{23}. **[2]**

(b) Solve the linear congruence $7n+11 \equiv 9 \pmod{23}$. **[3]**

(c) Let $N = 10a+b$ and $M = a+7b$, where a and b are integers and $0 \leqslant b \leqslant 9$.

(i) By considering $3N-7M$, prove that $23 \mid N$ if and only if $23 \mid M$. **[4]**

(ii) Use a procedure based on this result to show that $N = 711965$ is a multiple of 23. **[2]**

a) $23^3 \times 7 + 23^2 \times 1 + 23 \times 1 + 9 = 85730_{10}$

b) $7n + 11 \equiv 9 \bmod 23$

$7n \equiv -2 \bmod 23$

$7n \equiv 21 \bmod 23$

$7n \equiv 7(3) \bmod 23$

$7n \times 7^{-1} \equiv 7(3) \times 7^{-1} \bmod 23$

$n \equiv 3 \bmod 23$

Since $a \times a^{-1} \equiv 1 \bmod n$.

$7^{-1} \bmod 23$ exists since $hcf(7, 23) = 1$

for example, $6^{-1} \bmod 10$ doesn't exist since $hcf(6, 10) \neq 1$

c) $N = 10a + b$, $M = a + 7b$

$3N - 7M = 30a + 3b - 7a - 49b = 23a - 46b = 23(a - 2b)$ $(*)$

Want to show that $23 \mid N \iff 23 \mid M$

first $(\Rightarrow)$:

Suppose $23 \mid N$, then $23 \mid 3N - 23(a-2b)$

$a \mid b$ & $a \mid c \Rightarrow a \mid bx + cy$ for all integers x and y

$\Rightarrow 23 \mid 7M$

by rearranging $(*)$

$\Rightarrow 23 \mid 7$ or $23 \mid M$

if $p \mid ab$ then $p \mid a$ or $p \mid b$ (Euclid's lemma)

$\Rightarrow 23 \mid M$, since $23 \nmid 7$.

Now $(\Leftarrow)$: ***

Suppose $23 \mid M$, then $23 \mid 23(a-2b) + 7M$ — Same reason as before

$\Rightarrow 23 \mid 3N$ — " "

$\Rightarrow 23 \mid 3$ or $23 \mid N$ — " "

$\Rightarrow 23 \mid N$ since $23 \nmid 3$ — " "

Or $23 \mid 3N \Rightarrow 23 \mid N$ since $\gcd(23,3)=1$
(direct application of Euclid's lemma

$\boxed{///}$

ii) $N = 711965 = 10(71196) + 5$

$M_0 = 71196 + 7(5)$ — since $M = a + 7b$

$= 71231$

$= 10(7123) + \underline{1}$

$M_1 = 7123 + 7(1)$

$= 7130$

$= 10(713)$

$M_2 = 713 + 7(0)$

$= 713$

$= 10(71) + 3$

$M_3 = 71 + 7(3) = 92 = 4(23)$

So $23 \mid M_3 \Leftrightarrow 23 \mid M_2$

$\Leftrightarrow 23 \mid M_1$

$\Leftrightarrow 23 \mid M_0$ –

$\Leftrightarrow 23 \mid N$

$\boxed{//}$

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18. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

(a) Use the Euclidean Algorithm to determine the highest common factor h of 234 and 96
(3)

(b) Hence determine integers a and b such that

$$234a + 96b = h$$
(3)

(c) Solve the congruence equation

$$96x \equiv 36 \pmod{234}$$
(5)

a)
$234 = 96(2) + 42$
$96 = 42(2) + 12$
$42 = 12(3) + 6$
$12 = 6(2) + 0$

$\text{hcf}(234, 96) = 6$

b)
$6 = 42 - 12(3)$
$= 42 - (96 - 42(2))(3)$
$= 7(42) - 3(96)$
$= 7(234 - 96(2)) - 3(96)$
$6 = 7(234) - 17(96)$ $(*)$

$a = 7 \quad b = -17$

c)
$96x \equiv 36 \bmod 234$

if we reduce $(*)$ mod 234 we get

$-17(96) \equiv 6 \bmod 234$

$96x \equiv 36 \bmod 234$
$\times -17$ $\times -17$
$6x \equiv 90 \bmod 234 \implies x \equiv 15 \bmod 39$

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19 In this question, N is the number 26 132 652.

(a) Without dividing N by 13, explain why 13 is a factor of N. **[1]**

(b) Use standard divisibility tests to show that 36 is a factor of N. **[3]**

It is given that $N = 36 \times 725\,907$.

(c) Use the results of parts **(a)** and **(b)** to deduce that 13 is a factor of 725 907. **[2]**

a) by inspection - 26, 13, 26, 52 all divisible by 13

b) $36 = 6^2 = 2^2 \times 3^2$

$= 4 \times 9$

Need to check divisibility by 4 and 9

note the only reason this works is because $hcf(4,9) = 1$

$4 \mid 52$ since $4 \times 13 = 52$

$\Rightarrow 4 \mid N$

$2+6+1+3+2+6+5+2 = 27 = 9(3)$

$\Rightarrow 9 \mid N$

$\Rightarrow 36 \mid N$

c) $N = 36 \times 725907$

from a) we know $13 \mid N$

$\Rightarrow 13 \mid 36 \times 725907$

$\Rightarrow 13 \mid 36$ or $13 \mid 725907$ by Euclid's lemma

$\Rightarrow 13 \mid 725907$ since $13 \nmid 36$.

$\boxed{/\!/}$

20 For integers a and b, with $a \geqslant 0$ and $0 \leqslant b \leqslant 99$, the numbers M and N are such that

$$M = 100a + b \quad \text{and} \quad N = a - 9b.$$

(i) By considering the number $M + 2N$, show that $17 \mid M$ if and only if $17 \mid N$. **[4]**

(ii) Demonstrate step-by-step how an algorithm based on the result of part **(i)** can be used to show that $2\,058\,376\,813\,901$ is a multiple of 17. **[4]**

i) $M = 100a + b \qquad N = a - 9b$

$M + 2N = 102a - 17b = 17(6a - b) \qquad (*)$

wts $17 \mid M \iff 17 \mid N$

first $\Rightarrow$:

suppose $17 \mid M$, then $17 \mid 17(6a-b) - M$

Since if $a \mid b$ and $a \mid c$ then $a \mid bx + cy$ for all $x, y \in \mathbb{Z}$.

$\Rightarrow 17 \mid 2N$ by $(*)$

$\Rightarrow 17 \mid 2$ or $17 \mid N$ by Euclid's lemma

$\Rightarrow 17 \mid N$ since $17 \nmid 2$.

Now $\Leftarrow$:

Suppose $17 \mid N$, then $17 \mid 17(6a-b) - 2N$ '' ''

$\Rightarrow 17 \mid M$ '' ''

ii) $M = 2058376813901 = 100(20583768139) + 1$

$N_0 = 20583768139 - 9(1)$

$= 20583768130$

$= 100(205837681) + 30$

$N_1 = 205837681 - 9(30)$

$= 205837411$

$= 100(2058374) + 11$

$N_2 = 2058374 - 9(11)$ ***

$= 2058275$

$= 100(20582) + 75$

$N_3 = 20582 - 9(75)$

$= 19907$

$= 100(199) + 7$

$N_4 = 199 - 9(7)$

$= 136 = \underline{\underline{17(8)}}$

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21 Consider the integers a and b, where, for each integer n, $a = 7n+4$ and $b = 8n+5$.

Let $h = \text{hcf}(a, b)$.

(a) Determine all possible values of h. **[3]**

(b) Find all values of n for which a and b are **not** co-prime. **[2]**

a) $h \mid 7n+4$ & $h \mid 8n+5$

$\Rightarrow h \mid x(7n+4) + y(8n+5)$ for all $x, y \in \mathbb{Z}$ by closure of divisibility (not Bezout's Lemma)

take $x = 8, y = -7$

$\Rightarrow h \mid 32 - 35$

$\Rightarrow h \mid -3$ impossible

~~take~~ instead $x = -8, y = 7$

$\Rightarrow h \mid -32 + 35$

$\Rightarrow h \mid 3$

$\Rightarrow h = 1$ or 3

b) not coprime $\Rightarrow h = 3$

i.e. $7n + 4 \equiv 0 \bmod 3$ & $8n + 5 \equiv 0 \bmod 3$

$\Rightarrow n + 1 \equiv 0 \bmod 3$ (*)

$n \equiv 2 \bmod 3$

$2n + 2 \equiv 0 \bmod 3$

$2(n+1) \equiv 0 \bmod 3$ same as *

so $n \equiv 2 \bmod 3$

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22 **(a)** Determine all values of x for which $16x \equiv 5 \pmod{101}$. **[4]**

(b) Solve

(i) $95x \equiv 6 \pmod{101}$, **[2]**

(ii) $95x \equiv 5 \pmod{101}$. **[2]**

a) $16x \equiv 5 \bmod 101$ $(*)$

want to find the inverse of 16 mod 101

could try be a hero and 'brute force it', i.e. check $1, 2, 3, \dots$ and so on

OR (much better) use the Euclidean Algorithm

$$101 = 6 \times 16 + 5$$
$$16 = 3 \times 5 + 1$$

$$\Rightarrow 1 = 16 - 3(5)$$
$$= 16 - 3(101 - 6(16))$$
$$= 19(16) - 3(101)$$

$$\Rightarrow 16 \times 19 = 3(101) + 1$$
$$\equiv 1 \bmod 101$$

in other 'words' $16^{-1} \equiv 19 \bmod 101$

Multiply both sides of $(*)$ by 19

$$x \equiv 5 \times 19 \bmod 101$$
$$\equiv 95 \bmod 101$$

b) i) $95x \equiv 6 \bmod 101$

Method 1: Notice $95 \equiv -6 \bmod 101$

i.e. $-6x \equiv 6 \bmod 101$

$(-6)^{-1} \equiv (95)^{-1} \bmod 101$ exists since $\text{hcf}(95, 101) = 1$

$$-6 \times (-6)^{-1} x \equiv 6 \times (-6)^{-1} \bmod 101$$
$$x \equiv -1 \bmod 101$$
$$\equiv 100 \bmod 101$$

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Method 2 : standard method

$95x \equiv 6 \mod 101$

$19(5x) \equiv 6 \mod 101$

$\times 16$ $\times 16$

$5x \equiv 96 \mod 101$

$\times 81$ $\times 81$

$x \equiv 96 \times 81 \mod 101$

$\equiv -5 \times -20 \mod 101$

$\equiv 100 \mod 101$

Since $19 \times 16 \equiv 1 \mod 101$ from (a)

we don't need to use Euclidean Algorithm for 5^{-1}.

$5 \times 81 = 405 \equiv 1 \mod 101$

bii) $95x \equiv 5 \mod 101$

$19 \times 5x \equiv 5 \mod 101$

$\times 5^{-1}$ $\times 5^{-1}$

$19x \equiv 1 \mod 101$

$\times 16$ $\times 16$

$x \equiv 16 \mod 101$

5^{-1} exists since $hcf(5, 101) = 1$

Since $19 \times 16 \equiv 1 \mod 101$ by part (a)

****

23 Let $N = 10a + b$ and $M = a + 3b$, where a and b are integers such that $a \geqslant 1$ and $0 \leqslant b \leqslant 9$.

(a) Prove that $29 \mid N$ if and only if $29 \mid M$. **[5]**

(b) Use an iterative method based on the result of part **(a)** to show that 899364472 is a multiple of 29. **[3]**

a) Consider $N + 19M = 10a + b + 19a + 57b$
$= 29a + 58b = 29(a+2b)$ $(*)$

$a \mid b$ & $a \mid c \Rightarrow a \mid bx + cy$ for all $x, y \in \mathbb{Z}$

$(\Rightarrow)$: Suppose $29 \mid N$ then $29 \mid 29(a+2b) - N$

$\Rightarrow 29 \mid 19M$ by $(*)$

$\xRightarrow{\text{Euclid's Lemma}} 29 \mid 19$ or $29 \mid M$

$\Rightarrow 29 \mid M$ since $29 \nmid 19$

$(\Leftarrow)$: Suppose $29 \mid M$ then $29 \mid 29(a+2b) - 19M$ " "

$\Rightarrow 29 \mid N$ " "

$\boxed{//}$

b) $N = 10(89936447) + 2$

$M_0 = 89936447 + 3(2)$
$= 89936453$
$= 10(8993645) + 3$

$M_1 = 8993645 + 3(3)$
$= 8993654$
$= 10(899365) + 4$

$M_2 = 899365 + 3(4)$
$= 899377$
$= 10(89937) + 7$

$M_3 = 89937 + 3(7)$
$= 89958$
$= 10(8995) + 8$

$M_4 = 8995 + 3(8)$
$= 9019$
$= 10(901) + 9$

$M_5 = 901 + 3(9)$
$= 928$
$= 10(92) + 8$

$M_6 = 92 + 3(8) = 116 = 10(11) + 6$

$M_7 = 11 + 3(6) = 29 = 29(1)$

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24 **(a)** Let $a = 1071$ and $b = 67$.

(i) Find the unique integers q and r such that $a = bq + r$, where $q > 0$ and $0 \leqslant r < b$. **[1]**

(ii) Hence express the answer to **(a)(i)** in the form of a linear congruence modulo b. **[1]**

(b) Use the fact that $358 \times 715 - 239 \times 1071 = 1$ to prove that 715 and 1071 are co-prime. **[4]**

a) i) $1071 = 67(15) + 66$

ii) $1071 \equiv 67(15) + 66 \mod 67$

$\equiv 66 \mod 67$

$67 \equiv 0 \mod 67$

b) $358 \times 715 - 239 \times 1071 = 1$

let $h = hcf(715, 1071)$

$h \mid 715$ & $h \mid 1071$ (because it's a factor of 715 and 1071 by definition)

$\Rightarrow h \mid 358 \times 715 - 239 \times 1071$ (if $a \mid b$ and $a \mid c$ then $a \mid bx + cy$ for all $x, y \in \mathbb{Z}$)

$\Rightarrow h \mid 1$

$\Rightarrow h = 1$ because the only integer that goes into 1 is 1.

Note. -1 also does but hcf is always positive so $h = 1$.

$\Rightarrow$ 715 and 1071 are coprime.

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25 For positive integers n, let $f(n) = 1 + 2^n + 4^n$.

(a) **(i)** Given that n is a multiple of 3, but **not** of 9, use the division algorithm to write down the two possible forms that n can take. **[1]**

(ii) Show that when n is a multiple of 3, but **not** of 9, $f(n)$ is a multiple of 73. **[6]**

(b) Determine the value of $f(n)$, modulo 73, in the case when n is a multiple of 9. **[2]**

a i) $n = 9k+3$ or $n = 9k+6$ $k \in \mathbb{Z}$.

(ii) Case 1: $n = 9k+3$

$$f(n) = 1 + 2^{9k+3} + 4^{9k+3}$$
$$= 1 + 8(2)^{9k} + 64(4)^{9k}$$
$$= 1 + 8(2)^{9k} + 64((2)^{9k})^2$$
$$\equiv 1 + 8(1) + 64(1)^2 \mod 73$$
$$\equiv 1 + 8 + 64 \mod 73$$
$$\equiv 73 \mod 73$$
$$\equiv 0 \mod 73$$

$$2^{9k} \mod 73$$
$$\equiv (2^9)^k \mod 73$$
$$\equiv (8 \times 8 \times 8)^k \ 73$$
$$\equiv (64 \times 8)^k \mod 73$$
$$\equiv (-9 \times 8)^k \mod 73$$
$$\equiv (-72)^k \mod 73$$
$$\equiv (1)^k \mod 73$$
$$\equiv 1$$

Note that you could also 'brute force it' although this isn't ***** advised

Case 1 (alt)

$$f(n) = 1 + 2^{9k+3} + 4^{9k+3}$$

$$= 1 + 8(2)^{9k} + 64(4)^{9k}$$

$$\equiv 1 + 8(512)^{k} + 64(262144)^{k}$$

$$\equiv 1 + 8(1)^{k} + 64(1)^{k}$$

$$\equiv \ldots\ldots$$

$$\equiv 0 \mod 73$$

$$\frac{262144}{73} = 3591.01\ldots$$

$$\therefore 262144 = 3591(73) + 1$$

but this kind of defeats the point of the question as you need to use a calculator — unclear if marks would be awarded.

Case 2: $n = 9k+6$

$$f(n) = 1 + 2^{9k+6} + 4^{9k+6}$$

$$= 1 + 2^{6}(2)^{9k} + 4^{6}(4)^{9k}$$

$$\equiv 1 + 64(2)^{9k} + 64^{2}(4)^{9k}$$

$$\equiv 1 + 64 + 64^{2} \mod 73$$

$$\equiv 1 + 64 + (-9)^{2} \mod 73$$

$$\equiv 1 + 64 + 81 \mod 73$$

$$\equiv 1 + 64 + 8 \mod 73$$

$$\equiv 0 \mod 73$$

we have already shown $2^{9k} \equiv 4^{9k} \equiv 1 \mod 73$ in case 1

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b) let $a = 9k$

$$f(a) = 1 + 2^{9k} + 4^{9k}$$

$$\equiv 1 + 1 + 1 \mod 73$$ from part (a)ii)

$$\equiv 3 \mod 73$$

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26. (i) Determine all the possible integers a, where $a > 3$, such that

$$15 \equiv 3 \bmod a$$

(2)

(ii) Show that if p is prime, x is an integer and $x^2 \equiv 1 \bmod p$ then either

$$x \equiv 1 \bmod p \qquad \text{or} \qquad x \equiv -1 \bmod p$$

(3)

(iii) A company has £13 940 220 to share between 11 charities.

Without performing any division and showing all your working, decide if it is possible to share this money equally between the 11 charities.

(2)

i) $15 - 3 \equiv 0 \bmod a$

$12 \equiv 0 \bmod a$

$\Leftrightarrow \quad a \mid 12$

i.e. $a = 1, 2, 3, 4, 6, 12$

but $a > 3$ so $a = 4, 6$ or 12

ii) $x^2 \equiv 1 \bmod p$

$\Rightarrow \quad x^2 - 1 \equiv 0 \bmod p$

$\Rightarrow \quad p \mid x^2 - 1$

$\Rightarrow \quad p \mid (x-1)(x+1)$

$\overset{\text{Euclid's lemma}}{\Longrightarrow} \quad p \mid x-1$ or $p \mid x+1$

$\Rightarrow \quad x - 1 \equiv 0 \bmod p$ or $x + 1 \equiv 0 \bmod p$

$\Rightarrow \quad x \equiv 1 \bmod p$ or $x \equiv -1 \bmod p$

iii) $1 - 3 + 9 - 4 + 0 - 2 + 2 - 0 = 3$

$11 \nmid 3$ so it won't be possible

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27 **(i)** Let $N = 10a + b$ and $M = a - 5b$ where a and b are integers such that $a \geq 1$ and $0 \leq b \leq 9$. N is to be tested for divisibility by 17.

(a) Prove that $17 \mid N$ if and only if $17 \mid M$. **[5]**

(b) Demonstrate step-by-step how an algorithm based on these forms can be used to show that $17 \mid 4097$. **[2]**

(ii) **(a)** Show that, for $n \geq 2$, any number of the form 1001_n is composite. **[3]**

(b) Given that n is a positive even number, provide a counter-example to show that the statement "any number of the form 10001_n is prime" is false. **[3]**

i) a Consider $N + 7M = 10a + b + 7a - 35b$

$= 17a - 34b = 17(a - 2b)$ $\quad *$

$(\Rightarrow)$ suppose $17 \mid N$, then $17 \mid 17(a-2b) - N$

$\Rightarrow 17 \mid 7M$ by $(*)$

$\overset{\text{Euclid's lemma}}{\Longrightarrow}$ $17 \mid 7$ or $17 \mid M$

$\Rightarrow 17 \mid M$ since $17 \nmid 7$

$(\Leftarrow)$ Suppose $17 \mid M$ then $17 \mid 17(a-2b) - 7M$

$\Rightarrow 17 \mid N$ by $(*)$

i) b let $N = 4097 = 10(409) + 7$

$M_0 = 409 + 7(7)$

$= 458$

$= 10(45) + 8$

$M_1 = 45 + 7(8)$

$= 101$

$= 10(10) + 1$

$M_2 = 10 + 7(1)$

$= 17$

$17 \mid M_2$

$\Rightarrow 17 \mid M_1$

$\Rightarrow 17 \mid M$

$\Rightarrow 17 \mid N$

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ii) a $\quad 1001_n = n^3+1$

$$= (n+1)(n^2-n+1)$$

$n+1 \geq 3$ for $n \geq 2$. need to check n^2-n+1.

$$n^2-n+1 = \left(n-\tfrac{1}{2}\right)^2 + \tfrac{3}{4}$$

i.e. for $n \geq 2$, n^2-n+1 is increasing because $a = \underline{1}$ and turning point at $\left(\tfrac{1}{2}, \tfrac{3}{4}\right)$

so for $n \geq 2$, $n^2-n+1 \geq 4-2+1 = 3$.

so both factors are ≥ 3, $\therefore$ 1001_n is composite

ii) b $\quad 10001_n = n^4+1$

Unfortunately there's no 'nice' algebraic way of finding a counter example eg by factorising as $n^{2k}+1$ can't be factorised when $k \in \mathbb{Z}$.

So we check $n=2, n=4, \ldots$

using the FACT button on calculator helps show which is composite.

$$\underline{8^4+1 = 4097 = 17 \times 241}$$

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28 **(a)** The number N has the base-10 form $N = abba\ abba \ldots abba$, consisting of blocks of four digits, as shown, where a and b are integers such that $\leqslant a < 10$ and $0 \leqslant b < 10$.

Use a standard divisibility test to show that N is always divisible by 11. **[3]**

(b) The number M has the base-n form $M = cddc\ cddc \ldots cddc$, where $n > 11$ and c and d are integers such that $1 \leqslant c < n$ and $0 \leqslant d < n$.

Show that M is always divisible by a number of the form $k_1 n + k_2$, where k_1 and k_2 are integers to be determined. **[3]**

a) $(a-b+b-a)+(a-b+b-a)+\ldots\ldots+(a-b+b-a)$

$= 0$ since a's and b's cancel. (because they are in blocks of 4)

so $11 \mid 0 \implies 11 \mid N$.

b) one block of $cddc_n$ in base-10 is

let $x = cn^3 + dn^2 + dn + c$

want to factorise this. No obvious way via factor theorem
So split terms:

$c(n^3+1) + d(n^2+n)$

$= c(n+1)(n^2-n+1) + dn(n+1)$

$= (n+1)(c(n^2-n+1)+dn)$

$\therefore n+1 \mid cn^3+dn^2+dn+c$

i.e. $k_1 = 1$ and $k_2 = 1$

let $y = cddc\,cddc \ldots cddc = x + n^4x + n^8x + \ldots + n^{4m}x$
$= x(1+n^4+n^8+\ldots+n^{4m})$

so if $n+1 \mid x$, $n+1 \mid y$ $\quad x \mid y$ $\quad (1+n^4+\ldots+n^{4m})$ is an integer

$\square$

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29 **(a)** Let $f(n) = 2^{4n+3} + 3^{3n+1}$.

Use arithmetic modulo 11 to prove that $f(n) \equiv 0 \pmod{11}$ for all integers $n \geqslant 0$. **[4]**

(b) Use the standard test for divisibility by 11 to prove the following statements.

(i) $10^{33} + 1$ is divisible by 11 **[2]**

(ii) $10^{33} + 1$ is divisible by 121 **[4]**

a) $2^{4n+3} + 3^{3n+1} \equiv 8(2)^{4n} + 3(3)^{3n} \mod 11$

$\equiv 8(16)^n + 3(27)^n \mod 11$

$\equiv 8(5)^n + 3(5)^n \mod 11$

$\equiv 11(5)^n \mod 11$

$\equiv 0 \times (5)^n \mod 11$

$\equiv 0 \mod 11$

b) i) $10^{33}+1 = \underbrace{1000\ldots\ldots 000}_{32 \text{ zeros}}1$ ← not 33 since the last one becomes a one.

Divisibility by 11,

$1+0-0+0-0+\ldots\ldots+0-1 = 0$ which is a multiple of 11

So $10^{33}+1$ is a multiple of 11

P.T.O

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ii) $n^3+1 = (n+1)(n^2-n+1)$

$n^5+1 = (n+1)(n^4-n^3+n^2-n+1)$

$\vdots$

$n^{33}+1 = (n+1)(n^{32}-n^{31}+n^{30}-n^{29}+\dots-n+1)$

let $x = 10^{33}+1 = (10+1)(10^{32}-10^{31}+\dots-10+1)$

$= 11\left(\underbrace{(10^{32}+10^{30}+\dots+10^2+1)}_{a} - \underbrace{(10^{31}+10^{29}+\dots+10^3+10)}_{b}\right)$

$= 11(\underbrace{90909\cdots091}_{32 \text{ digits}})$

let this be a

$9-0+9-0+9-\dots+9-1$

$= 16\times 9 - 1$

$= 143 = 11(13)$

So $11 \mid a$ $\quad\therefore\ x = 11\times 11k$ for some $k\in\mathbb{Z}$

i.e. $x = 121k$

$\Rightarrow 121 \mid x$

$\square$

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30 Let $f(n)$ denote the base-n number 2121_n where $n \geqslant 3$.

(a) (i) For each $n \geqslant 3$, show that $f(n)$ can be written as the product of two positive integers greater than 1, $a(n)$ and $b(n)$, each of which is a function of n. **[2]**

(ii) Deduce that $f(n)$ is always composite. **[1]**

(b) Let h be the highest common factor of $a(n)$ and $b(n)$.

(i) Prove that h is either 1 or 5. **[4]**

(ii) Find a value of n for which $h = 5$. **[2]**

ai) $2121_n = (2n^3 + n^2 + 2n + 1)_{10}$

Notice that $a, b, c, d > 0$ so any solution must be negative

also notice that if $n = -1$, $2121_n = -2 + 1 - 2 + 1_{10} = -2_{10}$

and if I take values of $n < -1$, 2121_n will be even less than -2.

So I try a negative between 0 and -1. The most obvious to try is $-\frac{1}{2}$:

$$2\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) + 1 = -\frac{1}{4} + \frac{1}{4} - 1 + 1 = 0$$

$$\Rightarrow 2n^3 + n^2 + 2n + 1 = (2n+1)\underline{(n^2+1)}$$ using algebra.

ii) $f(n)$ is composite since $2n+1 \geqslant 7$ and $n^2+1 \geqslant 10$ and both are integers so $f(n)$ is not prime. (since $n \geqslant 3$)

b) $h = hcf\left((2n+1), (n^2+1)\right)$

$h \mid 2n+1$ & $h \mid n^2+1$

$\Rightarrow h \mid x(2n+1) + y(n^2+1)$

note if I take $x = n$ and $y = -2$, I get

$h \mid 2n^2 + n - 2n^2 - 2$

$\Rightarrow h \mid n - 2$ which doesn't help as it's in terms of n.

P.T.O

I need to take x to be a value that will be a coefficient ******
of 0 for n. The only way to do this is with a difference
of two squares

Take $x = 2n-1$ and $y = -4$!

$$h \mid 4n^2 - 1 - 4n^2 - 4 \Rightarrow h \mid -5$$

Instead take $x = 1-2n$ and $y = 4$

$$h \mid 1 - 4n^2 + 4n^2 + 4 \Rightarrow h \mid 5$$

so $h = 1$ or 5

b ii) $h = 5 \Rightarrow 2n+1 \equiv 0 \bmod 5$

$$2n \equiv 4 \bmod 5$$

$$n \equiv 2 \bmod 5$$

$n \geq 3$ so $\boxed{n = 7}$

note if $n = 7$, $n^2 + 1 = 50 \equiv 0 \bmod 5$

so $5 \mid n^2 + 1$ as well.