London Academy of Maths

Vectors

Worked Solutions

1 **(i)** Find a vector which is perpendicular to both $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -6 \\ 4 \end{pmatrix}$. **[2]**

(ii) The cartesian equation of a line is $\frac{x}{2} = y - 3 = 2z + 4$.

Express the equation of this line in vector form. **[3]**

i) $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} -3 \\ -6 \\ 4 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 1 & 3 & -2 \\ -3 & -6 & 4 \end{vmatrix}$

$= \underline{i}(12-12) - \underline{j}(4-6) + \underline{k}(-6+9)$

$= 2\underline{j} + 3\underline{k} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

ii) $\frac{x}{2} = y - 3 = 2z + 4 (= \lambda)$

$x = 2\lambda$

$y = 3 + \lambda$

$z = -2 + \frac{1}{2}\lambda$

so $\underline{r} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -\frac{1}{2} \end{pmatrix}$

2 Find, to the nearest degree, the angle between the vectors $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ -3 \end{pmatrix}$. **[3]**

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|} = \frac{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -3 \end{pmatrix}}{\sqrt{1^2+2^2}\,\sqrt{2^2+3^2+3^2}}$$

$$= \frac{4}{\sqrt{5}\,\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{22}}\right) = 68^\circ \text{ to nearest degree.}$$

3 **(a)** Determine whether the point (19, −12, 17) lies on the line $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$. **[3]**

Vectors **a** and **b** are given by $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix}$.

(b) **(i)** Find, in degrees, the angle between **a** and **b**. **[3]**

(ii) Find a vector which is perpendicular to both **a** and **b**. **[2]**

Suppose it does. Then,

a) $$\begin{pmatrix} 19 \\ -12 \\ 17 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

Equate x, y:

$$19 = 4 + 3\lambda \implies 15 = 3\lambda$$
$$\lambda = 5$$

$[y]$: $RHS = -2 + 5(-2) = -12 = LHS$

$[z]$: $RHS = 7 + 5(4) = 27 \neq 17 = LHS$

So the point doesn't lie on the line

b) i) $$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|} = \frac{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix}}{\sqrt{1^2+2^2+2^2}\,\sqrt{3^2+6^2+2^2}}$$

$$= \frac{-11}{21}$$

$$\theta = 121.6^\circ$$

ii) P.T.O

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b(ii) $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix} = \left| \begin{pmatrix} i & j & k \\ 1 & -2 & 2 \\ -3 & 6 & 2 \end{pmatrix} \right|$

$= -16\underline{i} - 8\underline{j}$

$= \begin{pmatrix} -16 \\ -8 \\ 0 \end{pmatrix}$

can rescale if you want

$\left(= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right)$

4 The vector **p**, all of whose components are positive, is given by $\mathbf{p} = \begin{pmatrix} a^2 \\ a-5 \\ 26 \end{pmatrix}$ where a is a constant.

You are given that **p** is perpendicular to the vector $\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$.

Determine the value of a. **[4]**

$$\begin{pmatrix} a^2 \\ a-5 \\ 26 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix} = 0$$

$$2a^2 + 6a - 30 - 78 = 0$$

$$a^2 + 3a - 54 = 0$$

$$(a+9)(a-6) = 0$$

$$a = -9 \quad \text{or} \quad a = 6$$

Now $p_x, p_y, p_z > 0$ so $\boxed{a = 6}$ since

$-9 - 5 = -14 < 0$.

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5 The lines l_1 and l_2 have the following equations.

$$l_1 : \mathbf{r} = \begin{pmatrix} 8 \\ -11 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}$$
$$l_2 : \mathbf{r} = \begin{pmatrix} -6 \\ 11 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$

(a) Show that l_1 and l_2 intersect. **[4]**

(b) Write down the point of intersection of l_1 and l_2. **[1]**

a) $$\begin{pmatrix} 8 \\ -11 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 11 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$

A $\quad 8 - 2\lambda = -6 - 3\mu$

B $\quad -11 + 5\lambda = 11 + \mu$

C $\quad -2 + 3\lambda = 8 - \mu$

B + C $\quad -13 + 8\lambda = 19$

$8\lambda = 32$

$\lambda = 4$

Sub $\lambda = 4$ into B / C : $\quad -11 + 20 = 11 + \mu$

$\Rightarrow \mu = -2$

sub both into A :

LHS $= 8 - 2(4) = 0 \qquad$ RHS $= -6 - 3(-2) = 0$

LHS = RHS

so they do intersect

b) Sub $\lambda = 4$ or $\mu = -2$ into l_1 or l_2 :

... $(0, 9, 10)$

6 **(a)** **(i)** Find $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$. **[1]**

(ii) State a geometrical relationship between the answer to part **(a)(i)** and the vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$. **[1]**

(iii) Verify the relationship stated in part **(a)(ii)**. **[2]**

(b) Find the angle between the vectors $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $4\mathbf{i}-\mathbf{j}+8\mathbf{k}$. **[3]**

ai) $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = \left|\begin{pmatrix} i & j & k \\ 1 & 2 & -1 \\ 3 & 5 & -2 \end{pmatrix}\right|$

$= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

ii) the answer to part a(i) is a vector perpendicular to both $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$

iii) $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 1 - 2 + 1 = 0$

$\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 3 - 5 + 2 = 0$

the dot product is zero in both cases so $\theta = 90^\circ$
so vectors are perpendicular.

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b)

$$\cos\theta = \frac{\begin{pmatrix}2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\8\end{pmatrix}}{\sqrt{2^2+2^2+1^2}\sqrt{4^2+1^2+8^2}}$$

$$= \frac{18}{27}$$

$$= \frac{2}{3}$$

$\theta = 48.2^\circ$ to 1 dp

7 **(a)** Find a vector which is perpendicular to both $3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. **[1]**

The equations of two lines are $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 11\mathbf{j} - 4\mathbf{k} + \mu(-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$.

(b) Show that the lines intersect, stating the point of intersection. **[5]**

a) $\begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 3 & -5 & -1 \\ 1 & 3 & -4 \end{vmatrix} = i(20+3) - j(-12+1) + k(9+5)$

$= \begin{pmatrix} 23 \\ 11 \\ 14 \end{pmatrix}$

b) $\underline{r} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\quad \underline{r} = \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$

equate x, y, z:

$2 + \lambda = 1 - \mu$ (A)

$3 - 2\lambda = 11 + 3\mu$ (B)

$3 + \lambda = -4 - 2\mu$ (C)

(A) − (C): $-1 = 5 + \mu \Rightarrow \mu = -6$

sub into (A): $2 + \lambda = 1 + 6$

$\Rightarrow \lambda = 5$

Now check with (B): $LHS = 3 - 2(5) = -7$

$RHS = 11 + 3(-6) = -7$ $\quad LHS = RHS$

so the lines do intersect.

sub $\lambda = 5$ into first line or $\mu = -6$ into second to find point of intersection:

$\boxed{(7, -7, 8)}$

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8 The lines L_1 and L_2 have the following equations.

$$L_1 : \mathbf{r} = \begin{pmatrix} -5 \\ 6 \\ 15 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ -2 \end{pmatrix}$$

$$L_2 : \mathbf{r} = \begin{pmatrix} 24 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

(a) Show that L_1 and L_2 intersect, giving the position vector of the point of intersection. **[5]**

(b) Find the equation of the line which intersects L_1 and L_2 and is perpendicular to both. Give your answer in cartesian form. **[3]**

a) $\begin{pmatrix} -5 \\ 6 \\ 15 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 24 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$

Equate x, y, z:

$-5 + 5\lambda = 24 + 3\mu \implies 5\lambda - 3\mu = 29$ (A)

$6 - 2\lambda = 1 + \mu \implies \mu + 2\lambda = 5$ (B)

$15 - 2\lambda = -5 - 4\mu \implies 2\lambda - 4\mu = 20$ (C)

(B) − (C): $5\mu = -15 \implies \mu = -3$

Sub $\mu = -3$ into (B) or (C) (but not (A)):

$-3 + 2\lambda = 5$

$2\lambda = 8$

$\lambda = 4$

Now check with (A):

$LHS = 5\lambda - 3\mu = 5(4) - 3(-3) = 29 = RHS$

So the lines intersect at $\begin{pmatrix} 15 \\ -2 \\ 7 \end{pmatrix}$ (by subbing in $\lambda = 4$ or $\mu = -3$ in either line)

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b) let L_3 be the line we want to find

A fixed point on L_3 is $(15, -2, 7)$ since it's on both L_1 & L_2 as it's the intersection point. The direction vector of L_3 is just the cross/vector product of the direction vectors of L_1 & L_2:

$$\begin{pmatrix} 5 \\ -2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} = \left| \begin{pmatrix} i & j & k \\ 5 & -2 & -2 \\ 3 & 1 & -4 \end{pmatrix} \right|$$

$$= 10\underline{i} + 14\underline{j} + 11\underline{k}$$

so $$\underline{r} = \begin{pmatrix} 15 \\ -2 \\ 7 \end{pmatrix} + t \begin{pmatrix} 10 \\ 14 \\ 11 \end{pmatrix}$$

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9 Find the acute angle between the lines with vector equations $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. **[3]**

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{1^2+2^2+1^2}\sqrt{3^2+1^2+2^2}}$$

$$= \frac{3+2+2}{\sqrt{6}\sqrt{14}}$$

$$= \frac{7}{\sqrt{6}\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{7}{\sqrt{6}\sqrt{14}}\right) = 40.2^\circ \quad \text{to } 1\text{dp.}$$

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10 **(a)** Given that the lines $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ k \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ meet, determine k. **[5]**

(b) **In this question you must show detailed reasoning.**

Find the acute angle between the two lines. **[4]**

a) Equate lines:

A $\quad -\lambda = -1 + 2\mu$

B $\quad 2 + \lambda = 2 + 3\mu$

C $\quad 2 + 3\lambda = k + 4\mu$

A + B $\quad 2 = 1 + 5\mu$

$\Rightarrow \mu = 1/5$

sub into A:

$-\lambda = -1 + 2/5$

$\Rightarrow \lambda = 3/5$

No need to check with C as we know they intersect.

Sub into C to find k:

$2 + 9/5 = k + 4/5$

$10 + 9 = 5k + 4$

$15 = 5k \quad \Rightarrow \quad \boxed{k = 3}$

b)

$$\cos\theta = \frac{\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}}{\sqrt{1^2+1^2+3^2}\sqrt{2^2+3^2+4^2}}$$

$$= \frac{-2+3+12}{\sqrt{11}\sqrt{29}}$$

$$= \frac{13}{\sqrt{11}\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{11}\sqrt{29}}\right) = 43.3^\circ \text{ to 1dp.}$$

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11 Vectors, **a**, **b** and **c**, are given by $\mathbf{a} = \mathbf{i} + (1-p)\mathbf{j} + (p+2)\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 14\mathbf{j} + (p-3)\mathbf{k}$ where p is a constant.

You are given that $\mathbf{a} \times \mathbf{b}$ is perpendicular to **c**.

Determine the possible values of p. **[6]**

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = 0$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 1-p & p+2 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \underline{i}\left((1-p)-(p+2)\right) - \underline{j}\left(1-2(p+2)\right) + \underline{k}\left(1-2(1-p)\right)$$

$$= \underline{i}\,(1-p-p-2) - \underline{j}\,(1-2p-4) + \underline{k}\,(1-2+2p)$$

$$= \begin{pmatrix} -1-2p \\ 3+2p \\ 2p-1 \end{pmatrix}$$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = 0 \Rightarrow \begin{pmatrix} -1-2p \\ 3+2p \\ 2p-1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 14 \\ p-3 \end{pmatrix} = 0$$

$$-1-2p + (3+2p)14 + (2p-1)(p-3) = 0$$

$$-1-2p + 42 + 28p + 2p^2 - 7p + 3 = 0$$

$$2p^2 + 19p + 44 = 0$$

$$(2p+11)(p+4) = 0$$

$$\boxed{p = -\frac{11}{2} \text{ or } p = -4}$$

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12 Determine whether the lines

$$\frac{x-1}{1}=\frac{y+2}{-1}=\frac{z+4}{2} \quad \text{and} \quad \frac{x+3}{2}=\frac{y-1}{3}=\frac{z-5}{4}$$

intersect or are skew. **[5]**

First convert to parametric equations

$$\frac{x-1}{1}=\frac{y+2}{-1}=\frac{z+4}{2}\ (=\lambda) \qquad \frac{x+3}{2}=\frac{y-1}{3}=\frac{z-5}{4}$$

$\Rightarrow$ $x = 1+\lambda$, $y = -2-\lambda$, $z = -4+2\lambda$

$\Rightarrow$ $x = -3+2\mu$, $y = 1+3\mu$, $z = 5+4\mu$

Equate x, y, z:

(A) $1+\lambda = -3+2\mu$

(B) $-2-\lambda = 1+3\mu$

(C) $-4+2\lambda = 5+4\mu$

(A) + (B) : $-1 = -2+5\mu$

$\Rightarrow \mu = 1/5$

sub into (A):

$1+\lambda = -3+2/5$

$\Rightarrow \lambda = -\frac{18}{5}$

sub into (C):

$LHS = -4+2\left(-\frac{18}{5}\right) = -\frac{56}{5}$

$RHS = 5+4\left(1/5\right) = \frac{29}{5}$ $\qquad LHS \neq RHS$

so lines don't intersect.

They are also not parallel since $\begin{pmatrix}2\\3\\4\end{pmatrix}$ is not a multiple of $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ (these are the direction vectors of the lines)

So the lines are skew

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13 The position vector of point A is $\mathbf{a} = -9\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$.
The line l passes through A and is perpendicular to $\mathbf{a}$.

(a) Determine the shortest distance between the origin, O, and l. **[2]**

l is also perpendicular to the vector $\mathbf{b}$ where $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

(b) Find a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. **[1]**

(c) Write down an equation of l in vector form. **[1]**

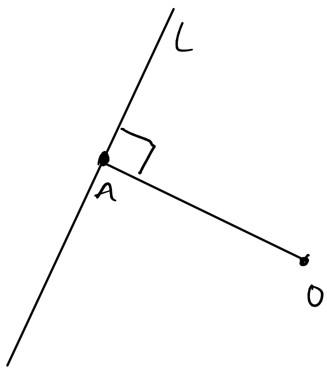
P is a point on l such that $PA = 2OA$.

(d) Find angle POA giving your answer to 3 significant figures. **[3]**

C is a point whose position vector, $\mathbf{c}$, is given by $\mathbf{c} = p\mathbf{a}$ for some constant p. The line m passes through C and has equation $\mathbf{r} = \mathbf{c} + \mu\mathbf{b}$. The point with position vector $9\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}$ lies on m.

(e) Find the value of p. **[3]**

a)



$$d = \sqrt{9^2 + 2^2 + 6^2}$$
$$= \sqrt{121}$$
$$= 11$$

b)

$$\underline{a} \times \underline{b} = \begin{pmatrix} -9 \\ 2 \\ 6 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \left| \begin{pmatrix} i & j & k \\ -9 & 2 & 6 \\ -2 & 1 & 1 \end{pmatrix} \right|$$

$$= -4i - 3j - 5j$$

$$= \begin{pmatrix} -4 \\ -3 \\ -5 \end{pmatrix}$$

c)

$$\underline{r} = \begin{pmatrix} -9 \\ 2 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ -3 \\ -5 \end{pmatrix}$$

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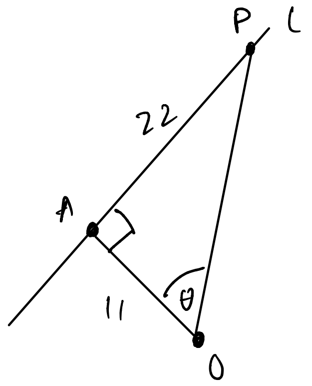
d)



we know $|\vec{OA}| = 11$ from a)

and therefore $|\vec{PA}| = 22$ since

$|\vec{PA}| = 2|\vec{OA}|$

So $\tan\theta = \frac{22}{11} = 2$

$\theta = 63.4°$ to 3 sf.

e) $\underline{r} = \underline{c} + \mu\underline{b}$

$$= p\begin{pmatrix}-9\\2\\6\end{pmatrix} + \mu\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

Now $\begin{pmatrix}9\\8\\-12\end{pmatrix}$ lies on this line so,

$$\begin{pmatrix}9\\8\\-12\end{pmatrix} = p\begin{pmatrix}-9\\2\\6\end{pmatrix} + \mu\begin{pmatrix}-2\\1\\1\end{pmatrix} \quad \text{for some } \mu$$

Equate x, y:

$9 = -9p - 2\mu$ (A)

$8 = 2p + \mu$ (B)

(C) = (B) × 2: $16 = 4p + 2\mu$

(A) + (C) $25 = -5p$

$\Rightarrow p = -5$

14 The line through points $A(8, -7, -2)$ and $B(11, -9, 0)$ is denoted by L_1.

(a) Find a vector equation for L_1. **[2]**

(b) Determine whether the point $(26, -19, -14)$ lies on L_1. **[2]**

The line L_2 passes through the origin, O, and intersects L_1 at the point C. The lines L_1 and L_2 are perpendicular.

(c) By using the fact that C lies on L_1, find a vector equation for L_2. **[4]**

(d) Hence find the shortest distance from O to L_1. **[2]**

5a) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

you could also use $\begin{pmatrix} 11 \\ -9 \\ 0 \end{pmatrix}$ here

So $\underline{r} = \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

b) $\begin{pmatrix} 26 \\ -19 \\ -14 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

$26 = 8 + 3\mu \Rightarrow \mu = 6$

$-19 = -7 - 2\mu$ Sub $\mu = 6$ in: $RHS = -7 - 2(6) = -19 = LHS$

$-14 = -2 - 2\mu$ Sub $\mu = 6$ in: $RHS = -2 - 2(6) = -14 = LHS$

So the point $(26, -19, -14)$ is on the line.

c) P.T.O

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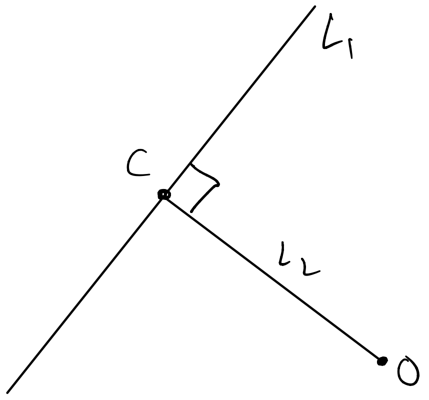
c)



C lies on L_1 so

$$\vec{OC} = \begin{pmatrix} 8+3\mu \\ -7-2\mu \\ -2+2\mu \end{pmatrix} \quad \text{for some } \mu \in \mathbb{R}.$$

$L_1 \perp L_2$ (by info given)

$$\text{so } \vec{OC} \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = 0$$

$$\Rightarrow 24 + 9\mu + 14 + 4\mu - 4 + 4\mu = 0$$

$$\Rightarrow 34 + 17\mu = 0$$

$$\Rightarrow \mu = -2$$

$$\text{so } \vec{OC} = \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix}$$

so L_2 has equation $\underline{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix}$

d) Shortest distance from O to L_1 is

$$|\vec{OC}| = \sqrt{2^2 + 3^2 + 6^2} = 7$$

because C lies on L_1 and L_2 and $\vec{OC} \perp L_1$.

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15 **(i)** Find the value of k such that $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\3\\k\end{pmatrix}$ are perpendicular. **[2]**

Two lines have equations $l_1: \mathbf{r} = \begin{pmatrix}3\\2\\7\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix}6\\5\\2\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$.

(ii) Find the point of intersection of l_1 and l_2. **[4]**

(iii) The vector $\begin{pmatrix}1\\a\\b\end{pmatrix}$ is perpendicular to the lines l_1 and l_2.

Find the values of a and b. **[5]**

i) $\begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-2\\3\\k\end{pmatrix} = 0$

$\Rightarrow -2 + 6 + k = 0$

$k = -4$

ii) $\begin{pmatrix}3\\2\\7\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}6\\5\\2\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$

$3 + \lambda = 6 + 2\mu \Rightarrow \lambda - 2\mu = 3$ (A)

$2 - \lambda = 5 + \mu \Rightarrow \mu + \lambda = -3$ (B)

$7 + 3\lambda = 2 - \mu \Rightarrow 3\lambda + \mu = -5$ (C)

(A) − (B) $-3\mu = 6$

$\mu = -2$

Sub $\mu = -2$ into (B):

$-2 + \lambda = -3$

$\lambda = -1$

by subbing into $\mu = -2$ or $\lambda = -1$ into either L_1 or L_2,

P.O.I $= (2, 3, 4)$

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iii) $\begin{pmatrix}1\\a\\b\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=0 \implies 1-a+3b=0$

$\implies 3b-a=-1$ B

$\begin{pmatrix}1\\a\\b\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=0 \implies 2+a-b=0$

$a-b=-2$ A

A + B $\quad 2b=-3$

$b=-3/2$

Sub into A :

$a=-7/2$

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16 Determine the acute angle between the line $\mathbf{r}=\begin{pmatrix}-\sqrt{3}\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\sqrt{3}\\-\sqrt{3}\end{pmatrix}$ and the y-axis. **[4]**

y axis: $\underline{r} = \mu\begin{pmatrix}0\\1\\0\end{pmatrix}$

so, $\cos\theta = \dfrac{\begin{pmatrix}1\\2\sqrt{3}\\-\sqrt{3}\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{\sqrt{1^2+(2\sqrt{3})^2+(\sqrt{3})^2}\sqrt{1^2}}$

$= \dfrac{2\sqrt{3}}{\sqrt{16}} = \dfrac{\sqrt{3}}{2}$

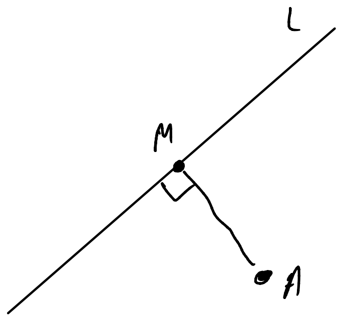
$\theta = \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = 30^\circ$

17 The line l has equations $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{2}$ and the point A is (7, 3, 7). M is the point where the perpendicular from A meets l.

(i) Find, in either order, the coordinates of M and the perpendicular distance from A to l. **[7]**

(ii) Find the coordinates of the point B on AM such that $\overrightarrow{AB} = 3\overrightarrow{BM}$. **[3]**

i) Vector form: $\underline{r} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$



again, here make sure not to fall into trap of calling M (x, y, z)

$M = (1+2\lambda,\ 1+3\lambda,\ -1+2\lambda)$ for some $\lambda \in \mathbb{R}$

$$\overrightarrow{AM} = \begin{pmatrix} 1+2\lambda \\ 1+3\lambda \\ -1+2\lambda \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2\lambda-6 \\ 3\lambda-2 \\ 2\lambda-8 \end{pmatrix}$$

$\overrightarrow{AM}$ is perpendicular to l, so

$$\begin{pmatrix} 2\lambda-6 \\ 3\lambda-2 \\ 2\lambda-8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 0$$

$$4\lambda - 12 + 9\lambda - 6 + 4\lambda - 16 = 0$$

$$17\lambda - 34 = 0$$

$$\boxed{\lambda = 2}$$

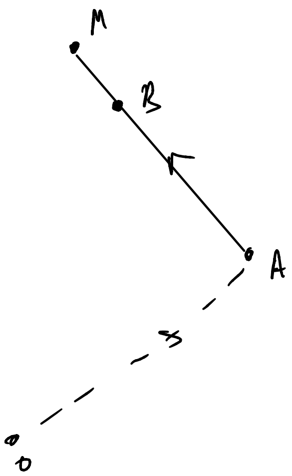
So $M = (5, 7, 3)$

$$\overrightarrow{AM} = \begin{pmatrix} 2(2)-6 \\ 3(2)-2 \\ 2(2)-8 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$$

$$|\overrightarrow{AM}| = \sqrt{2^2+4^2+4^2} = \sqrt{36} = \boxed{6}$$

***

ii)



$$\vec{AB} = 3\vec{BM}$$

$$\Rightarrow \quad \vec{AB} = \frac{3}{4}\vec{AM}$$

$$= \frac{3}{4}\begin{pmatrix}-2\\4\\-4\end{pmatrix} = \begin{pmatrix}-3/2\\3\\-3\end{pmatrix}$$

$$\vec{OB} = \vec{OA} + \vec{AB}$$

$$= \begin{pmatrix}7\\3\\7\end{pmatrix} + \begin{pmatrix}-3/2\\3\\-3\end{pmatrix} = \begin{pmatrix}11/2\\6\\4\end{pmatrix}$$

$$\Rightarrow \boxed{B = \left(11/2,\ 6,\ 4\right)}$$

18 Points A, B and C have coordinates $(4, 2, 0)$, $(1, 5, 3)$ and $(1, 4, -2)$ respectively. The line l passes through A and B.

(a) Find a cartesian equation for l. **[3]**

M is the point on l that is closest to C.

(b) Find the coordinates of M. **[4]**

(c) Find the exact area of the triangle ABC. **[4]**

a) $\vec{AB} = \begin{pmatrix}1\\5\\3\end{pmatrix} - \begin{pmatrix}4\\2\\0\end{pmatrix} = \begin{pmatrix}-3\\3\\3\end{pmatrix}$

so $r = \begin{pmatrix}4\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}-3\\3\\3\end{pmatrix}$

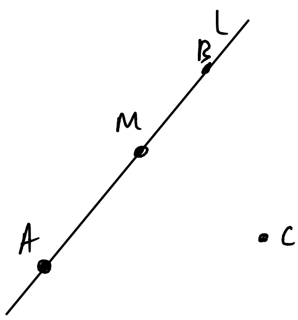
$= \begin{pmatrix}4\\2\\0\end{pmatrix} + t\begin{pmatrix}-1\\1\\1\end{pmatrix}$

Note what I have done here is called rescaling. I can only do this with direction vectors (and I also don't have to do it).

$\dfrac{x-4}{-1} = \dfrac{y-2}{1} = \dfrac{z}{1}$

$\iff \boxed{4 - x = y - 2 = z}$

b)



the classic mistake here is to call $M(x,y,z)$ but we can express M in terms of one unknown not 3:

$M = (4-t,\ 2+t,\ t)$ for some $t \in \mathbb{R}$

$\Rightarrow \vec{CM} = \begin{pmatrix}4-t\\2+t\\t\end{pmatrix} - \begin{pmatrix}1\\4\\-2\end{pmatrix} = \begin{pmatrix}3-t\\t-2\\t+2\end{pmatrix}$

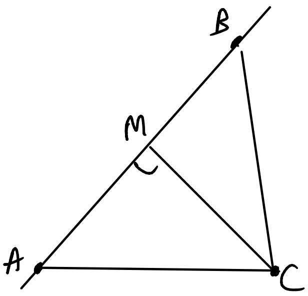
$\vec{CM} \cdot \begin{pmatrix}-1\\1\\1\end{pmatrix} = 0$ since $\vec{CM}$ is perpendicular to l.

$-(3-t) + t - 2 + t + 2 = 0$

$3t - 3 = 0 \quad \Rightarrow t = 1$

So $M = (3, 3, 1)$

c)



because $\vec{MC}$ is perpendicular to to $\vec{AB}$ ***

we can use $\frac{1}{2}bh$ to find area.

$$|\vec{AB}| = \sqrt{3^2+3^2+3^2} = \sqrt{27}$$

$$|\vec{MC}| = \sqrt{2^2+1^2+3^2} = \sqrt{14}$$

$$\text{Area} = \frac{1}{2}\sqrt{27}\sqrt{14} = \frac{3\sqrt{42}}{2}$$

****

19 Three intersecting lines L_1, L_2 and L_3 have equations

$$L_1\text{: } \frac{x}{2}=\frac{y}{3}=\frac{z}{1}, \quad L_2\text{: } \frac{x}{1}=\frac{y}{2}=\frac{z}{-4} \quad \text{and} \quad L_3\text{: } \frac{x-1}{1}=\frac{y-2}{1}=\frac{z+4}{5}.$$

Find the area of the triangle enclosed by these lines. **[9]**

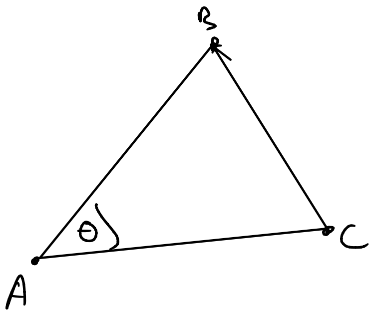
We need to first find intersections.

Convert all to vector form:

$$\underline{r}_1 = \lambda\begin{pmatrix}2\\3\\1\end{pmatrix} \qquad \underline{r}_2 = \mu\begin{pmatrix}1\\2\\-4\end{pmatrix} \qquad \underline{r}_3 = \begin{pmatrix}1\\2\\-4\end{pmatrix} + t\begin{pmatrix}1\\1\\5\end{pmatrix}$$

intersection of $\underline{r}_1$ & $\underline{r}_2$ has to be $A(0,0,0)$ since they both go through that point. By inspection $\underline{r}_2$ & $\underline{r}_3$ intersect at $B(1,2,-4)$ since $\mu=1$ and $t=0$ ensures this (can also find by solving simultaneous eqns)

$\underline{r}_1$ & $\underline{r}_3$ intersect at $C(2,3,1)$ again by inspection, since $\lambda=1$, $t=1$ ensures this.



To find area ABC, first find $|\vec{AB}|$, $|\vec{AC}|$:

$$|\vec{AB}| = \sqrt{1^2+2^2+4^2} = \sqrt{21}$$

$$|\vec{AC}| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$$

to find $\angle B\hat{A}C$, either use cosine rule by finding $|\vec{BC}|$ or better, use scalar product:

$$\cos\theta = \frac{\begin{pmatrix}1\\2\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\3\\1\end{pmatrix}}{\sqrt{21}\ \sqrt{14}} = \frac{4}{\sqrt{21}\sqrt{14}}$$

Area of $ABC = \frac{1}{2}\sqrt{21}\sqrt{14} \times \sin\left(\cos^{-1}\left(\frac{4}{\sqrt{21}\sqrt{14}}\right)\right) = \boxed{8.34}$ units^2 to 3sf

****

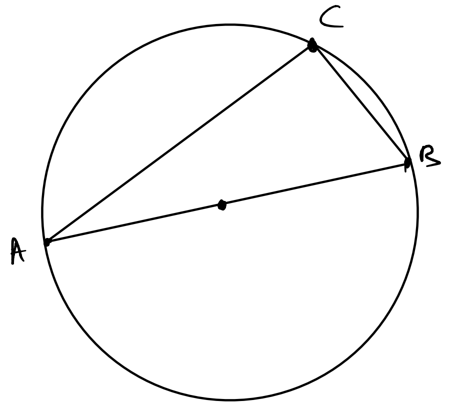
20 The line segment AB is a diameter of a sphere, S. The point C is **any** point on the surface of S.

(a) Explain why $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ for **all** possible positions of C. **[3]**

You are now given that A is the point $(11, 12, -14)$ and B is the point $(9, 13, 6)$.

(b) Given that the coordinates of C have the form $(2p, p, 1)$, where p is a constant, determine the coordinates of the possible positions of C. **[6]**

a)



using knowledge of circle theorems $A\hat{C}B = 90°$ since C is on the perimeter so ACB is an angle in a semi-circle (i.e it's a right angle). (1)

so $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ since $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are perpendicular since dot product equal zero. (2)

In the case where $C = A$ or $C = B$, $\overrightarrow{AC} = 0$ or $\overrightarrow{BC} = 0$ respectively, so $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ since a zero vector dotted with any vector is zero. (3)

b) $\overrightarrow{AC} = \begin{pmatrix} 2p-11 \\ p-12 \\ 15 \end{pmatrix}$ $\quad \overrightarrow{BC} = \begin{pmatrix} 2p-9 \\ p-13 \\ -5 \end{pmatrix}$

$\overrightarrow{AC} \cdot \overrightarrow{BC} = (2p-11)(2p-9) + (p-12)(p-13) + 15(-5) = 0$

$4p^2 - 40p + 99 + p^2 - 25p + 156 - 75 = 0$

$5p^2 - 65p + 180 = 0$

$p^2 - 13p + 36 = 0$

$(p-4)(p-9) = 0$

$p = 4$ or $p = 9$ $\Rightarrow$ $C = (8, 4, 1)$ or $(18, 9, 1)$

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21 The equations of two **intersecting** lines are

$$\mathbf{r} = \begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$

where a is a constant.

(a) Find a vector, **b**, which is perpendicular to both lines. **[2]**

(b) Show that $\mathbf{b}.\begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} = \mathbf{b}.\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$. **[2]**

(c) Hence, or otherwise, find the value of a. **[2]**

a) $$\underline{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} = \left| \begin{pmatrix} i & j & k \\ 2 & 2 & 1 \\ -3 & 1 & -1 \end{pmatrix} \right|$$

$$= -3\underline{i} - \underline{j} + 8\underline{k}$$

b) $$\begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$

for some $\mu, \lambda \in \mathbb{R}$ since we know they intersect

$$\Rightarrow \underline{b} \cdot \left(\begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right) = \underline{b} \cdot \left(\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} \right)$$

$$\Rightarrow \underline{b} \cdot \begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} + \lambda \underline{b} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \underline{b} \cdot \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \mu \underline{b} \cdot \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow b \cdot \begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} = \underline{b} \cdot \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$$

since $\underline{b}$ is perpendicular to both $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$ so the dot product of b with each vector is zero.

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c) $\begin{pmatrix} -3 \\ -1 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$

$\Rightarrow \quad 36 - a - 8 = -6 + 40$

$28 - a = 34$

$a = -6$

******

22 The points $P(3, 5, -21)$ and $Q(-1, 3, -16)$ are on the ceiling of a long straight underground tunnel. A ventilation shaft must be dug from the point M on the ceiling of the tunnel midway between P and Q to horizontal ground level (where the z-coordinate is 0). The ventilation shaft must be perpendicular to the tunnel.

The path of the ventilation shaft is modelled by the vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is the position vector of M.

You are given that $\mathbf{b} = \begin{pmatrix} 1 \\ s \\ t \end{pmatrix}$ where s and t are real numbers.

(a) Show that $s = 2.5t - 2$. **[3]**

(b) Show that at the point where the ventilation shaft reaches the ground $\lambda = \frac{c}{t}$, where c is a constant to be determined. **[3]**

(c) Using the results in parts **(a)** and **(b)**, determine the shortest possible length of the ventilation shaft. **[6]**

(d) Explain what the fact that $\mathbf{b} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \mathbf{0}$ means about the direction of the ventilation shaft. **[1]**

a) $\underline{b} \cdot \overrightarrow{PQ} = 0$ since $\underline{b}$ is perpendicula to $\overrightarrow{PQ}$

$$\begin{pmatrix} 1 \\ s \\ t \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix} = 0$$

$$-4 - 2s + 5t = 0$$

$$2s = 5t - 4$$

$$s = 2.5t - 2$$

b) Equation of ventilation shaft :

$$\underline{r} = \begin{pmatrix} 1 \\ 4 \\ -18.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2.5t - 2 \\ t \end{pmatrix}$$ by result of (a)

now at ground $z = 0$, so $-18.5 + \lambda t = 0$

$$\Rightarrow \lambda t = 18.5$$

$$\lambda = \frac{18.5}{t}$$

c) assuming the ventilation shaft stops completely at $z=0$,

Shortest distance $= |\vec{MX}|$ where X is the point where $z=0$. Now we know $M = (1, 4, -18.5)$ so just need X: by subbing in $\lambda = \frac{18.5}{t}$ we get

$$X = \left(1 + \frac{18.5}{t},\ 4 + \frac{18.5}{t}(2.5t-2),\ 0\right)$$

$$\text{so } \vec{MX} = \left(\frac{18.5}{t},\ \frac{18.5}{t}(2.5t-2),\ 18.5\right)$$

$$|\vec{MX}| = \sqrt{\left(\frac{18.5}{t}\right)^2 + \left(\frac{18.5}{t}(2.5t-2)\right)^2 + 18.5^2}$$

$$= \sqrt{\left(\frac{18.5}{t}\right)^2\left(1 + (2.5t-2)^2 + t^2\right)}$$

by factoring out $\left(\frac{18.5}{t}\right)^2$

$$= \sqrt{\left(\frac{18.5}{t}\right)^2\left(7.25t^2 - 10t + 5\right)}$$

$$= \sqrt{(18.5)^2\left(7.25 - 10t^{-1} + 5t^{-2}\right)}$$

let $f(t) = (18.5)^2\left(7.25 - 10t^{-1} + 5t^{-2}\right)$

to minimise $|\vec{MX}|$, we differentiate $f(t)$ and set it equal to 0.

$$f'(t) = (18.5)^2\left(10t^{-2} - 10t^{-3}\right) = 0$$

$$\Rightarrow \quad \times t^3\left(\frac{10}{t^2} - \frac{10}{t^3} = 0\right)\times t^3$$

$$10t - 10 = 0$$

$$t = 1$$

sub $t=1$ into $|\overrightarrow{MX}|$:

$$|\overrightarrow{MX}| = \sqrt{(18-5)^2 (7-25-10+5)} = \boxed{27.75}$$

d) In general $\underline{a} \times \underline{b} = \underline{0} \iff \underline{a}$ is parallel to $\underline{b}$

so if $\underline{b} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq 0$ this means that the ventilation shaft isn't perfectly vertical (or parallel with z-axis)