London Academy of Maths

Series - Method of Differences

Question Paper

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1. (a) Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions.

(2)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}$$

(4)

(c) Evaluate $\displaystyle\sum_{r=10}^{100} \frac{2}{(r+1)(r+3)}$, giving your answer to 3 significant figures.

(2)

Question 1 continued

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2. (a) Express $\dfrac{1}{(r+6)(r+8)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+6)(r+8)} = \frac{n(an+b)}{56(n+7)(n+8)}$$

where a and b are integers to be found.

(4)

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Question 2 continued

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3. (a) Express $\dfrac{1}{4r^2 - 1}$ in partial fractions.

(1)

(b) Hence prove that

$$\sum_{r=1}^{n} \frac{1}{4r^2 - 1} = \frac{n}{2n+1}$$

(3)

(c) Find the exact value of

$$\sum_{r=9}^{25} \frac{5}{4r^2 - 1}$$

(2)

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Question 3 continued

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4. (a) Express $\frac{1}{r(r+2)}$ in partial fractions.

(1)

(b) Hence show that $\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}$.

(5)

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Question 4 continued

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5. (a) Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions.

(2)

(b) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(an+b)}{6(n+2)(n+3)},$$

where a and b are constants to be found.

(6)

(c) Find the value of $\displaystyle\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)}$, to 5 decimal places.

(3)

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Question 5 continued

6. Given that

$$\frac{2n+1}{n^2(n+1)^2} \equiv \frac{A}{n^2} + \frac{B}{(n+1)^2}$$

(a) determine the value of A and the value of B

(1)

(b) Hence show that, for $n \geqslant 5$

$$\sum_{r=5}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n^2 + an + b}{c(n+1)^2}$$

where a, b and c are integers to be determined.

(4)

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Question 6 continued

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7. (a) Express $\dfrac{3}{(3r-1)(3r+2)}$ in partial fractions.
(2)

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}$$
(3)

(c) Evaluate $\displaystyle\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)}$, giving your answer to 3 significant figures.
(2)

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Question 7 continued

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8. (a) Express $\dfrac{2}{(2r+1)(2r+3)}$ in partial fractions.

(2)

(b) Using your answer to (a), find, in terms of n,

$$\sum_{r=1}^{n} \frac{3}{(2r+1)(2r+3)}$$

Give your answer as a single fraction in its simplest form.

(3)

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Question 8 continued

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9. (a) Express $\dfrac{2}{4r^2 - 1}$ in partial fractions.

(2)

(b) Hence use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2 - 1} = \frac{n}{2n+1}$$

(3)

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Question 9 continued

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10. (a) Express $\dfrac{2}{(r+2)(r+4)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}$$

(5)

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Question 10 continued

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11. (a) Express $\dfrac{1}{(r+3)(r+4)}$ in partial fractions.

(1)

(b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \frac{1}{(r+3)(r+4)} = \frac{n}{a(n+a)}$$

where a is a constant to be found.

(5)

(c) Find the exact value of $\displaystyle\sum_{r=15}^{30} \frac{1}{(r+3)(r+4)}$

(2)

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Question 11 continued

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Question 12 continued

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13. (a) Show that, for $r > 0$

$$r - 3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{r^3 - 7r - 5}{(r+1)(r+2)}$$

(2)

(b) Hence prove, using the method of differences, that

$$\sum_{r=1}^{n} \frac{r^3 - 7r - 5}{(r+1)(r+2)} = \frac{n(n^2 + an + b)}{2(n+2)}$$

where a and b are constants to be found.

(5)

Question 13 continued

14. Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{2}{(r+4)(r+6)} = \frac{n(an+b)}{30(n+5)(n+6)}$$

where a and b are integers to be determined.

(6)

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Question 14 continued

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15. (a) Express $\frac{1}{r(r+2)}$ in partial fractions.

(2)

(b) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(an+b)}{4(n+1)(n+2)}$$

where a and b are constants to be found.

(6)

(c) Hence show that

$$\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \frac{n(4n+5)}{4(n+1)(n+2)(2n+1)}$$

(3)

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Question 15 continued

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16. (a) Show that

$$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{1}{2(r+1)(r+2)} - \frac{1}{2(r+2)(r+3)}$$

(2)

(b) Hence, or otherwise, find

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)}$$

giving your answer as a single fraction in its simplest form.

(4)

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Question 16 continued

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17. (a) Show that, for $r > 0$

$$\frac{r+2}{r(r+1)} - \frac{r+3}{(r+1)(r+2)} = \frac{r+4}{r(r+1)(r+2)}$$

(2)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \frac{n(an+b)}{c(n+1)(n+2)}$$

where a, b and c are integers to be determined.

(4)

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Question 17 continued

18. (a) Express

$$\frac{1}{(2n-1)(2n+1)(2n+3)}$$

in partial fractions.

(2)

(b) Hence, using the method of differences, show that for all integer values of n,

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)(2r+3)} = \frac{n(n+2)}{a(2n+b)(2n+c)}$$

where a, b and c are integers to be determined.

(4)

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Question 18 continued

19. (a) Express

$$\frac{1}{(n+3)(n+5)}$$

in partial fractions.

(2)

(b) Hence, using the method of differences, show that for all positive integer values of n,

$$\sum_{r=1}^{n} \frac{1}{(r+3)(r+5)} = \frac{n(pn+q)}{40(n+4)(n+5)}$$

where p and q are integers to be determined.

(4)

(c) Use the answer to part (b) to determine, as a simplified fraction, the value of

$$\frac{1}{9\times 11}+\frac{1}{10\times 12}+\ldots+\frac{1}{24\times 26}$$

(2)

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Question 19 continued

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20. (a) Express $\dfrac{4r+2}{r(r+1)(r+2)}$ in partial fractions.

(3)

(b) Hence, using the method of differences, prove that

$$\sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} = \frac{n(an+b)}{2(n+1)(n+2)}$$

where a and b are constants to be found.

(5)

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Question 20 continued

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21. Prove that, for $n \in \mathbb{Z}$, $n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are inte. ers to be found.

(5)

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Question 21 continued

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22 (a) Show that

$$r^2(r+1)^2 - (r-1)^2\, r^2 \equiv 4r^3$$

(3)

Given that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$(1^3 + 2^3 + 3^3 + \ldots + n^3) = (1 + 2 + 3 + \ldots + n)^2$$

(4)

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Question 22 continued

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23. (a) Show that, for $r > 0$

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}$$

(1)

(b) Hence prove that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

(3)

(c) Show that, for $n \in \mathbb{N}$, $n > 1$

$$\sum_{r=n}^{3n} \frac{6r+3}{r^2(r+1)^2} = \frac{an^2+bn+c}{n^2(3n+1)^2}$$

where a, b and c are constants to be found.

(3)

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Question 23 continued

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24. (a) Show that

$$(r+1)^3 - (r-1)^3 \equiv 6r^2 + 2.$$

(2)

(b) Hence show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$

(5)

(c) Show that $\sum_{r=n}^{2n} r^2 = \frac{1}{6}n(n+1)(an+b)$, where a and b are constants to be found.

(4)

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Question 24 continued

25. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Show that, for $r \geqslant 2$

$$\frac{2}{\sqrt{r}+\sqrt{r-2}} = \sqrt{r} - \sqrt{r-2}$$

(2)

(b) Hence use the method of differences to determine

$$\sum_{r=2}^{n} \frac{2}{\sqrt{r}+\sqrt{r-2}}$$

giving your answer in simplest form.

(3)

(c) Hence show that

$$\sum_{r=4}^{50} \frac{2}{\sqrt{r}+\sqrt{r-2}} = A + B\sqrt{2} + C\sqrt{3}$$

where A, B and C are integers to be determined.

(2)

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Question 25 continued

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26. (a) Use the method of differences to prove that for $n > 2$

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right)$$

(4)

(b) Hence find the exact value of

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$$

Give your answer in the form $a\ln\left(\frac{b}{c}\right)$ where a, b and c are integers to be determined.

(3)

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Question 26 continued

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27. (a) Express $\dfrac{2}{r(r^2-1)}$ in partial fractions.

(3)

(b) Hence find, in terms of n,

$$\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$$

Give your answer in the form

$$\frac{n^2 + An + B}{Cn(n+1)}$$

where A, B and C are constants to be found.

(5)

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Question 27 continued

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28. (a) Write $\dfrac{3r+1}{r(r-1)(r+1)}$ in partial fractions.

(2)

(b) Hence find

$$\sum_{r=2}^{n} \frac{3r+1}{r(r-1)(r+1)} \qquad n \geqslant 2$$

giving your answer in the form

$$\frac{an^2 + bn + c}{2n(n+1)}$$

where a, b and c are integers to be determined.

(5)

(c) Hence determine the exact value of

$$\sum_{r=15}^{20} \frac{3r+1}{r(r-1)(r+1)}$$

(2)

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Question 28 continued

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29. (a) Show that

$$n^5 - (n-1)^5 \equiv 5n^4 - 10n^3 + 10n^2 - 5n + 1$$

(2)

(b) Hence, using the method of differences, show that for all integer values of n,

$$\sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1)(2n+1)(an^2 + bn + c)$$

where a, b and c are integers to be determined.

(7)

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Question 29 continued

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30. Given that $A > B > 0$, by letting $x = \arctan A$ and $y = \arctan B$

(a) prove that

$$\arctan A - \arctan B = \arctan\left(\frac{A-B}{1+AB}\right)$$

(3)

(b) Show that when $A = r + 2$ and $B = r$

$$\frac{A-B}{1+AB} = \frac{2}{(1+r)^2}$$

(2)

(c) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \arctan\left(\frac{2}{(1+r)^2}\right) = \arctan(n+p) + \arctan(n+q) - \arctan 2 - \frac{\pi}{4}$$

where p and q are integers to be determined.

(4)

(d) Hence, making your reasoning clear, determine

$$\sum_{r=1}^{\infty} \arctan\left(\frac{2}{(1+r)^2}\right)$$

giving the answer in the form $k\pi - \arctan 2$, where k is a constant.

(2)

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Question 30 continued