London Academy of Maths

Series - Method of Differences

Question Paper

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1 **(i)** Find $\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right)$. **[3]**

(ii) What does the sum in part **(i)** tend to as $n \to \infty$? Justify your answer. **[1]**

2 **(a)** Use the method of differences to show that $\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=1-\frac{1}{n+1}$. **[1]**

(b) Hence determine the following sums.

(i) $\sum_{r=1}^{99}\frac{1}{r}-\frac{1}{r+1}$ **[1]**

(ii) $\sum_{r=100}^{\infty}\frac{1}{r}-\frac{1}{r+1}$ **[3]**

3 By expressing $\frac{1}{r+1} - \frac{1}{r+2}$ as a single fraction, find $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$ in terms of n. **[4]**

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4 **In this question you must show detailed reasoning.**

Find $\displaystyle\sum_{r=1}^{100}\left(\frac{1}{r}-\frac{1}{r+2}\right)$, giving your answer correct to 4 decimal places. **[3]**

5 In this question you must show detailed reasoning.

Find $\sum_{r=2}^{50}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$, expressing the answer as an exact fraction. **[3]**

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6 **(i)** Show that $\dfrac{1}{3r-1} - \dfrac{1}{3r+2} \equiv \dfrac{3}{(3r-1)(3r+2)}$ for all integers r. [2]

(ii) Hence use the method of differences to find $\displaystyle\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)}$. [5]

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7 **(i)** Show that $\frac{1}{r^2} - \frac{1}{(r+2)^2} \equiv \frac{4(r+1)}{r^2(r+2)^2}$. **[2]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=1}^{n} \frac{4(r+1)}{r^2(r+2)^2}$. **[6]**

(iii) Find $\sum_{r=5}^{\infty} \frac{4(r+1)}{r^2(r+2)^2}$, giving your answer in the form $\frac{p}{q}$ where p and q are integers. **[2]**

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8 **(i)** Show that $\dfrac{1}{5r-2} - \dfrac{1}{5r+3} \equiv \dfrac{5}{(5r-2)(5r+3)}$ for all integers r. **[2]**

(ii) Hence use the method of differences to show that $\displaystyle\sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} = \frac{n}{3(5n+3)}$. **[4]**

9 Given that $\dfrac{3}{(3r-1)(3r+2)} \equiv \dfrac{1}{3r-1} - \dfrac{1}{3r+2}$, find $\displaystyle\sum_{r=1}^{20} \frac{1}{(3r-1)(3r+2)}$, giving your answer as an exact fraction. **[5]**

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10 **(i)** Show that $\dfrac{1}{2r+1} - \dfrac{1}{2r+3} \equiv \dfrac{2}{(2r+1)(2r+3)}$. **[2]**

(ii) Use the method of differences to find $\displaystyle\sum_{r=1}^{30} \frac{1}{(2r+1)(2r+3)}$, expressing your answer as a fraction. **[5]**

11 You are given that $\dfrac{4}{(4n-3)(4n+1)} \equiv \dfrac{1}{4n-3} - \dfrac{1}{4n+1}$. Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1}.$$

[6]

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13 In this question you must show detailed reasoning.

(a) By using partial fractions show that $\sum_{r=1}^{n} \frac{1}{r^2+3r+2} = \frac{1}{2} - \frac{1}{n+2}$. **[5]**

(b) Hence determine the value of $\sum_{r=1}^{\infty} \frac{1}{r^2+3r+2}$. **[2]**

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14 Use the result $\frac{1}{5r-1} - \frac{1}{5r+4} \equiv \frac{5}{(5r-1)(5r+4)}$ and the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{(5r-1)(5r+4)},$$

simplifying your answer. **[6]**

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15 **In this question you must show detailed reasoning.**

(a) Use partial fractions to show that $\displaystyle\sum_{r=5}^{n}\frac{3}{r^2+r-2}=\frac{37}{60}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}$. **[5]**

(b) Write down the value of $\displaystyle\lim_{n\to\infty}\left(\sum_{r=5}^{n}\frac{3}{r^2+r-2}\right)$. **[1]**

 **

16 **(a)** Express $\dfrac{1}{(2r-1)(2r+1)}$ in partial fractions. **[3]**

(b) Hence find $\displaystyle\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$, expressing the result as a single fraction. **[4]**

 **

17 **(i)** Show that $\frac{1}{2r-3} - \frac{1}{2r+1} = \frac{4}{4r^2-4r-3}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=2}^{n} \frac{4}{4r^2-4r-3}.$$ **[6]**

(iii) Show that $\sum_{r=2}^{\infty} \frac{4}{4r^2-4r-3} = \frac{4}{3}$. **[1]**

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18 **(i)** Show that $\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$. **[4]**

(iii) Find $\sum_{r=2}^{\infty} \frac{2r+1}{r^2(r+1)^2}$. **[2]**

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19 You are given that $\frac{3}{(5+3x)(2+3x)} \equiv \frac{1}{2+3x} - \frac{1}{5+3x}$.

(i) Use this result to find $\sum_{r=1}^{100} \frac{1}{(5+3r)(2+3r)}$, giving your answer as an exact fraction. **[5]**

(ii) Write down the limit to which $\sum_{r=1}^{n} \frac{1}{(5+3r)(2+3r)}$ converges as n tends to infinity. **[1]**

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20 **(i)** Show that $\frac{1}{r} - \frac{1}{r+2} \equiv \frac{2}{r(r+2)}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=1}^{n} \frac{2}{r(r+2)}$. **[6]**

(iii) Given that $\sum_{r=N+1}^{\infty} \frac{2}{r(r+2)} = \frac{11}{30}$, find the value of N. **[4]**

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21 **(i)** Show that $\dfrac{3}{r-1}-\dfrac{2}{r}-\dfrac{1}{r+1}\equiv\dfrac{4r+2}{r(r^2-1)}$. **[2]**

(ii) Hence find an expression, in terms of n, for $\displaystyle\sum_{r=2}^{n}\frac{4r+2}{r(r^2-1)}$. **[6]**

(iii) Hence find the value of $\displaystyle\sum_{r=4}^{\infty}\frac{4r+2}{r(r^2-1)}$. **[2]**

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22 **(i)** Verify that $\dfrac{4+r}{r(r+1)(r+2)} = \dfrac{2}{r} - \dfrac{3}{r+1} + \dfrac{1}{r+2}$. **[2]**

(ii) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{4+r}{r(r+1)(r+2)} = \frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}.$$ **[6]**

(iii) Write down the limit to which $\displaystyle\sum_{r=1}^{n} \frac{4+r}{r(r+1)(r+2)}$ converges as n tends to infinity. **[1]**

(iv) Find $\displaystyle\sum_{r=50}^{100} \frac{4+r}{r(r+1)(r+2)}$, giving your answer to 3 significant figures. **[3]**

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23 **(i)** Show that $\frac{r}{r+1} - \frac{r-1}{r} \equiv \frac{1}{r(r+1)}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)}.$$ **[4]**

(iii) Hence find $\sum_{r=n+1}^{\infty} \frac{1}{r(r+1)}$. **[2]**

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24 **(i)** Show that $\frac{1}{r} - \frac{3}{r+1} + \frac{2}{r+2} \equiv \frac{2-r}{r(r+1)(r+2)}$. **[2]**

(ii) Hence show that $\sum_{r=1}^{n} \frac{2-r}{r(r+1)(r+2)} = \frac{n}{(n+1)(n+2)}$. **[5]**

(iii) Find the value of $\sum_{r=2}^{\infty} \frac{2-r}{r(r+1)(r+2)}$. **[2]**

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25 **(i)** Show that $\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$. [2]

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!}.$$ [4]

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26 **(i)** Show that $\frac{1}{r-1} - \frac{1}{r+1} \equiv \frac{2}{r^2-1}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=2}^{n} \frac{2}{r^2-1}$. **[5]**

(iii) Find the value of $\sum_{r=1000}^{\infty} \frac{2}{r^2-1}$. **[3]**

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27 **(a)** By considering $(r+1)^3 - r^3$, find $\sum_{r=1}^{n}(3r^2+3r+1)$. **[3]**

(b) Use this result to find $\sum_{r=1}^{n} r(r+1)$, expressing your answer in fully factorised form. **[4]**

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28 **(a)** Using partial fractions and the method of differences, show that

$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\ldots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{an+b}{2(n+1)(n+2)},$$

where a and b are integers to be determined. **[5]**

(b) Deduce the sum to infinity of the series.

$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\ldots.$$ **[1]**

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29 **(i)** Express $\frac{1}{2r-1} - \frac{1}{2r+1}$ as a single fraction. **[2]**

(ii) Find how many terms of the series

$$\frac{2}{1\times 3} + \frac{2}{3\times 5} + \frac{2}{5\times 7} \quad \dots + \frac{2}{(2r-1)(2r+1)} + \dots$$

are needed for the sum to exceed 0.999 999. **[7]**

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30 **(i)** Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions. **[2]**

(ii) Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)}$, expressing your answer as a single fraction. **[5]**

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31 **(i)** Show that $\frac{1}{2r+1} - \frac{1}{2r+3} \equiv \frac{2}{(2r+1)(2r+3)}$. **[1]**

(ii) Hence find $\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)}$, giving your answer as a single fraction. **[6]**

(iii) Find $\sum_{r=n}^{\infty} \frac{1}{(2r+1)(2r+3)}$, giving your answer as a single fraction. **[3]**

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32 **(i)** Show that $\frac{1}{2r-1} - \frac{1}{2r+5} \equiv \frac{6}{(2r-1)(2r+5)}$. **[1]**

Hence find

(ii) $\sum_{r=2}^{30} \frac{6}{(2r-1)(2r+5)}$, giving your answer correct to 3 decimal places, **[5]**

(iii) $\sum_{r=2}^{\infty} \frac{6}{(2r-1)(2r+5)}$, giving your answer as a single fraction. **[1]**

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33 Using an algebraic method, determine the least value of n for which $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} \geqslant 0.49$. **[8]**

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34 **In this question you must show detailed reasoning.**

(a) Show that

$$\sum_{r=1}^{n} \frac{5r+6}{r^3+r^2} = \frac{a}{n+1} + b + c\sum_{r=1}^{n} \frac{1}{r^2}$$

where a, b and c are integers whose values are to be determined. **[6]**

You are given that $\sum_{r=1}^{\infty} \frac{1}{r^2}$ exists and is equal to $\frac{1}{6}\pi^2$.

(b) Show that $\sum_{r=1}^{\infty} \frac{5r+6}{r^3+r^2}$ exists and is equal to $(\pi-1)(\pi+1)$. **[2]**

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35 **(i)** Show that $\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2} \equiv \frac{2}{r(r+1)(r+2)}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}.$$ **[6]**

(iii) Show that $\sum_{r=n+1}^{\infty} \frac{2}{r(r+1)(r+2)} = \frac{1}{(n+1)(n+2)}$. **[3]**

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36 You are given that $\frac{1}{2r-1}-\frac{1}{2r+3}=\frac{4}{(2r-1)(2r+3)}$ for all integers r.

(i) Use the method of differences to show that

$$\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+3)}=k-\frac{n+1}{(2n+1)(2n+3)},$$

stating the value of k. **[6]**

(ii) The sum of the infinite series

$$\frac{1}{\left(2(n+1)-1\right)\left(2(n+1)+3\right)}+\frac{1}{\left(2(n+2)-1\right)\left(2(n+2)+3\right)}+\frac{1}{\left(2(n+3)-1\right)\left(2(n+3)+3\right)}+\dots$$

is $\frac{7}{195}$. Show that n satisfies $28n^2-139n-174=0$ and hence find the value of n. **[5]**

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37 You are given that $\dfrac{3}{4(2r-1)} - \dfrac{1}{2r+1} + \dfrac{1}{4(2r+3)} = \dfrac{2r+5}{(2r-1)(2r+1)(2r+3)}$.

(i) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)} = \frac{2}{3} - \frac{3}{4(2n+1)} + \frac{1}{4(2n+3)}.$$ **[6]**

(ii) Write down the limit to which $\sum_{r=1}^{n} \dfrac{2r+5}{(2r-1)(2r+1)(2r+3)}$ converges as n tends to infinity. **[1]**

(iii) Find the sum of the finite series

$$\frac{45}{39 \times 41 \times 43} + \frac{47}{41 \times 43 \times 45} + \frac{49}{43 \times 45 \times 47} + \dots + \frac{105}{99 \times 101 \times 103},$$

giving your answer to 3 significant figures. **[4]**

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38 **(i)** Show that $\dfrac{1}{\sqrt{r+2}+\sqrt{r}} \equiv \dfrac{\sqrt{r+2}-\sqrt{r}}{2}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r+2}+\sqrt{r}}.$$ **[6]**

(iii) State, giving a brief reason, whether the series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{\sqrt{r+2}+\sqrt{r}}$ converges. **[1]**

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39 **(i)** Find $\sum_{r=1}^{n} r(r^2+r-7)$, giving your answer in a fully factorised form. **[5]**

A sequence $u_0, u_1, u_2, \ldots$ is defined by

$$u_0 = 5,\ u_n = u_{n-1} + n^3 + n^2 - 7n \text{ for } n \geqslant 1.$$

(ii) By considering $\sum_{r=1}^{n}(u_r - u_{r-1})$, find a formula for u_n in terms of n. **[3]**

[You do not need to factorise your answer.]

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40 **In this question you must show detailed reasoning.**

It is given that $\sum_{r=k}^{98} \frac{5r+2}{r(r+1)(r+2)} = \frac{20539}{34650}$ for some k.

Determine the value of k. **[7]**