Series - Method of Differences

Worked Solutions

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1. (a) Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions.

(2)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}$$

(4)

(c) Evaluate $\displaystyle\sum_{r=10}^{100} \frac{2}{(r+1)(r+3)}$, giving your answer to 3 significant figures.

(2)

a) $\dfrac{2}{(r+1)(r+3)} \equiv \dfrac{A}{r+1} + \dfrac{B}{r+3}$

$2 \equiv A(r+3) + B(r+1)$

$r=-3: \quad 2 = -2B \Rightarrow B = -1$

$r=-1: \quad 2 = 2A \Rightarrow A = 1$

$\dfrac{2}{(r+1)(r+3)} \equiv \boxed{\dfrac{1}{r+1} - \dfrac{1}{r+3}}$ need to write it out explicitly.

b) $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$

$r=1: \quad \frac{1}{2} - \cancel{\frac{1}{4}}$

$r=2: \quad \frac{1}{3} - \cancel{\frac{1}{5}}$

$r=3: \quad \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$

$r=n-1: \quad \cancel{\frac{1}{n}} - \frac{1}{n+2}$

$r=n: \quad \cancel{\frac{1}{n+1}} - \frac{1}{n+3}$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$

$= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$

$= \dfrac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)}$

$= \dfrac{5n^2 + 25n + 30 - 6n - 18 - 6n - 12}{6(n+2)(n+3)}$

Question 1 continued

$$= \frac{5n^2 + 13n}{6(n+2)(n+3)}$$

$$= \frac{n(5n+13)}{6(n+2)(n+3)}$$

c) $\sum_{r=10}^{100} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{100} \frac{2}{(r+1)(r+3)} - \sum_{r=1}^{9} \frac{2}{(r+1)(r+3)}$

$$= \frac{100(5(100)+13)}{6(100+2)(100+3)} - \frac{9(5(9)+13)}{6(9+2)(9+3)}$$

$$= \frac{1425}{1751} - \frac{29}{44}$$

$= \boxed{0.155}$ to 3sf

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2. (a) Express $\dfrac{1}{(r+6)(r+8)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+6)(r+8)} = \frac{n(an+b)}{56(n+7)(n+8)}$$

where a and b are integers to be found.

(4)

a) $\dfrac{1}{(r+6)(r+8)} = \dfrac{A}{r+6} + \dfrac{B}{r+8}$

$1 = A(r+8) + B(r+6)$

$r=-6: \quad 1 = 2A \quad \Rightarrow A = \frac{1}{2}$

$r=-8: \quad 1 = -2B \quad \Rightarrow B = \frac{-1}{2}$

$$\boxed{\frac{1}{2(r+6)} - \frac{1}{2(r+8)}}$$

b) $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+6)(r+8)} = 2\sum_{r=1}^{n} \frac{1}{(r+6)(r+8)}$

$\displaystyle = 2\sum_{r=1}^{n}\left(\frac{1}{2(r+6)} - \frac{1}{2(r+8)}\right)$

$\displaystyle = \sum_{r=1}^{n}\left(\frac{1}{r+6} - \frac{1}{r+8}\right)$

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Question 2 continued

$r=1: \quad \frac{1}{7} - \cancel{\frac{1}{9}}$

$r=2: \quad \frac{1}{8} - \cancel{\frac{1}{10}}$

$r=3: \quad \cancel{\frac{1}{9}} - \cancel{\frac{1}{11}}$

$\vdots$

$r=n-2: \quad \cancel{\frac{1}{n+4}} - \cancel{\frac{1}{n+6}}$

$r=n-1: \quad \cancel{\frac{1}{n+5}} - \frac{1}{n+7}$

$r=n: \quad \cancel{\frac{1}{n+6}} - \frac{1}{n+8}$

$$\text{so} \quad \sum_{r=1}^{n} \frac{2}{(r+6)(r+8)} = \frac{1}{7} + \frac{1}{8} - \frac{1}{n+7} - \frac{1}{n+8}$$

$$= \frac{15}{56} - \frac{1}{n+7} - \frac{1}{n+8}$$

$$= \frac{15(n+7)(n+8) - 56(n+8) - 56(n+7)}{56(n+7)(n+8)}$$

$$= \frac{15n^2 + 225n + 840 - 56n - 448 - 56n - 392}{56(n+7)(n+8)}$$

$$= \frac{15n^2 + 113n}{56(n+7)(n+8)}$$

$$= \boxed{\frac{n(15n+113)}{56(n+7)(n+8)}}$$

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3. (a) Express $\dfrac{1}{4r^2-1}$ in partial fractions. **(1)**

(b) Hence prove that

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$$ **(3)**

(c) Find the exact value of

$$\sum_{r=9}^{25} \frac{5}{4r^2-1}$$ **(2)**

a) $\dfrac{1}{4r^2-1} = \dfrac{1}{(2r-1)(2r+1)} = \dfrac{A}{2r-1} + \dfrac{B}{2r+1}$

$1 \equiv A(2r+1) + B(2r-1)$

$r = \frac{1}{2}$: $1 = 2A \Rightarrow A = \frac{1}{2}$

$r = -\frac{1}{2}$: $1 = -2B \Rightarrow B = -\frac{1}{2}$

$$\boxed{\frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}}$$

b) $\displaystyle\sum_{r=1}^{n} \frac{1}{4r^2-1} = \sum_{r=1}^{n}\left(\frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}\right)$

$r=1$: $\dfrac{1}{2\times1} - \cancel{\dfrac{1}{2\times3}}$

$r=2$: $\cancel{\dfrac{1}{2\times3}} - \cancel{\dfrac{1}{2\times5}}$

$r=3$: $\cancel{\dfrac{1}{2\times5}} - \cancel{\dfrac{1}{2\times7}}$

$r=n-1$: $\cancel{\dfrac{1}{2(2n-3)}} - \cancel{\dfrac{1}{2(2n-1)}}$

$r=n$: $\cancel{\dfrac{1}{2(2n-1)}} - \dfrac{1}{2(2n+1)}$

$= \dfrac{1}{2} - \dfrac{1}{2(2n+1)}$

$= \dfrac{2n+1-1}{2(2n+1)}$

$= \dfrac{2n}{2(2n+1)}$

$$\boxed{= \frac{n}{2n+1}}$$

Question 3 continued

c) $$\sum_{r=9}^{25} \frac{5}{4r^2-1} = 5\sum_{r=9}^{25} \frac{1}{4r^2-1}$$

$$= 5\left(\sum_{r=1}^{25} \frac{1}{4r^2-1} - \sum_{r=1}^{8} \frac{1}{4r^2-1}\right)$$

$$= 5\left(\frac{25}{2(25)+1} - \frac{8}{2(8)+1}\right)$$

$$= \boxed{\frac{5}{51}}$$

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4. (a) Express $\dfrac{1}{r(r+2)}$ in partial fractions.

(1)

(b) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}$.

(5)

a) $\dfrac{1}{r(r+2)} \equiv \dfrac{A}{r} + \dfrac{B}{r+2}$

$1 \equiv A(r+2) + Br$

$r=-2: \quad 1 = -2B \quad \Rightarrow B = -1/2$

$r=0: \quad 1 = 2A \quad \Rightarrow A = 1/2$

$$\boxed{\frac{1}{2r} - \frac{1}{2(r+2)}}$$

b) $\displaystyle\sum_{r=1}^{n} \frac{4}{r(r+2)} = 4\sum_{r=1}^{n} \frac{1}{r(r+2)}$

$\displaystyle = 4\sum_{r=1}^{n}\left(\frac{1}{2r} - \frac{1}{2(r+2)}\right)$

$r=1: \quad \frac{1}{2} - \cancel{\frac{1}{6}}$

$r=2: \quad \frac{1}{4} - \cancel{\frac{1}{8}}$

$r=3: \quad \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}$

$r=4: \quad \cancel{\frac{1}{8}} - \cancel{\frac{1}{12}}$

$r=n-2: \quad \cancel{\frac{1}{2n-4}} - \cancel{\frac{1}{2n}}$

$r=n-1: \quad \cancel{\frac{1}{2n-2}} - \frac{1}{2n+2}$

$r=n: \quad \cancel{\frac{1}{2n}} - \frac{1}{2n+4}$

$= 4\left(\frac{1}{2} + \frac{1}{4} - \frac{1}{2n+2} - \frac{1}{2n+4}\right)$

$= 4\left(\frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right)$

$= 4\left(\frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}\right)$

$= \cancel{4}\left(\frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{\cancel{4}(n+1)(n+2)}\right)$

$= \dfrac{3n^2 + 5n}{(n+1)(n+2)} = \dfrac{n(3n+5)}{(n+1)(n+2)}$

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5. (a) Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions. **(2)**

(b) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(an+b)}{6(n+2)(n+3)},$$

where a and b are constants to be found. **(6)**

(c) Find the value of $\displaystyle\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)}$, to 5 decimal places. **(3)**

a) $\dfrac{2}{(r+1)(r+3)} = \dfrac{A}{r+1} + \dfrac{B}{r+3}$

$2 = A(r+3) + B(r+1)$

$r=-3:\quad 2 = -2B \quad \Rightarrow B = -1$

$r=-1:\quad 2 = 2A \quad \Rightarrow A = 1$

$\boxed{\dfrac{1}{r+1} - \dfrac{1}{r+3}}$

b) $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{n} \left(\frac{1}{r+1} - \frac{1}{r+3}\right)$

$r=1:\quad \frac{1}{2} - \cancel{\frac{1}{4}}$

$r=2:\quad \frac{1}{3} - \cancel{\frac{1}{5}}$

$r=3:\quad \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$

$r=4:\quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$

$r=n-2:\quad \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n+1}}$

$r=n-1:\quad \cancel{\frac{1}{n}} - \frac{1}{n+2}$

$r=n:\quad \cancel{\frac{1}{n+1}} - \frac{1}{n+3}$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$

$= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$

$= \dfrac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)}$

$= \dfrac{5n^2 + 25n + 30 - 6n - 18 - 6n - 12}{6(n+2)(n+3)}$

$= \dfrac{5n^2 + 13n}{6(n+2)(n+3)} = \boxed{\dfrac{n(5n+13)}{6(n+2)(n+3)}}$

Question 5 continued

$\boxed{a=5,\ b=13}$

c) $\displaystyle\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{30} \frac{2}{(r+1)(r+3)} - \sum_{r=1}^{20} \frac{2}{(r+1)(r+3)}$

$$= \frac{30(5(30)+13)}{6(30+2)(30+1)} - \frac{20(5(20)+13)}{6(20+2)(20+1)}$$

$$= \frac{815}{992} - \frac{565}{693}$$

$$= \boxed{0.00628} \text{ to 5dp}$$

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6. Given that

$$\frac{2n+1}{n^2(n+1)^2} \equiv \frac{A}{n^2} + \frac{B}{(n+1)^2}$$

(a) determine the value of A and the value of B

(1)

(b) Hence show that, for $n \geqslant 5$

$$\sum_{r=5}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n^2+an+b}{c(n+1)^2}$$

where a, b and c are integers to be determined.

(4)

a) $2n+1 = A(n+1)^2 + Bn^2$

$n=-1: \quad -1 = B$

$n=0: \quad 1 = A$

b) $\displaystyle\sum_{r=5}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=5}^{n}\left(\frac{1}{r^2} - \frac{1}{(r+1)^2}\right)$

no need to split it up as $\sum_{r=1}^{n} - \sum_{r=1}^{4}$ since we can just start method at $r=5$.

$r=5: \quad \frac{1}{5^2} - \cancel{\frac{1}{6^2}}$

$r=6: \quad \cancel{\frac{1}{6^2}} - \cancel{\frac{1}{7^2}}$

$r=7: \quad \cancel{\frac{1}{7^2}} - \cancel{\frac{1}{8^2}}$

$\vdots$

$r=n-1: \quad \cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{n^2}}$

$r=n: \quad \cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}$

$= \frac{1}{5^2} - \frac{1}{(n+1)^2}$

$= \frac{(n+1)^2 - 25}{25(n+1)^2}$

$= \frac{n^2+2n+1-25}{25(n+1)^2}$

$= \boxed{\frac{n^2+2n-24}{25(n+1)^2}}$ $\quad a=2,\ b=-24,\ c=25$

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7. (a) Express $\dfrac{3}{(3r-1)(3r+2)}$ in partial fractions.
(2)

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}$$
(3)

(c) Evaluate $\displaystyle\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)}$, giving your answer to 3 significant figures.
(2)

a) $\dfrac{3}{(3r-1)(3r+2)} \equiv \dfrac{A}{3r-1} + \dfrac{B}{3r+2}$

$3 \equiv A(3r+2) + B(3r-1)$

$r = -2/3 :\quad 3 = -3B \quad \Rightarrow B = -1$

$r = 1/3 :\quad 3 = 3A \quad \Rightarrow A = 1$

$\boxed{\dfrac{1}{3r-1} - \dfrac{1}{3r+2}}$ Have to write it out explicitly.

b) $\displaystyle\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+1)} = \sum_{r=1}^{n}\left(\frac{1}{3r-1} - \frac{1}{3r+2}\right)$

$r=1 :\quad \frac{1}{2} - \cancel{\frac{1}{5}}$

$r=2 :\quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}$

$r=3 :\quad \cancel{\frac{1}{8}} - \cancel{\frac{1}{11}}$

$\vdots$

$r=n-1 :\quad \cancel{\frac{1}{3n-4}} - \cancel{\frac{1}{3n-1}}$

$r=n :\quad \cancel{\frac{1}{3n-1}} - \frac{1}{3n+2}$

$= \frac{1}{2} - \frac{1}{3n+2}$

$= \dfrac{3n+2-2}{2(3n+2)}$

$= \dfrac{3n}{2(3n+2)}$

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Question 7 continued

c) $$\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)} = \sum_{r=1}^{1000} \frac{3}{(3r-1)(3r+2)} - \sum_{r=1}^{99} \frac{3}{(3r-1)(3r+2)}$$

$$= \frac{3(1000)}{2(3(1000)+2)} - \frac{3(99)}{2(3(99)+2)}$$

$$= \frac{750}{1501} - \frac{297}{598}$$

$$= \boxed{0.00301} \quad \text{to 3sf}$$

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8. (a) Express $\frac{2}{(2r+1)(2r+3)}$ in partial fractions. **(2)**

(b) Using your answer to (a), find, in terms of n,

$$\sum_{r=1}^{n} \frac{3}{(2r+1)(2r+3)}$$

Give your answer as a single fraction in its simplest form. **(3)**

a) $\frac{2}{(2r+1)(2r+3)} = \frac{A}{2r+1} + \frac{B}{2r+3}$

$2 = A(2r+3) + B(2r+1)$

$r = -\frac{3}{2}$: $2 = -2B \Rightarrow B = -1$

$r = -\frac{1}{2}$: $2 = 2A \Rightarrow A = 1$

$$\boxed{\frac{1}{2r+1} - \frac{1}{2r+3}}$$

b) $\sum_{r=1}^{n} \frac{3}{(2r+1)(2r+3)} = \frac{3}{2}\sum_{r=1}^{n} \frac{2}{(2r+1)(2r+3)} = \frac{3}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$

$r=1$: $\frac{1}{3} - \cancel{\frac{1}{5}}$

$r=2$: $\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$

$r=3$: $\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}$

$\vdots$

$r=n-1$: $\cancel{\frac{1}{2n-1}} - \cancel{\frac{1}{2n+1}}$

$r=n$: $\cancel{\frac{1}{2n+1}} - \frac{1}{2n+3}$

$= \frac{3}{2}\left(\frac{1}{3} - \frac{1}{2n+3}\right)$

$= \frac{3}{2}\left(\frac{2n+3-3}{3(2n+3)}\right)$

$= \frac{\cancel{3}}{\cancel{2}}\left(\frac{\cancel{2}n}{\cancel{3}(2n+3)}\right)$

$= \boxed{\frac{n}{2n+3}}$

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9. (a) Express $\dfrac{2}{4r^2 - 1}$ in partial fractions.

(2)

(b) Hence use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2 - 1} = \frac{n}{2n+1}$$

(3)

a) $\dfrac{2}{4r^2-1} \equiv \dfrac{2}{(2r-1)(2r+1)} \equiv \dfrac{A}{2r-1} + \dfrac{B}{2r+1}$

$2 \equiv A(2r+1) + B(2r-1)$

$r = -\tfrac{1}{2}$: $2 = -2B \Rightarrow B = -1$

$r = \tfrac{1}{2}$: $2 = 2A \Rightarrow A = 1$

$\boxed{\dfrac{1}{2r-1} - \dfrac{1}{2r+1}}$

b) $\displaystyle\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{1}{2}\sum_{r=1}^{n} \frac{2}{4r^2-1} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$

$r=1$: $\dfrac{1}{1} - \cancel{\dfrac{1}{3}}$

$r=2$: $\cancel{\dfrac{1}{3}} - \cancel{\dfrac{1}{5}}$

$r=3$: $\cancel{\dfrac{1}{5}} - \cancel{\dfrac{1}{7}}$

$\vdots$

$r=n-1$: $\cancel{\dfrac{1}{2n-3}} - \cancel{\dfrac{1}{2n-1}}$

$r=n$: $\cancel{\dfrac{1}{2n-1}} - \dfrac{1}{2n+1}$

$= \dfrac{1}{2}\left(1 - \dfrac{1}{2n+1}\right)$

$= \dfrac{1}{2}\left(\dfrac{2n+1-1}{2n+1}\right)$

$= \dfrac{1}{2}\left(\dfrac{2n}{2n+1}\right)$

$= \boxed{\dfrac{n}{2n+1}}$

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10. (a) Express $\dfrac{2}{(r+2)(r+4)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}$$

(5)

$$\frac{2}{(r+2)(r+4)} \equiv \frac{A}{r+2} + \frac{B}{r+4}$$

$$2 \equiv A(r+4) + B(r+2)$$

$r=-4$: $\quad 2 = -2B \quad \Rightarrow B = -1$

$r=-2$: $\quad 2 = 2A \quad \Rightarrow A = 1$

$$\boxed{\frac{1}{r+2} - \frac{1}{r+4}}$$

b) $$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \sum_{r=1}^{n}\left(\frac{1}{r+2} - \frac{1}{r+4}\right)$$

$r=1$: $\quad \frac{1}{3} - \cancel{\frac{1}{5}}$

$r=2$: $\quad \frac{1}{4} - \cancel{\frac{1}{6}}$

$r=3$: $\quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$

$r=4$: $\quad \cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}$

$r=n-2$: $\quad \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+2}}$

$r=n-1$: $\quad \cancel{\frac{1}{n+1}} - \frac{1}{n+3}$

$r=n$: $\quad \cancel{\frac{1}{n+2}} - \frac{1}{n+4}$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{7}{12} - \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{7(n+3)(n+4) - 12(n+4) - 12(n+3)}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 49n + 84 - 12n - 48 - 12n + 36}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 25n}{12(n+3)(n+4)}$$

$$= \frac{n(7n+25)}{12(n+3)(n+4)}$$

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11. (a) Express $\dfrac{1}{(r+3)(r+4)}$ in partial fractions. **(1)**

(b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \frac{1}{(r+3)(r+4)} = \frac{n}{a(n+a)}$$

where a is a constant to be found. **(5)**

(c) Find the exact value of $\displaystyle\sum_{r=15}^{30} \frac{1}{(r+3)(r+4)}$ **(2)**

a) $\dfrac{1}{(r+3)(r+4)} \equiv \dfrac{A}{r+3} + \dfrac{B}{r+4}$

$1 \equiv A(r+4) + B(r+3)$

$r=-4: \quad 1 = -B \quad \Rightarrow B=-1$

$r=-3: \quad 1 = A$

$\boxed{\dfrac{1}{r+3} - \dfrac{1}{r+4}}$

need to write this out explicitly.

b) $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+3)(r+4)} = \sum_{r=1}^{n}\left(\frac{1}{r+3} - \frac{1}{r+4}\right)$

$r=1: \quad \dfrac{1}{4} - \cancel{\dfrac{1}{5}} \qquad = \dfrac{1}{4} - \dfrac{1}{n+4}$

$r=2: \quad \cancel{\dfrac{1}{5}} - \cancel{\dfrac{1}{6}}$

$= \dfrac{n+4-4}{4(n+4)} = \boxed{\dfrac{n}{4(n+4)}}$

$r=3: \quad \cancel{\dfrac{1}{6}} - \cancel{\dfrac{1}{7}}$

$\boxed{a=4}$

$\vdots$

$r=n-1: \quad \cancel{\dfrac{1}{n+2}} - \cancel{\dfrac{1}{n+3}}$

$r=n: \quad \cancel{\dfrac{1}{n+3}} - \dfrac{1}{n+4}$

Question 11 continued

c) $\displaystyle\sum_{r=15}^{30} \frac{1}{(r+3)(r+4)} = \sum_{r=1}^{30} \frac{1}{(r+3)(r+4)} - \sum_{r=1}^{14} \frac{1}{(r+3)(r+4)}$

$$= \frac{30}{4(30+4)} - \frac{14}{4(14+4)}$$

$$= \frac{15}{68} - \frac{7}{36}$$

$$= \boxed{\frac{4}{153}}$$

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12. (a) Express $\dfrac{5r+4}{r(r+1)(r+2)}$ in partial fractions. **(4)**

(b) Hence, or otherwise, show that

$$\sum_{r=1}^{n} \frac{5r+4}{r(r+1)(r+2)} = \frac{7n^2+11n}{2(n+1)(n+2)}.$$

(5)

a) $\dfrac{5r+4}{r(r+1)(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2}$

$5r+4 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$

$r=0: \quad 4 = 2A \Rightarrow A = 2$

$r=-1: \quad -1 = -B \Rightarrow B = 1$

$r=-2: \quad -6 = 2C \Rightarrow C = -3$

$$\boxed{\frac{2}{r} + \frac{1}{r+1} - \frac{3}{r+2}}$$

b) $\displaystyle\sum_{r=1}^{n} \frac{5r+4}{r(r+1)(r+2)} = \sum_{r=1}^{n}\left(\frac{2}{r} + \frac{1}{r+1} - \frac{3}{r+2}\right)$

$r=1: \quad \frac{2}{1} + \frac{1}{2} - \cancel{\frac{3}{3}}$

$r=2: \quad \frac{2}{2} + \cancel{\frac{1}{3}} - \cancel{\frac{3}{4}}$

$r=3: \quad \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}} - \cancel{\frac{3}{5}}$

$r=4: \quad \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}} - \cancel{\frac{3}{6}}$

P.T.O

Question 12 continued

$r = n-2$: $\cancel{\frac{2}{n-2}} + \cancel{\frac{1}{n-1}} - \cancel{\frac{3}{n}}$

$r = n-1$: $\cancel{\frac{2}{n-1}} + \cancel{\frac{1}{n}} - \frac{3}{n+1}$

$r = n$: $\cancel{\frac{2}{n}} + \frac{1}{n+1} - \frac{3}{n+2}$

So $\sum_{r=1}^{n} \frac{5r+4}{r(r+1)(r+2)} = \frac{2}{1} + \frac{1}{2} + \frac{2}{2} - \frac{3}{n+1} + \frac{1}{n+1} - \frac{3}{n+2}$

$= \frac{7}{2} - \frac{2}{n+1} - \frac{3}{n+2}$

$= \frac{7(n+1)(n+2) - 4(n+2) - 6(n+1)}{2(n+1)(n+2)}$

$= \frac{7n^2 + 21n + 14 - 4n - 8 - 6n - 6}{2(n+1)(n+2)}$

$= \frac{7n^2 + 11n}{2(n+1)(n+2)}$

Leave blank

13. (a) Show that, for $r > 0$

$$r - 3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{r^3 - 7r - 5}{(r+1)(r+2)}$$

(2)

(b) Hence prove, using the method of differences, that

$$\sum_{r=1}^{n} \frac{r^3 - 7r - 5}{(r+1)(r+2)} = \frac{n(n^2 + an + b)}{2(n+2)}$$

where a and b are constants to be found.

(5)

a) $r - 3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{(r-3)(r+1)(r+2) + r+2 - (r+1)}{(r+1)(r+2)}$

$= \frac{(r^2 - 2r - 3)(r+2) + r + 2 - r - 1}{(r+1)(r+2)}$

$= \frac{r^3 - 7r - 6 + r + 2 - r - 1}{(r+1)(r+2)}$

$= \frac{r^3 - 7r - 5}{(r+1)(r+2)}$

b) $\sum_{r=1}^{n} \frac{r^3 - 7r - 5}{(r+1)(r+2)} = \sum_{r=1}^{n}\left(r - 3 + \frac{1}{r+1} - \frac{1}{r+2}\right)$

$= \sum_{r=1}^{n} r - 3\sum_{r=1}^{n} 1 + \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)$

$= \underbrace{\frac{1}{2}n(n+1)} - 3n + \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)$

see "Series - Standard formulae" if you're not sure about this

Question 13 continued

$$\frac{1}{2}n(n+1) - 3n + \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)$$

$r=1: \quad \frac{1}{2} - \cancel{\frac{1}{3}}$

$r=2: \quad \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$r=3: \quad \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$\quad / \quad /$

$r=n-1: \quad \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}$

$r=n: \quad \cancel{\frac{1}{n+1}} - \frac{1}{n+2}$

$$= \frac{1}{2}n(n+1) - 3n + \frac{1}{2} - \frac{1}{n+2}$$

$$= \frac{n}{2}\left(n+1-6\right) + \frac{n+2-2}{2(n+2)}$$

$$= \frac{n}{2}(n-5) + \frac{n}{2(n+2)}$$

$$= \frac{n}{2(n+2)}\left((n-5)(n+2) + 1\right)$$

$$= \frac{n}{2(n+2)}\left(n^2 - 3n - 10 + 1\right)$$

$$= \frac{n}{2(n+2)}\left(n^2 - 3n - 9\right)$$

$$\boxed{= \frac{n(n^2-3n-9)}{2(n+2)} \qquad a=-3,\ b=-9}$$

14. Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{2}{(r+4)(r+6)} = \frac{n(an+b)}{30(n+5)(n+6)}$$

where a and b are integers to be determined.

(6)

Need partial fractions:

$$\frac{2}{(r+4)(r+6)} = \frac{A}{r+4} + \frac{B}{r+6}$$

$$2 = A(r+6) + B(r+4)$$

$r=-6$: $\quad 2 = -2B \Rightarrow B = -1$

$r=-4$: $\quad 2 = 2A \Rightarrow A = 1$

$$\sum_{r=1}^{n} \frac{2}{(r+4)(r+6)} = \sum_{r=1}^{n} \left(\frac{1}{r+4} - \frac{1}{r+6} \right)$$

$r=1$: $\frac{1}{5} - \cancel{\frac{1}{7}}$

$r=2$: $\frac{1}{6} - \cancel{\frac{1}{8}}$

$r=3$: $\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}$

$r=4$: $\cancel{\frac{1}{8}} - \cancel{\frac{1}{10}}$

$r=n-2$: $\cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+4}}$

$r=n-1$: $\cancel{\frac{1}{n+3}} - \frac{1}{n+5}$

$r=n$: $\cancel{\frac{1}{n+4}} - \frac{1}{n+6}$

$$= \frac{1}{5} + \frac{1}{6} - \frac{1}{n+5} - \frac{1}{n+6}$$

$$= \frac{11}{30} - \frac{1}{n+5} - \frac{1}{n+6}$$

$$= \frac{11(n+5)(n+6) - 30(n+6) - 30(n+5)}{30(n+5)(n+6)}$$

$$= \frac{11n^2 + 121n + 330 - 30n - 180 - 30n - 150}{30(n+5)(n+6)}$$

$$= \frac{11n^2 + 61n}{30(n+5)(n+6)}$$

$$= \boxed{\frac{n(11n+61)}{30(n+5)(n+6)} \qquad a=11,\ b=61}$$

Leave blank

15. (a) Express $\dfrac{1}{r(r+2)}$ in partial fractions.

(2)

(b) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(an+b)}{4(n+1)(n+2)}$$

where a and b are constants to be found.

(6)

(c) Hence show that

$$\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \frac{n(4n+5)}{4(n+1)(n+2)(2n+1)}$$

(3)

a) $\dfrac{1}{r(r+2)} \equiv \dfrac{A}{r} + \dfrac{B}{r+2}$

$1 = A(r+2) + Br$

$r=-2: \quad 1 = -2B \quad \Rightarrow B = -1/2$

$r=0: \quad 1 = 2A \quad \Rightarrow A = 1/2$

$\dfrac{1}{r(r+2)} \equiv \boxed{\dfrac{1}{2r} - \dfrac{1}{2r+4}}$

b) $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)} = \sum_{r=1}^{n}\left(\frac{1}{2r} - \frac{1}{2r+4}\right)$

$r=1: \quad \dfrac{1}{2} - \cancel{\dfrac{1}{6}}$

$r=2: \quad \dfrac{1}{4} - \cancel{\dfrac{1}{8}}$

$r=3: \quad \cancel{\dfrac{1}{6}} - \cancel{\dfrac{1}{10}}$

$r=4: \quad \cancel{\dfrac{1}{8}} - \cancel{\dfrac{1}{12}}$

$= \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{2(n-1)+4} - \dfrac{1}{2n+4}$

$= \dfrac{3}{4} - \dfrac{1}{2n+2} - \dfrac{1}{2n+4}$

$= \dfrac{3}{4} - \dfrac{1}{2(n+1)} - \dfrac{1}{2(n+2)}$

Question 15 continued

$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$$

$$= \boxed{\frac{n(3n+5)}{4(n+1)(n+2)} \quad ; \quad a = 3, b = 5}$$

c) $$\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \sum_{r=1}^{2n} \frac{1}{r(r+2)} - \sum_{r=1}^{n} \frac{1}{r(r+2)}$$

$$= \frac{2n(6n+5)}{4(2n+1)(2n+2)} - \frac{n(3n+5)}{4(n+1)(n+2)}$$

$$= \frac{{}^{1}\cancel{2}n(6n+5)}{4\cancel{8}(2n+1)(n+1)} - \frac{n(3n+5)}{4(n+1)(n+2)}$$

$$= \frac{n(6n+5)(n+2) - n(3n+5)(2n+1)}{4(n+1)(n+2)(2n+1)}$$

$$= \frac{n(6n^2 + 17n + 10 - 6n^2 - 13n - 5)}{4(n+1)(n+2)(2n+1)}$$

$$= \frac{n(4n + 5)}{4(n+1)(n+2)(2n+1)}$$

Note, you can also just do a method of differences with $r = n+1$ to $r = 2n$.

Leave blank

16. (a) Show that

$$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{1}{2(r+1)(r+2)} - \frac{1}{2(r+2)(r+3)}$$

(2)

(b) Hence, or otherwise, find

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)}$$

giving your answer as a single fraction in its simplest form.

(4)

a) $RHS = \dfrac{1}{2(r+1)(r+2)} - \dfrac{1}{2(r+2)(r+3)}$

$= \dfrac{r+3-(r+1)}{2(r+1)(r+2)(r+3)} = \dfrac{2}{2(r+1)(r+2)(r+3)}$

$= \dfrac{1}{(r+1)(r+2)(r+3)} = LHS.$

b) $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \sum_{r=1}^{n}\left(\frac{1}{2(r+1)(r+2)} - \frac{1}{2(r+2)(r+3)}\right)$

$r=1: \quad \dfrac{1}{2\times2\times3} - \cancel{\dfrac{1}{2\times3\times4}}$

$r=2: \quad \cancel{\dfrac{1}{2\times3\times4}} - \cancel{\dfrac{1}{2\times4\times5}}$

$r=3: \quad \cancel{\dfrac{1}{2\times4\times5}} - \cancel{\dfrac{1}{2\times5\times6}}$

$\vdots$

$r=n-1: \quad \cancel{\dfrac{1}{2n(n+1)}} - \cancel{\dfrac{1}{2(n+1)(n+2)}}$

$r=n: \quad \cancel{\dfrac{1}{2(n+1)(n+2)}} - \dfrac{1}{2(n+2)(n+3)}$

$= \dfrac{1}{2\times2\times3} - \dfrac{1}{2(n+2)(n+3)}$

$= \dfrac{1}{12} - \dfrac{1}{2(n+2)(n+3)}$

$= \dfrac{(n+2)(n+3) - 6}{12(n+2)(n+3)}$

$= \dfrac{n^2+5n}{12(n+2)(n+3)}$

$= \boxed{\dfrac{n(n+5)}{12(n+2)(n+3)}}$

Leave blank

17. (a) Show that, for $r > 0$

$$\frac{r+2}{r(r+1)} - \frac{r+3}{(r+1)(r+2)} = \frac{r+4}{r(r+1)(r+2)}$$

(2)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \frac{n(an+b)}{c(n+1)(n+2)}$$

where a, b and c are integers to be determined.

(4)

a) $\frac{r+2}{r(r+1)} - \frac{r+3}{(r+1)(r+2)} = \frac{(r+2)^2 - r(r+3)}{r(r+1)(r+2)}$

$= \frac{r^2+4r+4-r^2-3r}{r(r+1)(r+2)}$

$= \frac{r+4}{r(r+1)(r+2)}$

b) $\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \sum_{r=1}^{n}\left(\frac{r+2}{r(r+1)} - \frac{r+3}{(r+1)(r+2)}\right)$

$r=1: \quad \frac{3}{1\times 2} - \cancel{\frac{4}{2\times 3}}$

$r=2: \quad \cancel{\frac{4}{2\times 3}} - \cancel{\frac{5}{3\times 4}}$

$r=3: \quad \cancel{\frac{5}{3\times 4}} - \cancel{\frac{6}{4\times 5}}$

$\vdots$

$r=n-1: \quad \cancel{\frac{n+1}{(n-1)n}} - \cancel{\frac{n+2}{n(n+1)}}$

$r=n: \quad \cancel{\frac{n+2}{n(n+1)}} - \frac{n+3}{(n+1)(n+2)}$

$= \frac{3}{2} - \frac{n+3}{(n+1)(n+2)}$

$= \frac{3(n+1)(n+2) - 2(n+3)}{2(n+1)(n+2)}$

$= \frac{3n^2+9n+6-2n-6}{2(n+1)(n+2)}$

$= \frac{3n^2+7n}{2(n+1)(n+2)}$

$= \boxed{\frac{n(3n+7)}{2(n+1)(n+2)} \quad a=3,\ b=7,\ c=2}$

18. (a) Express

$$\frac{1}{(2n-1)(2n+1)(2n+3)}$$

in partial fractions.

(2)

(b) Hence, using the method of differences, show that for all integer values of n,

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)(2r+3)} = \frac{n(n+2)}{a(2n+b)(2n+c)}$$

where a, b and c are integers to be determined.

(4)

a) $\frac{1}{(2n-1)(2n+1)(2n+3)} \equiv \frac{A}{2n-1} + \frac{B}{2n+1} + \frac{C}{2n+3}$

$1 \equiv A(2n+1)(2n+3) + B(2n-1)(2n+3) + C(2n-1)(2n+1)$

$n = \frac{1}{2}$: $1 = 8A \Rightarrow A = \frac{1}{8}$

$n = -\frac{1}{2}$: $1 = -4B \Rightarrow B = -\frac{1}{4}$

$n = -\frac{3}{2}$: $1 = 8C \Rightarrow C = \frac{1}{8}$

$\frac{1}{8(2n-1)} - \frac{1}{4(2n+1)} + \frac{1}{8(2n+3)}$

b) $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)(2r+3)} = \sum_{r=1}^{n}\left(\frac{1}{8(2r-1)} - \frac{1}{4(2r+1)} + \frac{1}{8(2r+3)}\right)$

P.T.O.

Question 18 continued

$r=1: \quad \frac{1}{8\times 1} - \frac{1}{4\times 3} + \cancel{\frac{1}{8\times 5}}$

$r=2: \quad \frac{1}{8\times 3} - \cancel{\frac{1}{4\times 5}} + \cancel{\frac{1}{8\times 7}}$

$r=3: \quad \cancel{\frac{1}{8\times 5}} - \cancel{\frac{1}{4\times 7}} + \cancel{\frac{1}{8\times 9}}$

$r=4: \quad \cancel{\frac{1}{8\times 7}} - \cancel{\frac{1}{4\times 9}} + \cancel{\frac{1}{8\times 11}}$

$r=n-2: \quad \cancel{\frac{1}{8(2n-5)}} - \cancel{\frac{1}{4(2n-3)}} + \cancel{\frac{1}{8(2n-1)}}$

$r=n-1: \quad \cancel{\frac{1}{8(2n-3)}} - \cancel{\frac{1}{4(2n-1)}} + \frac{1}{8(2n+1)}$

$r=n: \quad \cancel{\frac{1}{8(2n-1)}} - \frac{1}{4(2n+1)} + \frac{1}{8(2n+3)}$

so sum $= \frac{1}{8} - \frac{1}{12} + \frac{1}{24} + \frac{1}{8(2n+1)} - \frac{1}{4(2n+1)} + \frac{1}{8(2n+3)}$

$= \frac{1}{12} - \frac{1}{8(2n+1)} + \frac{1}{8(2n+3)}$

$= \frac{2(2n+1)(2n+3) - 3(2n+3) + 3(2n+1)}{24(2n+1)(2n+3)}$

$= \frac{8n^2 + 16n + 6 - 6n - 9 + 6n + 3}{24(2n+1)(2n+3)}$

$= \frac{8n^2 + 16n}{24(2n+1)(2n+3)} = \frac{8n(n+2)}{24(2n+1)(2n+3)} = \boxed{\frac{n(n+2)}{3(2n+1)(2n+3)}}$

19. (a) Express

$$\frac{1}{(n+3)(n+5)}$$

in partial fractions.

(2)

(b) Hence, using the method of differences, show that for all positive integer values of n,

$$\sum_{r=1}^{n} \frac{1}{(r+3)(r+5)} = \frac{n(pn+q)}{40(n+4)(n+5)}$$

where p and q are integers to be determined.

(4)

(c) Use the answer to part (b) to determine, as a simplified fraction, the value of

$$\frac{1}{9\times 11} + \frac{1}{10\times 12} + \ldots + \frac{1}{24\times 26}$$

(2)

a) $\dfrac{1}{(n+3)(n+5)} = \dfrac{A}{n+3} + \dfrac{B}{n+5}$

$1 = A(n+5) + B(n+3)$

$n = -3:\quad 1 = 2A \Rightarrow A = \frac{1}{2}$

$n = -5:\quad 1 = -2B \Rightarrow B = -\frac{1}{2}$

$\dfrac{1}{(n+3)(n+5)} = \boxed{\dfrac{1}{2(n+3)} - \dfrac{1}{2(n+5)}}$

Need to write this out explicitly.

b) $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+3)(r+5)} = \sum_{r=1}^{n} \left(\frac{1}{2(r+3)} - \frac{1}{2(r+5)} \right)$

Question 19 continued

$r=1: \quad \dfrac{1}{2\times 4} - \cancel{\dfrac{1}{2\times 6}}$

$r=2: \quad \dfrac{1}{2\times 5} - \cancel{\dfrac{1}{2\times 7}}$

$r=3: \quad \cancel{\dfrac{1}{2\times 6}} - \cancel{\dfrac{1}{2\times 8}}$

$r=4: \quad \cancel{\dfrac{1}{2\times 7}} - \cancel{\dfrac{1}{2\times 9}}$

$\vdots \qquad \vdots \qquad \vdots$

$r=n-2: \quad \cancel{\dfrac{1}{2(n+1)}} - \cancel{\dfrac{1}{2(n+3)}}$

$r=n-1: \quad \cancel{\dfrac{1}{2(n+2)}} - \dfrac{1}{2(n+4)}$

$r=n: \quad \cancel{\dfrac{1}{2(n+3)}} - \dfrac{1}{2(n+5)}$

$$\sum_{r=1}^{n} \frac{1}{(r+3)(r+5)}$$

$$= \frac{1}{8} + \frac{1}{10} - \frac{1}{2(n+4)} - \frac{1}{2(n+5)}$$

$$= \frac{9}{40} - \frac{1}{2(n+4)} - \frac{1}{2(n+5)}$$

$$= \frac{9(n+4)(n+5) - 20(n+5) - 20(n+4)}{40(n+4)(n+5)}$$

$$= \frac{9n^2 + 81n + 180 - 20n - 100 - 20n - 80}{40(n+4)(n+5)}$$

$$= \frac{9n^2 + 41n}{40(n+4)(n+5)}$$

$$= \boxed{\frac{n(9n+41)}{40(n+4)(n+5)} \qquad p = 9,\ q = 41}$$

c) $\dfrac{1}{9\times 11} + \dfrac{1}{10\times 12} + \dots + \dfrac{1}{24\times 26}$

$$= \sum_{r=6}^{21} \frac{1}{(r+3)(r+5)} \qquad \begin{array}{ll} r+3 = 9 & \Rightarrow r = 6 \\ r+3 = 24 & \Rightarrow r = 21 \end{array}$$

$$= \sum_{r=1}^{21} \frac{1}{(r+3)(r+5)} - \sum_{r=1}^{5} \frac{1}{(r+3)(r+5)}$$

$$= \frac{21(9(21)+41)}{40(21+4)(21+5)} - \frac{5(9(5)+41)}{40(5+4)(5+5)}$$

$$= \frac{483}{2600} - \frac{43}{360} = \boxed{\frac{194}{2925}}$$

Leave blank

20. (a) Express $\dfrac{4r+2}{r(r+1)(r+2)}$ in partial fractions.

(3)

(b) Hence, using the method of differences, prove that

$$\sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} = \frac{n(an+b)}{2(n+1)(n+2)}$$

where a and b are constants to be found.

(5)

a) $\dfrac{4r+2}{r(r+1)(r+2)} \equiv \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2}$

$4r+2 \equiv A(r+1)(r+2) + Br(r+2) + Cr(r+1)$

$r=0: \quad 2 = 2A \quad \Rightarrow A = 1$

$r=-1: \quad -2 = -B \quad \Rightarrow B = 2$

$r=-2: \quad -6 = 2C \quad \Rightarrow C = -3$

$$\boxed{\frac{1}{r} + \frac{2}{r+1} - \frac{3}{r+2}}$$

b) $\displaystyle\sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} = \sum_{r=1}^{n}\left(\frac{1}{r} + \frac{2}{r+1} - \frac{3}{r+2}\right)$

$r=1: \quad \dfrac{1}{1} + \dfrac{2}{2} - \cancel{\dfrac{3}{3}}$

$r=2: \quad \dfrac{1}{2} + \cancel{\dfrac{2}{3}} - \cancel{\dfrac{3}{4}}$

$r=3: \quad \cancel{\dfrac{1}{3}} + \cancel{\dfrac{2}{4}} - \cancel{\dfrac{3}{5}}$

$r=4: \quad \cancel{\dfrac{1}{4}} + \cancel{\dfrac{2}{5}} - \cancel{\dfrac{3}{6}}$

Question 20 continued

$r = n-2: \quad \cancel{\frac{1}{n-2}} + \cancel{\frac{2}{n-1}} - \cancel{\frac{3}{n}}$

$r = n-1: \quad \cancel{\frac{1}{n-1}} + \cancel{\frac{2}{n}} - \frac{3}{n+1}$

$r = n: \quad \cancel{\frac{1}{n}} + \frac{2}{n+1} - \frac{3}{n+2}$

$$\text{So } \sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} = \frac{1}{1} + \frac{2}{2} + \frac{1}{2} - \frac{3}{n+1} + \frac{2}{n+1} - \frac{3}{n+2}$$

$$= \frac{5}{2} - \frac{1}{n+1} - \frac{3}{n+2}$$

$$= \frac{5(n+1)(n+2) - 2(n+2) - 6(n+1)}{2(n+1)(n+2)}$$

$$= \frac{5n^2 + 15n + 10 - 2n - 4 - 6n - 6}{2(n+1)(n+2)}$$

$$= \frac{5n^2 + 7n}{2(n+1)(n+2)}$$

$$= \boxed{\frac{n(5n+7)}{2(n+1)(n+2)}}$$

***

21. Prove that, for $n \in \mathbb{Z}$, $n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found.

(5)

$$\frac{1}{(r+1)(r+2)(r+3)} = \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}$$

$$1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)$$

$r=-1:\quad 1 = 2A \quad \Rightarrow A = \frac{1}{2}$

$r=-2:\quad 1 = -B \quad B = -1$

$r=-3:\quad 1 = 2C \quad \Rightarrow C = \frac{1}{2}$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \sum_{r=0}^{n} \left(\frac{1}{2(r+1)} - \frac{1}{r+2} + \frac{1}{2(r+3)} \right)$$

$r=0:\quad \frac{1}{2\times 1} - \frac{1}{2} + \cancel{\frac{1}{2\times 3}}$

$r=1:\quad \frac{1}{2\times 2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{2\times 4}}$

$r=2:\quad \cancel{\frac{1}{2\times 3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{2\times 5}}$

$r=3:\quad \cancel{\frac{1}{2\times 4}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{2\times 6}}$

$r=n-2:\quad \cancel{\frac{1}{2(n-1)}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{2(n+1)}}$

$r=n-1:\quad \cancel{\frac{1}{2n}} - \cancel{\frac{1}{n+1}} + \frac{1}{2(n+2)}$

$r=n:\quad \cancel{\frac{1}{2(n+1)}} - \frac{1}{n+2} + \frac{1}{2(n+3)}$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2(n+2)} - \frac{1}{n+2} + \frac{1}{2(n+3)}$$

$$= \frac{1}{4} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)}$$

$$= \frac{(n+2)(n+3) - 2(n+3) + 2(n+2)}{4(n+2)(n+3)}$$

$$= \frac{n^2+5n+6-2n-6+2n+4}{4(n+2)(n+3)}$$

$$= \frac{n^2+5n+4}{4(n+2)(n+3)}$$

$$= \frac{(n+1)(n+4)}{4(n+2)(n+3)}$$

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22 (a) Show that

$$r^2(r+1)^2 - (r-1)^2\, r^2 \equiv 4r^3$$

(3)

Given that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$(1^3 + 2^3 + 3^3 + \ldots + n^3) = (1 + 2 + 3 + \ldots + n)^2$$

(4)

a) $r^2(r+1)^2 - (r-1)^2 r^2$

$= r^2\left((r+1)^2 - (r-1)^2\right)$

$= r^2\,(r+1+r-1)(r+1-r+1)$ ← $a^2-b^2 = (a+b)(a-b)$

$= r^2(2r)2$

$= 4r^3$

You could also just expand and simplify.

b) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \sum_{r=1}^{n} r^3$

$= \frac{1}{4}\sum_{r=1}^{n} 4r^3$

$= \frac{1}{4}\sum_{r=1}^{n}\left(r^2(r+1)^2 - (r-1)^2 r^2\right)$

$r=1:\quad 1^2\times\cancel{2^2} - 0^2\times 1^2$ P.T.O

$r=2:\quad 2^2\times\cancel{3^2} - \cancel{1^2\times 2^2}$

$r=3:\quad 3^2\times\cancel{4^2} - \cancel{2^2\times 3^2}$

$r=n-1:\quad \cancel{(n-1)^2\times n^2} - \cancel{(n-2)^2\times(n-1)}$

$r=n:\quad n^2\times(n+1)^2 - \cancel{(n-1)^2\times n^2}$

Question 22 continued

$\frac{1}{4}\sum_{r=1}^{n}\left(r^2(r+1)^2-(r-1)^2r^2\right)$

$=\frac{1}{4}\left(n^2(n+1)^2-1^2\times 0^2\right)$

$=\frac{1}{4}\left(n^2(n+1)^2\right)$

$=\left(\frac{1}{2}n(n+1)\right)^2$

$=\left(\sum_{r=1}^{n}r\right)^2$

$=\left(1+2+3+\ldots+n\right)^2$ $\square$

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23. (a) Show that, for $r > 0$

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}$$

(1)

(b) Hence prove that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

(3)

(c) Show that, for $n \in \mathbb{N}$, $n > 1$

$$\sum_{r=n}^{3n} \frac{6r+3}{r^2(r+1)^2} = \frac{an^2 + bn + c}{n^2(3n+1)^2}$$

where a, b and c are constants to be found.

(3)

a) $\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \frac{r^2+2r+1-r^2}{r^2(r+1)^2}$

$= \frac{2r+1}{r^2(r+1)^2}$

b) $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n} \frac{1}{r^2} - \frac{1}{(r+1)^2}$

$r=1: \quad \frac{1}{1^2} - \cancel{\frac{1}{2^2}}$

$r=2: \quad \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}$

$r=3: \quad \cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}$

$r=n-1: \quad \cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{n^2}}$

$r=n: \quad \cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}$

$= \frac{1}{1^2} - \frac{1}{(n+1)^2}$

$= \frac{(n+1)^2 - 1}{(n+1)^2}$

$= \frac{n^2+2n+1-1}{(n+1)^2}$

$= \frac{n^2+2n}{(n+1)^2}$

$= \frac{n(n+2)}{(n+1)^2}$

***

Question 23 continued

c) usually I would write it out like this and use my answer from part b), but this will actually lead to a pretty lengthy expansion:

$$\sum_{r=n}^{3n} \frac{6r+3}{r^2(r+1)^2} = 3\left(\sum_{r=1}^{3n} \frac{2r+1}{r^2(r+1)^2} - \sum_{r=1}^{n-1} \frac{2r+1}{r^2(r+1)^2}\right)$$

Instead, just appy method of differences again:

$$3\sum_{r=n}^{3n} \frac{2r+1}{r^2(r+1)^2} = 3\sum_{r=n}^{3n} \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$r=n$: $\frac{1}{n^2} - \cancel{\frac{1}{(n+1)^2}}$

$r=n+1$: $\cancel{\frac{1}{(n+1)^2}} - \cancel{\frac{1}{(n+2)^2}}$

$r=n+2$: $\cancel{\frac{1}{(n+2)^2}} - \cancel{\frac{1}{(n+3)^2}}$

$r=3n-1$ $\cancel{\frac{1}{(3n-1)^2}} - \cancel{\frac{1}{(3n)^2}}$

$r=3n$ $\cancel{\frac{1}{(3n)^2}} - \frac{1}{(3n+1)^2}$

$$= 3\left(\frac{1}{n^2} - \frac{1}{(3n+1)^2}\right)$$

$$= 3\left(\frac{(3n+1)^2 - n^2}{n^2(3n+1)^2}\right)$$

$$= \frac{3(9n^2+6n+1-n^2)}{n^2(3n+1)^2}$$

$$= \boxed{\frac{24n^2+18n+3}{n^2(3n+1)^2}}$$

$\boxed{a=24,\ b=18,\ c=3}$

Leave blank

24. (a) Show that

$$(r+1)^3 - (r-1)^3 \equiv 6r^2 + 2.$$

(2)

(b) Hence show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$

(5)

(c) Show that $\sum_{r=n}^{2n} r^2 = \frac{1}{6}n(n+1)(an+b)$, where a and b are constants to be found.

(4)

a) $(r+1)^3 - (r-1)^3 \equiv r^3 + 3r^2 + 3r + 1 - (r^3 - 3r^2 + 3r - 1)$

$\equiv 6r^2 + 2$

b) $\sum_{r=1}^{n} (6r^2 + 2) = \sum_{r=1}^{n} \left((r+1)^3 - (r-1)^3\right)$

$r=1: \quad \cancel{2^3} \quad -0^3$

$r=2: \quad \cancel{3^3} \quad -1^3$

$r=3: \quad \cancel{4^3} \quad -\cancel{2^3}$

$r=4: \quad \cancel{5^3} \quad -\cancel{3^3}$

$\vdots$

$r=n-2: \quad \cancel{(n-1)^3} \quad -\cancel{(n-3)^3}$

$r=n-1: \quad n^3 \quad -\cancel{(n-2)^3}$

$r=n: \quad (n+1)^3 \quad -\cancel{(n-1)^3}$

$6\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2 = -0^3 - 1^3 + n^3 + (n+1)^3$

$6\sum_{r=1}^{n} r^2 + 2n = n^3 + n^3 + 3n^2 + 3n + 1 - 1$

$6\sum_{r=1}^{n} r^2 = 2n^3 + 3n^2 + n$

$\sum_{r=1}^{n} r^2 = \frac{1}{6} n (2n^2 + 3n + 1)$

$= \frac{1}{6} n (n+1)(2n+1)$

***

Question 24 continued

c) $\displaystyle\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2$

$= \frac{1}{6} 2n(2n+1)(4n+1) - \frac{1}{6}(n-1)(n-1+1)(2(n-1)+1)$

$= \frac{1}{6} 2n(2n+1)(4n+1) - \frac{1}{6}(n-1)n(2n-1)$

$= \frac{1}{6} n\left(2(2n+1)(4n+1) - (n-1)(2n-1)\right)$

$= \frac{1}{6} n\left(16n^2 + 12n + 2 - 2n^2 + 3n - 1\right)$

$= \frac{1}{6} n\left(14n^2 + 15n + 1\right)$

$= \boxed{\frac{1}{6} n(n+1)(14n+1) \quad a = 14,\ b = 1}$

25. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Show that, for $r \geqslant 2$

$$\frac{2}{\sqrt{r}+\sqrt{r-2}} = \sqrt{r} - \sqrt{r-2}$$

(2)

(b) Hence use the method of differences to determine

$$\sum_{r=2}^{n} \frac{2}{\sqrt{r}+\sqrt{r-2}}$$

giving your answer in simplest form.

(3)

(c) Hence show that

$$\sum_{r=4}^{50} \frac{2}{\sqrt{r}+\sqrt{r-2}} = A + B\sqrt{2} + C\sqrt{3}$$

where A, B and C are integers to be determined.

(2)

a) $$\frac{2}{\sqrt{r}+\sqrt{r-2}} = \frac{2}{\sqrt{r}+\sqrt{r-2}} \times \frac{\sqrt{r}-\sqrt{r-2}}{\sqrt{r}-\sqrt{r-2}}$$

$$= \frac{2(\sqrt{r}-\sqrt{r-2})}{r-(r-2)}$$

$$= \frac{\cancel{2}(\sqrt{r}-\sqrt{r-2})}{\cancel{2}} = \sqrt{r} - \sqrt{r-2}$$

b) $$\sum_{r=2}^{n} \frac{2}{\sqrt{r}+\sqrt{r-2}} = \sum_{r=2}^{n} (\sqrt{r} - \sqrt{r-2})$$

P.T.O.

Question 25 continued

$r=2: \quad \cancel{\sqrt{2}} \quad -\sqrt{0}$

$r=3: \quad \cancel{\sqrt{3}} \quad -\sqrt{1}$

$r=4: \quad \cancel{\sqrt{4}} - \cancel{\sqrt{2}}$

$r=5: \quad \cancel{\sqrt{5}} - \cancel{\sqrt{3}}$

$\vdots$

$r=n-2: \quad \cancel{\sqrt{n-2}} - \cancel{\sqrt{n-4}}$

$r=n-1: \quad \sqrt{n-1} - \cancel{\sqrt{n-3}}$

$r=n: \quad \sqrt{n} - \cancel{\sqrt{n-2}}$

$$\sum_{r=1}^{n}\left(\sqrt{r}-\sqrt{r-2}\right)$$

$$= \sqrt{n} + \sqrt{n-1} - \sqrt{1} - \sqrt{0}$$

$$= \boxed{\sqrt{n} + \sqrt{n-1} - 1}$$

c) $$\sum_{r=4}^{50} \frac{2}{\sqrt{r}+\sqrt{r-2}} = \sum_{r=2}^{50} \frac{2}{\sqrt{r}+\sqrt{r-2}} - \sum_{r=2}^{3} \frac{2}{\sqrt{r}+\sqrt{r-2}}$$

$$= \sqrt{50} + \sqrt{49} - 1 - \left(\sqrt{3} + \sqrt{2} - 1\right)$$

$$= 5\sqrt{2} + 7 - 1 - \left(\sqrt{3} + \sqrt{2} - 1\right)$$

$$= \boxed{7 + 4\sqrt{2} - \sqrt{3}}$$

$\boxed{A = 7,\ B = 4,\ C = -1}$

****

26. (a) Use the method of differences to prove that for $n > 2$

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right)$$

(4)

(b) Hence find the exact value of

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$$

Give your answer in the form $a\ln\left(\frac{b}{c}\right)$ where a, b and c are integers to be determined.

(3)

a) $\sum_{r=1}^{n} \ln\left(\frac{r+1}{r-1}\right) = \sum_{r=1}^{n} \left(\ln(r+1) - \ln(r-1)\right)$

$= \ln(n+1) + \ln(n) - \ln(2)$ $\ln(1)=0$

$= \ln\left(\frac{n(n+1)}{2}\right)$

$r=2: \ \ln(\cancel{3}) - \ln(1)$

$r=3: \ \ln(\cancel{4}) - \ln(2)$

$r=4: \ \ln(\cancel{5}) - \ln(\cancel{3})$

$r=5: \ \ln(\cancel{6}) - \ln(\cancel{4})$

$\vdots$

$r=n-2: \ \ln(\cancel{n-1}) - \ln(\cancel{n-3})$

$r=n-1: \ \ln(n) - \ln(\cancel{n-2})$

$r=n: \ \ln(n+1) - \ln(\cancel{n-1})$

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Question 26 continued

b) $\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$

$= 35 \sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)$ $\qquad \ln(A^k) = k\ln(A)$

$= 35\left(\sum_{r=1}^{100} \ln\left(\frac{r+1}{r-1}\right) - \sum_{r=1}^{50} \ln\left(\frac{r+1}{r-1}\right)\right)$

$= 35\left(\ln\left(\frac{100\times101}{2}\right) - \ln\left(\frac{50\times51}{2}\right)\right)$

$= 35 \ln\left(\frac{100\times101\times\cancel{2}}{\cancel{2}\times50\times51}\right)$

$$\boxed{= 35\ln\left(\frac{202}{51}\right) \qquad a = 35,\ b = 202,\ c = 51}$$

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27. (a) Express $\dfrac{2}{r(r^2-1)}$ in partial fractions.

(3)

(b) Hence find, in terms of n,

$$\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$$

Give your answer in the form

$$\frac{n^2 + An + B}{Cn(n+1)}$$

where A, B and C are constants to be found.

(5)

a) $\dfrac{2}{r(r^2-1)} = \dfrac{2}{r(r-1)(r+1)} = \dfrac{A}{r} + \dfrac{B}{r-1} + \dfrac{C}{r+1}$

$2 \equiv A(r+1)(r-1) + Br(r+1) + Cr(r-1)$

$r=0: \quad 2 = -A \quad \Rightarrow A = -2$

$r=1: \quad 2 = 2B \quad \Rightarrow B = 1$

$r=-1: \quad 2 = 2C \quad \Rightarrow C = 1$

$$\boxed{\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}}$$

I've written it in this order to help with part (b). This is just a trick that you can pick up with practice

b) $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r^2-1)} = \frac{1}{2}\sum_{r=1}^{n} \frac{2}{r(r^2-1)}$

$\displaystyle = \frac{1}{2}\left(\sum_{r=1}^{n}\left(\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}\right)\right)$

$r=2: \quad \frac{1}{1} - \frac{2}{2} + \cancel{\frac{1}{3}}$

$r=3: \quad \frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}}$

$r=4: \quad \cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}}$

$r=5: \quad \cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{6}}$

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Question 27 continued

$r=n-2: \quad \cancel{\frac{1}{n-3}} - \cancel{\frac{2}{n-2}} + \cancel{\frac{1}{n-1}}$

$r=n-1: \quad \cancel{\frac{1}{n-2}} - \cancel{\frac{2}{n-1}} + \frac{1}{n}$

$r=n \quad : \quad \cancel{\frac{1}{n-1}} - \frac{2}{n} + \frac{1}{n+1}$

$$\text{so} \sum_{r=1}^{n} \frac{1}{r(r^2-1)} = \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left(\frac{n(n+1) - 2(n+1) + 2n}{2n(n+1)}\right)$$

$$= \frac{n^2+n-2n-2+2n}{4n(n+1)}$$

$$= \boxed{\frac{n^2+n-2}{4n(n+1)} \qquad A=1, \; B=-2, \; C=4}$$

****

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28. (a) Write $\dfrac{3r+1}{r(r-1)(r+1)}$ in partial fractions.

(2)

(b) Hence find

$$\sum_{r=2}^{n} \frac{3r+1}{r(r-1)(r+1)} \qquad n \geqslant 2$$

giving your answer in the form

$$\frac{an^2+bn+c}{2n(n+1)}$$

where a, b and c are integers to be determined.

(5)

(c) Hence determine the exact value of

$$\sum_{r=15}^{20} \frac{3r+1}{r(r-1)(r+1)}$$

(2)

a) $\dfrac{3r+1}{r(r-1)(r+1)} = \dfrac{A}{r} + \dfrac{B}{r-1} + \dfrac{C}{r+1}$

$3r+1 = A(r-1)(r+1) + Br(r+1) + Cr(r-1)$

$r=0: \quad 1 = -A \quad \Rightarrow A = -1$

$r=1: \quad 4 = 2B \quad \Rightarrow B = 2$

$r=-1: \quad -2 = 2C \quad \Rightarrow C = -1$

$$\boxed{\frac{2}{r-1} - \frac{1}{r} - \frac{1}{r+1}}$$

✓ Similar to another question in that there's this small trick or ordering it in a way that will cancel nicely. The key is to do it in ascending order of the denominators.

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Question 28 continued

b) $$\sum_{r=2}^{n} \frac{3r+1}{r(r-1)(r+1)} = \sum_{r=2}^{n}\left(\frac{2}{r-1} - \frac{1}{r} - \frac{1}{r+1}\right)$$

$r=2: \quad \frac{2}{1} - \frac{1}{2} - \cancel{\frac{1}{3}}$

$r=3: \quad \frac{2}{2} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$r=4: \quad \cancel{\frac{2}{3}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$r=5: \quad \cancel{\frac{2}{4}} - \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

$\vdots$

$r=n-2: \quad \cancel{\frac{2}{n-3}} - \cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n-1}}$

$r=n-1: \quad \cancel{\frac{2}{n-2}} - \cancel{\frac{1}{n-1}} - \frac{1}{n}$

$r=n: \quad \cancel{\frac{2}{n-1}} - \frac{1}{n} - \frac{1}{n+1}$

$$= 2 - \frac{1}{2} + 1 - \frac{1}{n} - \frac{1}{n} - \frac{1}{n-1}$$

$$= \frac{5}{2} - \frac{2}{n} - \frac{1}{n-1}$$

$$= \frac{5n(n+1) - 4(n+1) - 2n}{2n(n-1)}$$

$$= \frac{5n^2+5n-4n-4-2n}{2n(n+1)}$$

$$= \boxed{\frac{5n^2-n-4}{2n(n+1)}}$$

$\boxed{a=5,\ b=-1,\ c=-4}$

c) $$\sum_{r=15}^{20} \frac{3r+1}{r(r-1)(r+1)} = \sum_{r=1}^{20} \frac{3r+1}{r(r-1)(r+1)} - \sum_{r=1}^{14} \frac{3r+1}{r(r-1)(r+1)}$$

$$= \frac{5(20)^2-20-4}{2(20)(20+1)} - \frac{5(14)^2-14-4}{2(14)(14+1)}$$

$$= \frac{247}{105} - \frac{481}{210}$$

$$= \boxed{\frac{13}{210}}$$

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29. (a) Show that

$$n^5 - (n-1)^5 \equiv 5n^4 - 10n^3 + 10n^2 - 5n + 1$$

(2)

(b) Hence, using the method of differences, show that for all integer values of n,

$$\sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1)(2n+1)(an^2 + bn + c)$$

where a, b and c are integers to be determined.

(7)

a) $n^5 - (n-1)^5 = n^5 - (n^5 - 5n^4 + 10n^3 - 10n^2 + 5n - 1)$

$= 5n^4 - 10n^3 + 10n^2 - 5n + 1$

b) $\sum_{r=1}^{n} (5r^4 - 10r^3 + 10r^2 - 5r + 1) = \sum_{r=1}^{n} (r^5 - (r-1)^5)$

$r=1$: $\cancel{1^5} - 0^5$

$r=2$ $\cancel{2^5} - \cancel{1^5}$

$r=3$: $\cancel{3^5} - \cancel{2^5}$

$\vdots$

$r=n-1$ $\cancel{(n-1)^5} - \cancel{(n-2)^5}$

$r=n$ $n^5 - \cancel{(n-1)^5}$

So $n^5 - 0^5 = 5\sum_{r=1}^{n} r^4 - 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$n^5 = 5\sum_{r=1}^{n} r^4 - 10\frac{1}{4}n^2(n+1)^2 + 10\frac{1}{6}n(n+1)(2n+1) - \frac{5}{2}n(n+1) + n$

$\Rightarrow 5\sum_{r=1}^{n} r^4 = n^5 + \frac{5}{2}n^2(n+1)^2 - \frac{5}{3}n(n+1)(2n+1) + \frac{5}{2}n(n+1) - n$

$5\sum_{r=1}^{n} r^4 = n(n^4 - 1) + \frac{1}{6}n(n+1)\left(15n(n+1) - 10(2n+1) + 15\right)$

$= n(n^2-1)(n^2+1) + \frac{1}{6}n(n+1)\left(15n^2 + 15n - 20n - 10 + 15\right)$

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Question 29 continued

$$= n(n^2-1)(n^2+1) + \frac{1}{6}n(n+1)(15n^2+15n-20n-10+15)$$

$$= n(n-1)(n+1)(n^2+1) + \frac{1}{6}n(n+1)(15n^2-5n+5)$$

$$= \frac{1}{6}n(n+1)\left(6(n-1)(n^2+1) + 15n^2-5n+5\right)$$

$$= \frac{1}{6}n(n+1)(6n^3-6n^2+6n-6+15n^2-5n+5)$$

$$= \frac{1}{6}n(n+1)(6n^3+9n^2+n-1)$$

$$= \frac{1}{6}n(n+1)(6n^3+3n^2+6n^2+n-1)$$

$$= \frac{1}{6}n(n+1)\left(3n^2(2n+1)+(3n-1)(2n+1)\right)$$

$$= \frac{1}{6}n(n+1)(2n+1)(3n^2+3n-1)$$

The reason I've done this is to be able to take out $(2n+1)$. I could also just try factorise the cubic with polynomial long division or other methods

so $\displaystyle\sum_{r=1}^{n} r^4 = \boxed{\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)}$

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30. Given that $A > B > 0$, by letting $x = \arctan A$ and $y = \arctan B$

(a) prove that

$$\arctan A - \arctan B = \arctan\left(\frac{A-B}{1+AB}\right)$$

(3)

(b) Show that when $A = r + 2$ and $B = r$

$$\frac{A-B}{1+AB} = \frac{2}{(1+r)^2}$$

(2)

(c) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \arctan\left(\frac{2}{(1+r)^2}\right) = \arctan(n+p) + \arctan(n+q) - \arctan 2 - \frac{\pi}{4}$$

where p and q are integers to be determined.

(4)

(d) Hence, making your reasoning clear, determine

$$\sum_{r=1}^{\infty} \arctan\left(\frac{2}{(1+r)^2}\right)$$

giving the answer in the form $k\pi - \arctan 2$, where k is a constant.

(2)

a) let $x = \arctan(A)$ & $y = \arctan B$,

we know, $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

by the compound angle formula.

$\Rightarrow \arctan(\tan(x-y)) = \arctan\left(\dfrac{\tan x - \tan y}{1+\tan x \tan y}\right)$

Now $\tan x = A$ and $\tan y = B$,

So $x - y = \arctan\left(\dfrac{A-B}{1+AB}\right)$

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Question 30 continued

$$\Rightarrow \arctan A - \arctan B = \arctan\left(\frac{A-B}{1+AB}\right) \quad \square$$

since $x = \arctan(A)$ and $y = \arctan(B)$.

b) if $A = r+2$ and $B = r$,

$$\frac{A-B}{1+AB} = \frac{r+2-r}{1+(r+2)r} = \frac{2}{1+2r+r^2} = \frac{2}{(1+r)^2}$$

c) $$\sum_{r=1}^{n} \arctan\left(\frac{2}{(1+r)^2}\right) = \sum_{r=1}^{n}\left(\arctan(r+2) - \arctan(r)\right)$$

by (a)

$r=1$: $\cancel{\arctan(3)}$ $-\arctan(1)$

$r=2$: $\arctan(\cancel{4})$ $-\arctan(2)$

$r=3$: $\cancel{\arctan(5)} - \cancel{\arctan(3)}$

$r=4$: $\cancel{\arctan(6)} - \cancel{\arctan(4)}$

$\vdots$

$r=n-2$: $\cancel{\arctan(n)} - \arctan(\cancel{n-2})$

$r=n-1$: $\arctan(n+1)$ $-\arctan(\cancel{n-1})$

$r=n$: $\arctan(n+2)$ $-\cancel{\arctan(n)}$

$\arctan(1) = \pi/4$

So $$\sum_{r=1}^{n} \arctan\left(\frac{2}{(1+r)^2}\right)$$

$$= \boxed{\arctan(n+2) + \arctan(n+1) - \arctan(2) - \pi/4}$$

$p = 2,\ q = 1$

Question 30 continued

d) $\displaystyle\sum_{r=1}^{\infty} \arctan\left(\frac{2}{(1+r)^2}\right)$

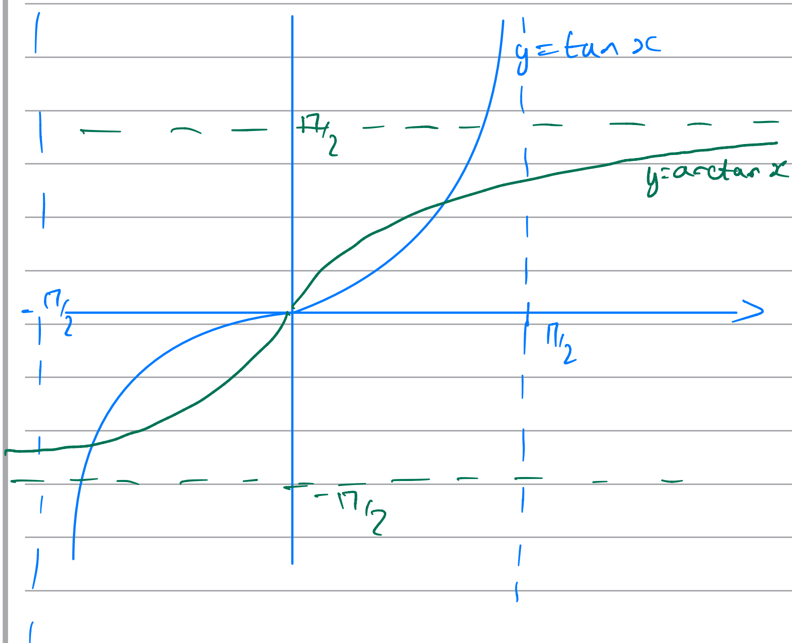
$$= \lim_{n\to\infty}\left(\arctan(n+2)+\arctan(n+1)\right) - \arctan(2) - \pi/4$$

$$= \frac{\pi}{2} + \frac{\pi}{2} - \arctan(2) - \frac{\pi}{4}$$

$$= \boxed{\frac{3\pi}{4} - \arctan(2), \quad k = \frac{3}{4}}$$

$\lim_{n\to\infty}\left(\arctan(n)\right) = \pi/4$

by considering the graph of $\tan x$ and subsequently $\arctan x$:



$y = \tan x$ and $y = \arctan x$

So they are reflections of eachother in the line $y = x$