London Academy of Maths

Series - Method of Differences

Worked Solutions

1 **(i)** Find $\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right)$. **[3]**

(ii) What does the sum in part **(i)** tend to as $n \to \infty$? Justify your answer. **[1]**

i)

$r=1$: $\frac{1}{1} - \frac{1}{1+2} = \boxed{1} - \cancel{\frac{1}{3}}$

$r=2$: $\frac{1}{2} - \frac{1}{2+2} = \boxed{\frac{1}{2}} - \cancel{\frac{1}{4}}$

$r=3$: $\frac{1}{3} - \frac{1}{3+2} = \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$

$r=4$: $\frac{1}{4} - \frac{1}{4+2} = \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$

$\vdots$

$r=n$: $\cancel{\frac{1}{n}} - \boxed{\frac{1}{n+2}}$

$$\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \boxed{\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}}$$

$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{2(n+1)(n+2)}$$

$$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{2(n+1)(n+2)}$$

$$= \frac{3n^2+5n}{2(n+1)(n+2)} = \frac{n(3n+5)}{2(n+1)(n+2)}$$

This isn't required because the question doesn't ask for the answer as a single fraction, but is good practice nonetheless!

b) as $n \to \infty$, $\frac{1}{n+1} \to 0$ and $\frac{1}{n+2} \to 0$

so $\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right) \to \frac{3}{2}$ (as $n \to \infty$).

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2 **(a)** Use the method of differences to show that $\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=1-\frac{1}{n+1}$. **[1]**

(b) Hence determine the following sums.

(i) $\sum_{r=1}^{99}\frac{1}{r}-\frac{1}{r+1}$ **[1]**

(ii) $\sum_{r=100}^{\infty}\frac{1}{r}-\frac{1}{r+1}$ **[3]**

a)

$r=1: \quad \frac{1}{1} - \cancel{\frac{1}{2}}$

$r=2: \quad \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$\vdots$

$r=n: \quad \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$$\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right) = \frac{1}{1} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

b) i) $\sum_{r=1}^{99}\left(\frac{1}{r}-\frac{1}{r+1}\right) = 1 - \frac{1}{99+1} = \boxed{\frac{99}{100}}$

ii) $\sum_{r=100}^{\infty}\left(\frac{1}{r}-\frac{1}{r+1}\right) = \sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{r+1}\right) - \sum_{r=1}^{99}\left(\frac{1}{r}-\frac{1}{r+1}\right)$

$$= 1 - \frac{99}{100}$$

$$= \boxed{\frac{1}{100}}$$

Since $\frac{1}{n+1} \to 0$ as $n \to \infty$

3 By expressing $\frac{1}{r+1}-\frac{1}{r+2}$ as a single fraction, find $\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}$ in terms of n. **[4]**

$$\frac{1}{r+1}-\frac{1}{r+2}=\frac{r+2-(r+1)}{(r+1)(r+2)}$$

$$=\frac{1}{(r+1)(r+2)}$$

$$\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$$

$r=1$: $\frac{1}{2}-\cancel{\frac{1}{3}}$

$r=2$: $\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}$

$r=3$: $\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}$

$r=n-1$ $\cancel{\frac{1}{n}}-\cancel{\frac{1}{n+1}}$

$r=n$ $\cancel{\frac{1}{n+1}}-\frac{1}{n+2}$

$$=\boxed{\frac{1}{2}-\frac{1}{n+2}}$$

don't need to write as a single fraction as per question.

4 In this question you must show detailed reasoning.

Find $\sum_{r=1}^{100}\left(\frac{1}{r}-\frac{1}{r+2}\right)$, giving your answer correct to 4 decimal places. **[3]**

$$\sum_{r=1}^{100}\left(\frac{1}{r}-\frac{1}{r+2}\right) = 1+\frac{1}{2}-\frac{1}{101}-\frac{1}{102}$$

$r=1$: $\frac{1}{1}-\cancel{\frac{1}{3}}$ $= \boxed{1.4803}$ to 4 dp.

$r=2$: $\frac{1}{2}-\cancel{\frac{1}{4}}$

$r=3$: $\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}$

$r=4$: $\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}$

$\vdots$

$r=99$: $\cancel{\frac{1}{99}}-\frac{1}{101}$

$r=100$ $\cancel{\frac{1}{100}}-\frac{1}{102}$

5 **In this question you must show detailed reasoning.**

Find $\sum_{r=2}^{50}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$, expressing the answer as an exact fraction. **[3]**

$$\sum_{r=2}^{50}\left(\frac{1}{r-1}-\frac{1}{r+1}\right) = 1+\frac{1}{2}-\frac{1}{50}-\frac{1}{51} = \boxed{\frac{1862}{1275}}$$

$r=2: \quad \frac{1}{1} - \cancel{\frac{1}{3}}$

$r=3: \quad \frac{1}{2} - \cancel{\frac{1}{4}}$

$r=4: \quad \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$

$\vdots$

$r=49: \quad \cancel{\frac{1}{48}} - \frac{1}{50}$

$r=50: \quad \cancel{\frac{1}{49}} - \frac{1}{51}$

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6 **(i)** Show that $\dfrac{1}{3r-1} - \dfrac{1}{3r+2} \equiv \dfrac{3}{(3r-1)(3r+2)}$ for all integers r. [2]

(ii) Hence use the method of differences to find $\displaystyle\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)}$. [5]

i) $$\frac{1}{3r-1} - \frac{1}{3r+2} = \frac{3r+2-(3r-1)}{(3r-1)(3r+2)}$$

$$= \frac{3}{(3r-1)(3r+2)}$$

ii) $$\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)} = \frac{1}{3}\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)}$$

$$= \frac{1}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-1} - \frac{1}{3r+2}\right)$$

$r=1$: $\frac{1}{2} - \cancel{\frac{1}{5}}$

$r=2$: $\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}$

$r=3$: $\cancel{\frac{1}{8}} - \cancel{\frac{1}{11}}$

$r=4$: $\cancel{\frac{1}{11}} - \cancel{\frac{1}{14}}$

$\vdots$

$r=n-1$: $\cancel{\frac{1}{3n-4}} - \cancel{\frac{1}{3n-1}}$

$r=n$: $\cancel{\frac{1}{3n-1}} - \frac{1}{3n+2}$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right)$$

$$= \boxed{\frac{1}{6} - \frac{1}{3(3n+2)}}$$

Nothing about simplifying answer in question, so no extra work needed.

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7 **(i)** Show that $\dfrac{1}{r^2} - \dfrac{1}{(r+2)^2} \equiv \dfrac{4(r+1)}{r^2(r+2)^2}$. **[2]**

(ii) Hence find an expression, in terms of n, for $\displaystyle\sum_{r=1}^{n} \frac{4(r+1)}{r^2(r+2)^2}$. **[6]**

(iii) Find $\displaystyle\sum_{r=5}^{\infty} \frac{4(r+1)}{r^2(r+2)^2}$, giving your answer in the form $\dfrac{p}{q}$ where p and q are integers. **[2]**

i) $$\frac{1}{r^2} - \frac{1}{(r+2)^2} = \frac{(r+2)^2 - r^2}{r^2(r+2)^2} = \frac{r^2+4r+4-r^2}{r^2(r+2)^2}$$

$$= \frac{4(r+1)}{r^2(r+2)^2}$$

ii) $$\sum_{r=1}^{n} \frac{4(r+1)}{r^2(r+2)^2} = \sum_{r=1}^{n}\left(\frac{1}{r^2} - \frac{1}{(r+2)^2}\right)$$

$r=1$: $\boxed{\dfrac{1}{1^2}} - \cancel{\dfrac{1}{3^2}}$

$r=2$: $\boxed{\dfrac{1}{2^2}} - \cancel{\dfrac{1}{4^2}}$

$r=3$: $\cancel{\dfrac{1}{3^2}} - \cancel{\dfrac{1}{5^2}}$

$r=4$: $\cancel{\dfrac{1}{4^2}} - \cancel{\dfrac{1}{6^2}}$

⋮

$r=n-1$: $\cancel{\dfrac{1}{(n-1)^2}}$ $\boxed{- \dfrac{1}{(n+1)^2}}$

$r=n$: $\cancel{\dfrac{1}{n^2}}$ $\boxed{- \dfrac{1}{(n+2)^2}}$

$$= \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}$$

$$= \boxed{\frac{5}{4} - \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}}$$

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(iii) $$\sum_{r=5}^{\infty} \frac{4(r+1)}{r^2(r+2)^2} = \sum_{r=1}^{\infty} \frac{4(r+1)}{r^2(r+2)^2} - \sum_{r=1}^{4} \frac{4(r+1)}{r^2(r+2)^2}$$

$$= \frac{5}{4} - \Big(\underbrace{\frac{5}{4} - \frac{1}{5^2} - \frac{1}{6^2}}_{\text{by ii)}}\Big)$$

as $n \to \infty$, $\frac{1}{n^2} \to 0$ and $\frac{1}{(n+2)^2} \to 0$.

$$= \frac{1}{5^2} + \frac{1}{6^2}$$

$$= \boxed{\frac{61}{900}}$$

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8 **(i)** Show that $\frac{1}{5r-2} - \frac{1}{5r+3} \equiv \frac{5}{(5r-2)(5r+3)}$ for all integers r. **[2]**

(ii) Hence use the method of differences to show that $\sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} = \frac{n}{3(5n+3)}$. **[4]**

i) $\frac{1}{5r-2} - \frac{1}{5r+3} = \frac{5r+3-(5r-2)}{(5r-2)(5r+3)}$

$= \frac{5}{(5r-2)(5r+3)}$

ii) $\sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} = \frac{1}{5}\sum_{r=1}^{n} \frac{5}{(5r-2)(5r+3)}$

$= \frac{1}{5}\sum_{r=1}^{n}\left(\frac{1}{5r-2} - \frac{1}{5r+3}\right)$

$r=1: \quad \frac{1}{3} - \cancel{\frac{1}{8}}$

$r=2: \quad \cancel{\frac{1}{8}} - \cancel{\frac{1}{13}}$

$r=3: \quad \cancel{\frac{1}{13}} - \cancel{\frac{1}{18}}$

$\vdots$

$r=n-1 \quad \cancel{\frac{1}{5n-7}} - \cancel{\frac{1}{5n-2}}$

$r=n \quad \cancel{\frac{1}{5n-2}} - \frac{1}{5n+3}$

$= \frac{1}{5}\left(\frac{1}{3} - \frac{1}{5n+3}\right)$

$= \frac{1}{5}\left(\frac{5n+3-3}{3(5n+3)}\right)$

$= \frac{1}{\cancel{5}}\left(\frac{\cancel{5}n}{3(5n+3)}\right)$

$= \boxed{\frac{n}{3(5n+3)}}$

**

9 Given that $\dfrac{3}{(3r-1)(3r+2)} \equiv \dfrac{1}{3r-1} - \dfrac{1}{3r+2}$, find $\displaystyle\sum_{r=1}^{20} \frac{1}{(3r-1)(3r+2)}$, giving your answer as an exact fraction. **[5]**

$$\sum_{r=1}^{20} \frac{1}{(3r-1)(3r+2)} = \frac{1}{3}\sum_{r=1}^{20} \frac{3}{(3r-1)(3r+2)}$$

$$= \frac{1}{3}\sum_{r=1}^{20}\left(\frac{1}{3r-1} - \frac{1}{3r+2}\right)$$

$r=1$: $\frac{1}{2} - \cancel{\frac{1}{5}}$

$r=2$: $\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}$

$r=3$: $\cancel{\frac{1}{8}} - \cancel{\frac{1}{11}}$

$\vdots$

$r=19$: $\cancel{\frac{1}{56}} - \cancel{\frac{1}{59}}$

$r=20$: $\cancel{\frac{1}{59}} - \frac{1}{62}$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{62}\right)$$

$$= \boxed{\frac{5}{31}}$$

**

10 **(i)** Show that $\dfrac{1}{2r+1} - \dfrac{1}{2r+3} \equiv \dfrac{2}{(2r+1)(2r+3)}$. **[2]**

(ii) Use the method of differences to find $\displaystyle\sum_{r=1}^{30} \frac{1}{(2r+1)(2r+3)}$, expressing your answer as a fraction. **[5]**

i) $\dfrac{1}{2r+1} - \dfrac{1}{2r+3} = \dfrac{2r+3-(2r+1)}{(2r+1)(2r+3)} = \dfrac{2}{(2r+1)(2r+3)}$

ii) $\displaystyle\sum_{r=1}^{30} \frac{1}{(2r+1)(2r+3)} = \frac{1}{2}\sum_{r=1}^{30} \frac{2}{(2r+1)(2r+3)}$

$\displaystyle = \frac{1}{2}\sum_{r=1}^{30}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$

$\displaystyle = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{63}\right)$

$= \boxed{\dfrac{10}{63}}$

$r=1$: $\dfrac{1}{3} - \cancel{\dfrac{1}{5}}$

$r=2$: $\cancel{\dfrac{1}{5}} - \cancel{\dfrac{1}{7}}$

$r=3$: $\cancel{\dfrac{1}{7}} - \cancel{\dfrac{1}{9}}$

$r=29$: $\cancel{\dfrac{1}{59}} - \cancel{\dfrac{1}{61}}$

$r=30$: $\cancel{\dfrac{1}{61}} - \dfrac{1}{63}$

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11 You are given that $\frac{4}{(4n-3)(4n+1)} \equiv \frac{1}{4n-3} - \frac{1}{4n+1}$. Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1}.$$ **[6]**

$$\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{1}{4}\sum_{r=1}^{n} \frac{4}{(4r-3)(4r+1)}$$

$$= \frac{1}{4}\sum_{r=1}^{n}\left(\frac{1}{4r-3} - \frac{1}{4r+1}\right)$$

$r=1: \quad \frac{1}{1} - \cancel{\frac{1}{5}}$

$r=2: \quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}$

$r=3: \quad \cancel{\frac{1}{9}} - \cancel{\frac{1}{13}}$

$\vdots$

$r=n-1: \quad \cancel{\frac{1}{4n-7}} - \cancel{\frac{1}{4n-3}}$

$r=n: \quad \cancel{\frac{1}{4n-3}} - \frac{1}{4n+1}$

$$= \frac{1}{4}\left(1 - \frac{1}{4n+1}\right)$$

$$= \frac{1}{4}\left(\frac{4n+1-1}{4n+1}\right)$$

$$= \frac{1}{\cancel{4}}\left(\frac{\cancel{4}n}{4n+1}\right)$$

$$= \boxed{\frac{n}{4n+1}}$$

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12 Use the identity $\frac{1}{2r+3} - \frac{1}{2r+5} \equiv \frac{2}{(2r+3)(2r+5)}$ and the method of differences to find $\sum_{r=1}^{n} \frac{1}{(2r+3)(2r+5)}$, expressing your answer as a single fraction. **[5]**

$$\sum_{r=1}^{n} \frac{1}{(2r+3)(2r+5)} = \frac{1}{2}\sum_{r=1}^{n} \frac{2}{(2r+3)(2r+5)}$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+3} - \frac{1}{2r+5}\right)$$

$r=1: \quad \frac{1}{5} - \cancel{\frac{1}{7}}$

$r=2: \quad \cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}$

$r=3: \quad \cancel{\frac{1}{9}} - \cancel{\frac{1}{11}}$

$\vdots$

$r=n-1: \quad \cancel{\frac{1}{2n+1}} - \cancel{\frac{1}{2n+3}}$

$r=n: \quad \cancel{\frac{1}{2n+3}} - \frac{1}{2n+5}$

$$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{2n+5}\right)$$

$$= \frac{1}{2}\left(\frac{2n+5-5}{2n+5}\right)$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}n}{2n+5}\right)$$

$$= \boxed{\frac{n}{2n+5}}$$

**

13 In this question you must show detailed reasoning.

(a) By using partial fractions show that $\sum_{r=1}^{n} \frac{1}{r^2+3r+2} = \frac{1}{2} - \frac{1}{n+2}$. **[5]**

(b) Hence determine the value of $\sum_{r=1}^{\infty} \frac{1}{r^2+3r+2}$. **[2]**

$$\frac{1}{r^2+3r+2} = \frac{1}{(r+2)(r+1)} = \frac{A}{r+2} + \frac{B}{r+1}$$

$$\Rightarrow 1 = A(r+1) + B(r+2)$$

$r=-1: \quad 1 = B$

$r=-2: \quad 1 = -A \Rightarrow A = -1$

so $$\sum_{r=1}^{n} \frac{1}{r^2+3r+2} = \sum_{r=1}^{n} \left(\frac{-1}{r+2} + \frac{1}{r+1} \right)$$

$r=1: \quad \frac{-1}{1+2} + \frac{1}{1+1} = \cancel{\frac{-1}{3}} + \frac{1}{2}$

$r=2: \quad \frac{-1}{2+2} + \frac{1}{2+1} = \cancel{\frac{-1}{4}} + \cancel{\frac{1}{3}}$

$r=3: \quad \frac{-1}{3+2} + \frac{1}{3+1} = \cancel{\frac{-1}{5}} + \cancel{\frac{1}{4}}$

$\vdots \qquad \vdots \qquad \vdots$

$r=n: \quad \frac{-1}{n+2} + \cancel{\frac{1}{n+1}}$

$$\Rightarrow \sum_{r=1}^{n} \left(\frac{-1}{r+2} + \frac{1}{r+1} \right) = \frac{1}{2} - \frac{1}{n+2}$$

b) as $n \to \infty$, $\frac{1}{n+2} \to 0$

so as $n \to \infty$, $\sum_{r=1}^{n} \frac{1}{r^2+3r+2} \to \frac{1}{2}$

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14 Use the result $\frac{1}{5r-1} - \frac{1}{5r+4} \equiv \frac{5}{(5r-1)(5r+4)}$ and the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{(5r-1)(5r+4)},$$

simplifying your answer. **[6]**

$$\frac{1}{5r-1} - \frac{1}{5r+4} = \frac{5r+4-(5r-1)}{(5r-1)(5r+4)} = \frac{5}{(5r-1)(5r+4)}$$

$$\sum_{r=1}^{n} \frac{1}{(5r-1)(5r+4)} = \frac{1}{5}\sum_{r=1}^{n} \frac{5}{(5r-1)(5r+4)}$$

$$= \frac{1}{5}\sum_{r=1}^{n}\left(\frac{1}{5r-1} - \frac{1}{5r+4}\right)$$

$r=1$: $\frac{1}{4} - \cancel{\frac{1}{9}}$

$r=2$: $\cancel{\frac{1}{9}} - \cancel{\frac{1}{14}}$

$r=3$: $\cancel{\frac{1}{14}} - \cancel{\frac{1}{19}}$

$\vdots$

$r=n-1$: $\cancel{\frac{1}{5n-6}} - \cancel{\frac{1}{5n-1}}$

$r=n$: $\cancel{\frac{1}{5n-1}} - \frac{1}{5n+4}$

$$= \frac{1}{5}\left(\frac{1}{4} - \frac{1}{5n+4}\right)$$

$$= \frac{1}{5}\left(\frac{5n+4-4}{4(5n+4)}\right)$$

$$= \frac{1}{\cancel{5}}\left(\frac{\cancel{5}n}{4(5n+4)}\right)$$

$$= \boxed{\frac{n}{4(5n+4)}}$$

15 In this question you must show detailed reasoning.

(a) Use partial fractions to show that $\sum_{r=5}^{n} \frac{3}{r^2+r-2} = \frac{37}{60} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$. **[5]**

(b) Write down the value of $\lim_{n\to\infty}\left(\sum_{r=5}^{n} \frac{3}{r^2+r-2}\right)$. **[1]**

a) $\frac{3}{r^2+r-2} = \frac{3}{(r+2(r-1)} \equiv \frac{A}{r+2} + \frac{B}{r-1}$

$\Rightarrow 3 \equiv A(r-1) + B(r+2)$

$r=1: \quad 3 = 3B \quad \Rightarrow B = 1$

$r=-2: \quad 3 = -3A \quad \Rightarrow A = -1$

$\sum_{r=5}^{n} \frac{3}{r^2+r-2} = \sum_{r=5}^{n} \frac{-1}{r+2} + \frac{1}{r-1}$

$r=5: \quad \cancel{-\frac{1}{7}} + \frac{1}{4}$

$r=6: \quad \cancel{-\frac{1}{8}} + \frac{1}{5}$

$r=7: \quad \cancel{-\frac{1}{9}} + \frac{1}{6}$

$r=8: \quad \cancel{-\frac{1}{10}} + \cancel{\frac{1}{7}}$

$r=9: \quad \cancel{-\frac{1}{11}} + \cancel{\frac{1}{8}}$

$\vdots$

$r=n: \quad \boxed{-\frac{1}{n+2}} + \cancel{\frac{1}{n-1}}$

$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$

$= \frac{37}{60} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$

b) as $n \to \infty$, $\frac{1}{n} \to 0$, $\frac{1}{n+1} \to 0$ and $\frac{1}{n+2} \to 0$

so $\sum_{r=5}^{\infty} \frac{3}{r^2+r-2} = \frac{37}{60}$

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16 **(a)** Express $\dfrac{1}{(2r-1)(2r+1)}$ in partial fractions. **[3]**

(b) Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$, expressing the result as a single fraction. **[4]**

a) $$\frac{1}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

$$\Rightarrow 1 \equiv A(2r+1) + B(2r-1)$$

$$r = -\tfrac{1}{2}: \quad 1 = -2B \quad \Rightarrow B = -\tfrac{1}{2}$$

$$r = \tfrac{1}{2}: \quad 1 = 2A \quad \Rightarrow A = \tfrac{1}{2}$$

$$\frac{1}{(2r-1)(2r+1)} \equiv \boxed{\frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}}$$

remember to actually write it out otherwise you'd lose the last mark.

b) $$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{n} \frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}$$

$r=1$: $\dfrac{1}{2} - \cancel{\dfrac{1}{6}}$

$r=2$: $\cancel{\dfrac{1}{6}} - \cancel{\dfrac{1}{10}}$

$r=3$: $\cancel{\dfrac{1}{10}} - \cancel{\dfrac{1}{14}}$

$\vdots$

$r=n-1$: $\cancel{\dfrac{1}{2(2n-3)}} - \cancel{\dfrac{1}{2(2n-1)}}$

$r=n$: $\cancel{\dfrac{1}{2(2n-1)}} - \dfrac{1}{2(2n+1)}$

$$= \frac{1}{2} - \frac{1}{2(2n+1)}$$

$$= \frac{2n+1-1}{2(2n+1)}$$

$$= \frac{2n}{2(2n+1)} = \boxed{\frac{n}{2n+1}}$$

**

17 **(i)** Show that $\frac{1}{2r-3} - \frac{1}{2r+1} = \frac{4}{4r^2-4r-3}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=2}^{n} \frac{4}{4r^2-4r-3}.$$ **[6]**

(iii) Show that $\sum_{r=2}^{\infty} \frac{4}{4r^2-4r-3} = \frac{4}{3}$. **[1]**

i) $\frac{1}{2r-3} - \frac{1}{2r+1} = \frac{2r+1-(2r-3)}{(2r-3)(2r+1)}$

$= \frac{1+3}{4r^2-4r-3} = \frac{4}{4r^2-4r-3}$

ii) $\sum_{r=2}^{n} \frac{4}{4r^2-4r-3} = \sum_{r=2}^{n}\left(\frac{1}{2r-3} - \frac{1}{2r+1}\right)$

$r=2: \quad \boxed{\frac{1}{1}} - \cancel{\frac{1}{5}}$

$r=3: \quad \boxed{\frac{1}{3}} - \cancel{\frac{1}{7}}$

$r=4: \quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}$

$\vdots$

$r=n-1: \quad \cancel{\frac{1}{2n-5}} - \boxed{\frac{1}{2n-1}}$

$r=n: \quad \cancel{\frac{1}{2n-3}} - \boxed{\frac{1}{2n+1}}$

$= 1 + \frac{1}{3} - \frac{1}{2n-1} - \frac{1}{2n+1}$

$= \frac{4}{3} - \frac{1}{2n-1} - \frac{1}{2n+1}$

iii) as $n \to \infty$, $\frac{1}{2n-1} \to 0$ and $\frac{1}{2n+1} \to 0$

So $\sum_{r=2}^{\infty} \frac{4}{4r^2-4r-3} = \frac{4}{3}$

**

18 **(i)** Show that $\dfrac{1}{r^2} - \dfrac{1}{(r+1)^2} \equiv \dfrac{2r+1}{r^2(r+1)^2}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\displaystyle\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$. **[4]**

(iii) Find $\displaystyle\sum_{r=2}^{\infty} \frac{2r+1}{r^2(r+1)^2}$. **[2]**

i) $\dfrac{1}{r^2} - \dfrac{1}{(r+1)^2} = \dfrac{(r+1)^2 - r^2}{r^2(r+1)^2}$

$= \dfrac{r^2+2r+1-r^2}{r^2(r+1)^2} = \dfrac{2r+1}{r^2(r+1)^2}$

ii) $\displaystyle\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n}\left(\frac{1}{r^2} - \frac{1}{(r+1)^2}\right)$

$r=1: \quad \dfrac{1}{1^2} - \cancel{\dfrac{1}{2^2}}$

$r=2: \quad \cancel{\dfrac{1}{2^2}} - \cancel{\dfrac{1}{3^2}}$

$r=3 \quad \cancel{\dfrac{1}{3^2}} - \cancel{\dfrac{1}{4^2}}$

$\vdots$

$r=n-1: \quad \cancel{\dfrac{1}{(n-1)^2}} - \cancel{\dfrac{1}{n^2}}$

$r=n \quad \cancel{\dfrac{1}{n^2}} - \dfrac{1}{(n+1)^2}$

$= \dfrac{1}{1^2} - \dfrac{1}{(n+1)^2}$

$= \boxed{1 - \dfrac{1}{(n+1)^2}}$

iii) $\displaystyle\sum_{r=2}^{\infty} \frac{2r+1}{r^2(r+1)^2} = 1$

Since as $n \to \infty$, $\dfrac{1}{(n+1)^2} \to 0$

 **

19 You are given that $\dfrac{3}{(5+3x)(2+3x)} \equiv \dfrac{1}{2+3x} - \dfrac{1}{5+3x}$.

(i) Use this result to find $\displaystyle\sum_{r=1}^{100} \frac{1}{(5+3r)(2+3r)}$, giving your answer as an exact fraction. **[5]**

(ii) Write down the limit to which $\displaystyle\sum_{r=1}^{n} \frac{1}{(5+3r)(2+3r)}$ converges as n tends to infinity. **[1]**

i) $$\sum_{r=1}^{100} \frac{1}{(5+3r)(2+3r)} = \frac{1}{3}\sum_{r=1}^{100} \frac{3}{(5+3r)(2+3r)}$$

$$= \frac{1}{3}\sum_{r=1}^{100}\left(\frac{1}{2+3r} - \frac{1}{5+3r}\right)$$

$$= \frac{1}{3}\left(\frac{1}{5} - \frac{1}{305}\right)$$

$$= \frac{4}{61}$$

$r=1$: $\frac{1}{5} - \cancel{\frac{1}{8}}$

$r=2$: $\cancel{\frac{1}{8}} - \cancel{\frac{1}{11}}$

$r=3$: $\cancel{\frac{1}{11}} - \cancel{\frac{1}{14}}$

$\vdots$

$r=99$: $\cancel{\frac{1}{299}} - \cancel{\frac{1}{302}}$

$r=100$: $\cancel{\frac{1}{302}} - \frac{1}{305}$

ii) $$\sum_{r=1}^{n} \frac{1}{(5+3r)(2+3r)} = \frac{1}{3}\left(\frac{1}{5} - \frac{1}{5+3n}\right)$$

as $n \to \infty$, $\frac{1}{5+3n} \to 0$

so $$\sum_{r=1}^{\infty} \frac{1}{(5+3r)(2+3r)} = \frac{1}{3}\left(\frac{1}{5}\right) = \boxed{\frac{1}{15}}$$

***

20 **(i)** Show that $\frac{1}{r} - \frac{1}{r+2} \equiv \frac{2}{r(r+2)}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=1}^{n} \frac{2}{r(r+2)}$. **[6]**

(iii) Given that $\sum_{r=N+1}^{\infty} \frac{2}{r(r+2)} = \frac{11}{30}$, find the value of N. **[4]**

i) $\frac{1}{r} - \frac{1}{r+2} = \frac{r+2-r}{r(r+2)} = \frac{2}{r(r+2)}$

ii) $\sum_{r=1}^{n} \frac{2}{r(r+2)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+2}\right)$

$r=1: \boxed{\frac{1}{1}} - \cancel{\frac{1}{3}}$

$r=2: \boxed{\frac{1}{2}} - \cancel{\frac{1}{4}}$

$r=3: \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$

$r=4: \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$

$\vdots$

$r=n-1: \cancel{\frac{1}{n-1}} \boxed{- \frac{1}{n+1}}$

$r=n: \cancel{\frac{1}{n}} \boxed{- \frac{1}{n+2}}$

$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$

$= \boxed{\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}}$

iii) $\sum_{r=N+1}^{\infty} \frac{2}{r(r+2)} = \sum_{r=1}^{\infty} \frac{2}{r(r+2)} - \sum_{r=1}^{N} \frac{2}{r(r+2)}$

$= \frac{3}{2} - \left(\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right)$

Since $\frac{1}{n+1} \to 0$ and $\frac{1}{n+2} \to 0$ as $n \to \infty$.

P.T.O

***

$$\frac{1}{N+1} + \frac{1}{N+2} = \frac{11}{30}$$

$$\frac{N+2+N+1}{(N+1)(N+2)} = \frac{11}{30}$$

$$30(2N+3) = 11(N+1)(N+2)$$

$$60N + 90 = 11N^2 + 33N + 22$$

$$11N^2 - 27N - 68 = 0$$

$$N = 4 \quad \text{or} \quad N = \frac{-17}{11}$$

$$N \in \mathbb{Z}^+ \quad \text{so} \quad \boxed{N = 4}$$

 ***

21 **(i)** Show that $\dfrac{3}{r-1}-\dfrac{2}{r}-\dfrac{1}{r+1}\equiv\dfrac{4r+2}{r(r^2-1)}$. **[2]**

(ii) Hence find an expression, in terms of n, for $\displaystyle\sum_{r=2}^{n}\frac{4r+2}{r(r^2-1)}$. **[6]**

(iii) Hence find the value of $\displaystyle\sum_{r=4}^{\infty}\frac{4r+2}{r(r^2-1)}$. **[2]**

i) $$\frac{3}{r-1}-\frac{2}{r}-\frac{1}{r+1}=\frac{3(r+1)r-2(r-1)(r+1)-(r-1)r}{(r-1)r(r+1)}$$

$$=\frac{3r^2+3r-2r^2+2-r^2+r}{r(r^2-1)}$$

$$=\frac{4r+2}{r(r^2-1)}$$

ii) $$\sum_{r=2}^{n}\frac{4r+2}{r(r^2-1)}=\sum_{r=2}^{n}\left(\frac{3}{r-1}-\frac{2}{r}-\frac{1}{r+1}\right)$$

$r=2$: $\dfrac{3}{1}-\dfrac{2}{2}-\cancel{\dfrac{1}{3}}$

$r=3$: $\dfrac{3}{2}-\cancel{\dfrac{2}{3}}-\cancel{\dfrac{1}{4}}$

$r=4$: $\cancel{\dfrac{3}{3}}-\cancel{\dfrac{2}{4}}-\cancel{\dfrac{1}{5}}$

$r=5$: $\cancel{\dfrac{3}{4}}-\cancel{\dfrac{2}{5}}-\cancel{\dfrac{1}{6}}$

$\vdots$

$r=n-1$: $\cancel{\dfrac{3}{n-2}}-\cancel{\dfrac{2}{n-1}}-\dfrac{1}{n}$

$r=n$: $\cancel{\dfrac{3}{n-1}}-\dfrac{2}{n}-\dfrac{1}{n+1}$

$$=\frac{3}{1}-\frac{2}{2}+\frac{3}{2}-\frac{1}{n}-\frac{2}{n}-\frac{1}{n+1}$$

$$=\boxed{\frac{7}{2}-\frac{3}{n}-\frac{1}{n+1}}$$

***

iii) $$\sum_{r=4}^{\infty} \frac{4r+2}{r(r^2-1)} = \sum_{r=2}^{\infty} \frac{4r+2}{r(r^2-1)} - \sum_{r=2}^{3} \frac{4r+2}{r(r^2-1)}$$

$$= \frac{7}{2} - \underbrace{\left(\frac{7}{2} - \frac{3}{3} - \frac{1}{3+1}\right)}_{n=3}$$

as $n \to \infty$, $\frac{3}{n} \to 0$ and $\frac{1}{n+1} \to 0$

$$= 1 + \frac{1}{4} = \boxed{\frac{5}{4}}$$

22 **(i)** Verify that $\frac{4+r}{r(r+1)(r+2)} = \frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2}$. **[2]**

(ii) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{4+r}{r(r+1)(r+2)} = \frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}.$$ **[6]**

(iii) Write down the limit to which $\sum_{r=1}^{n} \frac{4+r}{r(r+1)(r+2)}$ converges as n tends to infinity. **[1]**

(iv) Find $\sum_{r=50}^{100} \frac{4+r}{r(r+1)(r+2)}$, giving your answer to 3 significant figures. **[3]**

i) $\text{RHS} = \frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2}$

$= \frac{2(r+1)(r+2) - 3r(r+2) + r(r+1)}{r(r+1)(r+2)}$

$= \frac{2r^2 + 6r + 4 - 3r^2 - 6r + r^2 + r}{r(r+1)(r+2)}$

$= \frac{r+4}{r(r+1)(r+2)} = \text{LHS}$

ii) $\sum_{r=1}^{n} \frac{4+r}{r(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2} \right)$

$r=1: \quad \frac{2}{1} - \frac{3}{2} + \cancel{\frac{1}{3}}$

$r=2: \quad \frac{2}{2} - \cancel{\frac{3}{3}} + \cancel{\frac{1}{4}}$

$r=3: \quad \cancel{\frac{2}{3}} - \cancel{\frac{3}{4}} + \cancel{\frac{1}{5}}$

$r=4: \quad \cancel{\frac{2}{4}} - \cancel{\frac{3}{5}} + \cancel{\frac{1}{6}}$

$\vdots$

$r=n-1: \quad \cancel{\frac{2}{n-1}} - \cancel{\frac{3}{n}} + \frac{1}{n+1}$

$r=n: \quad \cancel{\frac{2}{n}} - \frac{3}{n+1} + \frac{1}{n+2}$

$= \frac{2}{1} - \frac{3}{2} + \frac{2}{2} + \frac{1}{n+1} - \frac{3}{n+1} + \frac{1}{n+2}$

$= \frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}$

***

iii) as $n \to \infty$, $-\frac{2}{n+1} \to 0$ and $\frac{1}{n+2} \to 0$

So $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{4+r}{r(r+1)(r+2)} = \frac{3}{2}$

iv) $\sum_{r=50}^{100} \frac{4+r}{r(r+1)(r+2)} = \sum_{r=1}^{100} \frac{4+r}{r(r+1)(r+2)} - \sum_{r=1}^{49} \frac{4+r}{r(r+1)(r+2)}$

$= \cancel{\frac{3}{2}} - \frac{2}{100+1} + \frac{1}{100+2} - \left(\cancel{\frac{3}{2}} - \frac{2}{49+1} + \frac{1}{49+2}\right)$

$= -\frac{2}{101} + \frac{1}{102} + \frac{2}{50} - \frac{1}{51}$

$= \boxed{0.0104}$ to 3sf

***

23 **(i)** Show that $\frac{r}{r+1} - \frac{r-1}{r} \equiv \frac{1}{r(r+1)}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)}.$$ **[4]**

(iii) Hence find $\sum_{r=n+1}^{\infty} \frac{1}{r(r+1)}$. **[2]**

i) $\frac{r}{r+1} - \frac{r-1}{r} = \frac{r^2-(r-1)(r+1)}{r(r+1)}$

$= \frac{r^2-(r^2-1)}{r(r+1)}$

$= \frac{r^2-r^2+1}{r(r+1)} = \frac{1}{r(r+1)}$

ii) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n(n+1)}$

$= \sum_{r=1}^{n} \frac{1}{r(r+1)}$

$= \sum_{r=1}^{n} \left(\frac{r}{r+1} - \frac{r-1}{r}\right)$

$r=1$: $\cancel{\frac{1}{2}} - \frac{0}{1}$ $=0$ $\qquad = \frac{n}{n+1}$

$r=2$: $\cancel{\frac{2}{3}} - \cancel{\frac{1}{2}}$

$r=3$: $\cancel{\frac{3}{4}} - \cancel{\frac{2}{3}}$

$\vdots$

$r=n-1$: $\cancel{\frac{n-1}{n}} - \cancel{\frac{n-2}{n-1}}$

$r=n$: $\frac{n}{n+1} - \cancel{\frac{n-1}{n}}$

***

$$\sum_{r=n+1}^{\infty} \frac{1}{r(r+1)} = \sum_{r=1}^{\infty} \frac{1}{r(r+1)} - \sum_{r=1}^{n} \frac{1}{r(r+1)}$$

as $n \to \infty \quad \frac{n}{n+1} \to 1$

So $\displaystyle\sum_{r=n+1}^{\infty} \frac{1}{r(r+1)} = 1 - \frac{n}{n+1}$

24 **(i)** Show that $\dfrac{1}{r} - \dfrac{3}{r+1} + \dfrac{2}{r+2} \equiv \dfrac{2-r}{r(r+1)(r+2)}$. **[2]**

(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2-r}{r(r+1)(r+2)} = \frac{n}{(n+1)(n+2)}$. **[5]**

(iii) Find the value of $\displaystyle\sum_{r=2}^{\infty} \frac{2-r}{r(r+1)(r+2)}$. **[2]**

i) $$\text{LHS} = \frac{1}{r} - \frac{3}{r+1} + \frac{2}{r+2} \equiv \frac{(r+1)(r+2) - 3r(r+2) + 2r(r+1)}{r(r+1)(r+2)}$$

$$\equiv \frac{r^2 + 3r + 2 - 3r^2 - 6r + 2r^2 + 2r}{r(r+1)(r+2)}$$

$$\equiv \frac{2-r}{r(r+1)(r+2)} = \text{RHS}$$

ii) $$\sum_{r=1}^{n} \frac{2-r}{r(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{3}{r+1} + \frac{2}{r+2} \right)$$

$r=1$: $\dfrac{1}{1} - \dfrac{3}{2} + \cancel{\dfrac{2}{3}}$

$r=2$: $\dfrac{1}{2} - \cancel{\dfrac{3}{3}} + \cancel{\dfrac{2}{4}}$

$r=3$: $\cancel{\dfrac{1}{3}} - \cancel{\dfrac{3}{4}} + \cancel{\dfrac{2}{5}}$

$r=4$: $\cancel{\dfrac{1}{4}} - \cancel{\dfrac{3}{5}} + \cancel{\dfrac{2}{6}}$

$\vdots$

$r=n-1$: $\cancel{\dfrac{1}{n-1}} - \cancel{\dfrac{3}{n}} + \dfrac{2}{n+1}$

$r=n$: $\cancel{\dfrac{1}{n}} - \dfrac{3}{n+1} + \dfrac{2}{n+2}$

$$= 1 - \frac{3}{2} + \frac{1}{2} + \frac{2}{n+1} - \frac{3}{n+1} + \frac{2}{n+2}$$

$$= \frac{-1}{n+1} + \frac{2}{n+2}$$

$$= \frac{-(n+2) + 2(n+1)}{(n+1)(n+2)}$$

$$= \frac{-n - 2 + 2n + 2}{(n+1)(n+2)}$$

$$= \frac{n}{(n+1)(n+2)}$$

***

(ii) $$\sum_{r=1}^{n} \frac{2-r}{r(r+1)(r+2)} = \frac{n}{(n+1)(n+2)}$$

$$\sum_{r=2}^{\infty} \frac{2-r}{r(r+1)(r+2)} = \sum_{r=1}^{\infty} \frac{2-r}{r(r+1)(r+2)} - \sum_{r=1}^{1} \frac{2-r}{r(r+1)(r+2)}$$

$$= 0 - \frac{1}{2\times 3}$$

$$= \boxed{-\frac{1}{6}}$$

Since $\frac{n}{(n+1)(n+2)} \longrightarrow 0$ as $n \to \infty$

***

25 **(i)** Show that $\dfrac{1}{r!} - \dfrac{1}{(r+1)!} = \dfrac{r}{(r+1)!}$. [2]

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}.$$ [4]

i) $\dfrac{1}{r!} - \dfrac{1}{(r+1)!} = \dfrac{1}{r!} - \dfrac{1}{(r+1)r!}$

$= \dfrac{r+1-1}{(r+1)\times r!} = \dfrac{r}{(r+1)!}$

ii) $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{n}{(n+1)!} = \displaystyle\sum_{r=1}^{n} \frac{r}{(r+1)!}$

$= \displaystyle\sum_{r=1}^{n} \frac{1}{r!} - \frac{1}{(r+1)!}$

$r=1: \quad \dfrac{1}{1!} - \cancel{\dfrac{1}{2!}}$

$r=2: \quad \cancel{\dfrac{1}{2!}} - \cancel{\dfrac{1}{3!}}$

$r=3: \quad \cancel{\dfrac{1}{3!}} - \cancel{\dfrac{1}{4!}}$

$\vdots$

$r=n-1: \quad \cancel{\dfrac{1}{(n-1)!}} - \cancel{\dfrac{1}{n!}}$

$r=n: \quad \cancel{\dfrac{1}{n!}} - \dfrac{1}{(n+1)!}$

$= \boxed{1 - \dfrac{1}{(n+1)!}}$

no need to express as a single fraction.

***

26 **(i)** Show that $\frac{1}{r-1} - \frac{1}{r+1} \equiv \frac{2}{r^2-1}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=2}^{n} \frac{2}{r^2-1}$. **[5]**

(iii) Find the value of $\sum_{r=1000}^{\infty} \frac{2}{r^2-1}$. **[3]**

i) $$\frac{1}{r-1} - \frac{1}{r+1} = \frac{r+1-(r-1)}{(r-1)(r+1)} = \frac{2}{r^2-1}$$

ii) $$\sum_{r=2}^{n} \frac{2}{r^2-1} = \sum_{r=2}^{n}\left(\frac{1}{r-1} - \frac{1}{r+1}\right)$$

$r=2$: $\boxed{\frac{1}{1}} - \cancel{\frac{1}{3}}$

$r=3$: $\boxed{\frac{1}{2}} - \cancel{\frac{1}{4}}$

$r=4$: $\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$

$\vdots$

$r=n-1$: $\cancel{\frac{1}{n-2}}\ \boxed{-\frac{1}{n}}$

$r=n$: $\cancel{\frac{1}{n-1}}\ \boxed{-\frac{1}{n+1}}$

$$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

iii) $$\sum_{r=1000}^{\infty} \frac{2}{r^2-1} = \sum_{r=2}^{\infty} \frac{2}{r^2-1} - \sum_{r=2}^{999} \frac{2}{r^2-1}$$

$$= \frac{3}{2} - \left(\frac{3}{2} - \frac{1}{999} - \frac{1}{1000}\right)$$

Since $\frac{1}{n} \to 0$ and $\frac{1}{n+1} \to 0$ as $n \to \infty$.

$$= \frac{1}{999} + \frac{1}{1000} = \boxed{\frac{1999}{999000}}$$

***

27 **(a)** By considering $(r+1)^3 - r^3$, find $\sum_{r=1}^{n}(3r^2+3r+1)$. **[3]**

(b) Use this result to find $\sum_{r=1}^{n} r(r+1)$, expressing your answer in fully factorised form. **[4]**

a) $(r+1)^3 - r^3 = r^3 + 3r^2 + 3r + 1 - r^3$

$= 3r^2 + 3r + 1$

$$\sum_{r=1}^{n}(3r^2+3r+1) = \sum_{r=1}^{n}\left((r+1)^3 - r^3\right)$$

$r=1: \quad \cancel{2^3} - 1^3$

$r=2: \quad \cancel{3^3} - \cancel{2^3}$

$r=3: \quad \cancel{4^3} - \cancel{3^3}$

$\vdots$

$r=n: \quad (n+1)^3 - \cancel{n^3}$

$= \boxed{(n+1)^3 - 1}$

$= n^3 + 3n^2 + 3n + 1 - 1$

$= \boxed{n^3 + 3n^2 + 3n}$

both acceptable.

b) $\sum_{r=1}^{n}(3r^2 + 3r + 1) = n^3 + 3n^2 + 3n$

$3\sum_{r=1}^{n}(r^2 + r) + \sum_{r=1}^{n} 1 = n^3 + 3n^2 + 3n$

$3\sum_{r=1}^{n}\left(r(r+1)\right) + n = n^3 + 3n^2 + 3n$

$3\sum_{r=1}^{n} r(r+1) = n^3 + 3n^2 + 2n$

$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}(n^3 + 3n^2 + 2n)$

$= \frac{n}{3}(n^2 + 3n + 2)$

$= \boxed{\frac{n}{3}(n+1)(n+2)}$

***

28 **(a)** Using partial fractions and the method of differences, show that

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\ldots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{an+b}{2(n+1)(n+2)},$$

where a and b are integers to be determined. **[5]**

(b) Deduce the sum to infinity of the series.

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\ldots.$$ **[1]**

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\ldots+\frac{1}{n(n+2)}=\sum_{r=1}^{n}\frac{1}{r(r+2)}$$

$$\frac{1}{r(r+2)}\equiv\frac{A}{r}+\frac{B}{r+2}$$

$$1=A(r+2)+Br$$

$r=-2:\quad 1=-2B \quad \Rightarrow B=-\frac{1}{2}$

$r=0:\quad 1=2A \quad \Rightarrow A=\frac{1}{2}$

$$\sum_{r=1}^{n}\frac{1}{r(r+2)}=\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{2(r+2)}\right)=\frac{1}{2}+\frac{1}{4}-\frac{1}{2n+2}-\frac{1}{2n+4}$$

$r=1:\quad \frac{1}{2}-\cancel{\frac{1}{6}}$

$r=2:\quad \frac{1}{4}-\cancel{\frac{1}{8}}$

$r=3:\quad \cancel{\frac{1}{6}}-\cancel{\frac{1}{10}}$

$r=4:\quad \cancel{\frac{1}{8}}-\cancel{\frac{1}{12}}$

$\vdots$

$r=n-1:\quad \cancel{\frac{1}{2n-2}}-\frac{1}{2n+2}$

$r=n:\quad \cancel{\frac{1}{2n}}-\frac{1}{2n+4}$

$$=\frac{3}{4}-\left(\frac{2n+4+2n+2}{(2n+2)(2n+4)}\right)$$

$$=\frac{3}{4}-\frac{4n+6}{2(n+1)2(n+2)}$$

$$=\frac{3}{4}-\frac{\cancel{2}(2n+3)}{2(n+1)\cancel{2}(n+2)}$$

$$\boxed{=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)} \qquad a=2,\ b=3}$$

***

b) $\frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$

as $n \to \infty$, $\frac{2n+3}{2(n+1)(n+2)} \to 0$

Since $\deg(2n+3) = 1$

and $\deg\{2(n+1)(n+2)\} = 2$

if degree of numerator is less than denominator

limit is 0 since denominator is growing at

a faster rate.

$$\boxed{\text{So} \quad \sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{3}{4}}$$

***

29 **(i)** Express $\frac{1}{2r-1} - \frac{1}{2r+1}$ as a single fraction. **[2]**

(ii) Find how many terms of the series

$$\frac{2}{1\times3} + \frac{2}{3\times5} + \frac{2}{5\times7} \quad \ldots + \frac{2}{(2r-1)(2r+1)} + \ldots$$

are needed for the sum to exceed 0.999 999. **[7]**

i) $\frac{2r+1-(2r-1)}{(2r-1)(2r+1)} = \frac{2}{(2r-1)(2r+1)}$

ii) $\sum_{r=1}^{n} \frac{2}{(2r-1)(2r+1)} > 0.999999$

$\sum_{r=1}^{n} \left(\frac{1}{2r-1} - \frac{1}{2r+1}\right) = 1 - \frac{1}{2n+1}$

$r=1: \quad \frac{1}{1} - \cancel{\frac{1}{3}}$

$r=2: \quad \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$

$r=3: \quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$

$\vdots$

$r=n: \quad \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}$

$\Rightarrow 1 - \frac{1}{2n+1} > 0.999999$

$\frac{1}{2n+1} < 0.000001$

$\frac{1{,}000{,}000}{2n+1} < 1$

$1{,}000{,}000 < 2n+1$ (Since $2n+1>0$)

$n > 49999.5$

so $n = \boxed{500{,}000}$

***

30 **(i)** Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions. **[2]**

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}$, expressing your answer as a single fraction. **[5]**

i) $$\frac{2}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$$

$$\Rightarrow \quad 2 \equiv A(r+3) + B(r+1)$$

$r=-3$: $\quad 2 = -2B \Rightarrow B = -1$

$r=-1$: $\quad 2 = 2A \Rightarrow A = 1$

so $$\frac{2}{(r+1)(r+3)} = \boxed{\frac{1}{r+1} - \frac{1}{r+3}}$$

ii) $$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{1}{2}\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)}$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$$

$r=1$: $\frac{1}{2} - \cancel{\frac{1}{4}}$

$r=2$: $\frac{1}{3} - \cancel{\frac{1}{5}}$

$r=3$: $\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$

$\vdots$

$r=n-1$: $\cancel{\frac{1}{n}} - \frac{1}{n+2}$

$r=n$: $\cancel{\frac{1}{n+1}} - \frac{1}{n+3}$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)}\right)$$

$$= \frac{5n^2 + 25n + 30 - 6n - 18 - 6n - 12}{12(n+2)(n+3)}$$

$$= \boxed{\frac{5n^2+13n}{12(n+2)(n+3)}} = \boxed{\frac{n(5n+13)}{12(n+2)(n+3)}}$$

***

31 **(i)** Show that $\dfrac{1}{2r+1}-\dfrac{1}{2r+3}\equiv\dfrac{2}{(2r+1)(2r+3)}$. **[1]**

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}$, giving your answer as a single fraction. **[6]**

(iii) Find $\displaystyle\sum_{r=n}^{\infty}\frac{1}{(2r+1)(2r+3)}$, giving your answer as a single fraction. **[3]**

i) $\dfrac{1}{2r+1}-\dfrac{1}{2r+3}=\dfrac{2r+3-(2r+1)}{(2r+1)(2r+3)}=\dfrac{2}{(2r+1)(2r+3)}$

ii) $\displaystyle\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}=\frac{1}{2}\sum_{r=1}^{n}\frac{2}{(2r+1)(2r+3)}$

$\displaystyle=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$

$r=1:\ \dfrac{1}{3}-\cancel{\dfrac{1}{5}}$

$r=2:\ \cancel{\dfrac{1}{5}}-\cancel{\dfrac{1}{7}}$

$r=3:\ \cancel{\dfrac{1}{7}}-\cancel{\dfrac{1}{9}}$

$\vdots$

$r=n-1:\ \cancel{\dfrac{1}{2n-1}}-\cancel{\dfrac{1}{2n+1}}$

$r=n:\ \cancel{\dfrac{1}{2n+1}}-\dfrac{1}{2n+3}$

$=\dfrac{1}{2}\left(\dfrac{1}{3}-\dfrac{1}{2n+3}\right)$

$=\dfrac{1}{2}\left(\dfrac{2n+3-3}{3(2n+3)}\right)$

$=\dfrac{1}{\cancel{2}}\dfrac{\cancel{2}n}{3(2n+3)}=\boxed{\dfrac{n}{3(2n+3)}}$

iii) $$\sum_{r=n}^{\infty} \frac{1}{(2r+1)(2r+3)} = \sum_{r=1}^{\infty} \frac{1}{(2r+1)(2r+3)} - \sum_{r=1}^{n-1} \frac{1}{(2r+1)(2r+3)} \quad ***$$

$$= \frac{1}{6} - \frac{n-1}{3(2(n-1)+3)}$$

Since $\frac{n}{3(2n+3)} = \frac{n}{6n+9} \to \frac{1}{6}$ as $n \to \infty$

$$= \frac{1}{6} - \frac{n-1}{3(2n+1)}$$

$$= \frac{2n+1 - 2(n-1)}{6(2n+1)}$$

$$= \frac{3}{6(2n+1)} = \boxed{\frac{1}{2(2n+1)}} = \boxed{\frac{1}{4n+2}}$$

***

32 **(i)** Show that $\dfrac{1}{2r-1}-\dfrac{1}{2r+5}\equiv\dfrac{6}{(2r-1)(2r+5)}$. **[1]**

Hence find

(ii) $\displaystyle\sum_{r=2}^{30}\frac{6}{(2r-1)(2r+5)}$, giving your answer correct to 3 decimal places, **[5]**

(iii) $\displaystyle\sum_{r=2}^{\infty}\frac{6}{(2r-1)(2r+5)}$, giving your answer as a single fraction. **[1]**

i) $\dfrac{1}{2r-1}-\dfrac{1}{2r+5}=\dfrac{2r+5-(2r-1)}{(2r-1)(2r+5)}$

$=\dfrac{6}{(2r-1)(2r+5)}$

ii) $\displaystyle\sum_{r=2}^{30}\frac{6}{(2r-1)(2r+5)}=\sum_{r=2}^{30}\left(\frac{1}{2r-1}-\frac{1}{2r+5}\right)$

$r=2:\quad \frac{1}{3}-\cancel{\frac{1}{9}}$

$r=3:\quad \frac{1}{5}-\cancel{\frac{1}{11}}$

$r=4:\quad \frac{1}{7}-\cancel{\frac{1}{13}}$

$r=5:\quad \cancel{\frac{1}{9}}-\cancel{\frac{1}{15}}$

$r=6:\quad \cancel{\frac{1}{11}}-\cancel{\frac{1}{17}}$

$\vdots$

$r=28:\quad \cancel{\frac{1}{55}}-\frac{1}{61}$

$r=29:\quad \cancel{\frac{1}{57}}-\frac{1}{63}$

$r=30:\quad \cancel{\frac{1}{59}}-\frac{1}{65}$

$=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}-\frac{1}{61}-\frac{1}{63}-\frac{1}{65}$

$=\boxed{0.629}$ to 3dp

(ii) $\displaystyle\sum_{r=2}^{n} \frac{6}{(2r-1)(2r+5)} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2n+1} - \frac{1}{2n+3} - \frac{1}{2n+5}$

as $n \to \infty$, $\frac{1}{2n+1} \to 0$, $\frac{1}{2n+3} \to 0$ and $\frac{1}{2n+5} \to 0$

so $\displaystyle\sum_{r=2}^{\infty} \frac{6}{(2r-1)(2r+5)} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \boxed{\frac{71}{105}}$

****

33 Using an algebraic method, determine the least value of n for which $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} \geqslant 0.49$. **[8]**

$$\frac{1}{(2r-1)(2r+1)} \equiv \frac{A}{2r-1} + \frac{B}{2r+1}$$

$$\Rightarrow \quad 1 \equiv A(2r+1) + B(2r-1)$$

$r = -\frac{1}{2}$: $\quad 1 = -2B \Rightarrow B = -\frac{1}{2}$

$r = \frac{1}{2}$: $\quad 1 = 2A \Rightarrow A = \frac{1}{2}$

So $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{n} \frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}$

$r=1$: $\frac{1}{2(2-1)} - \frac{1}{2(2+1)} = \frac{1}{2} - \cancel{\frac{1}{6}}$

$r=2$: $\frac{1}{2(4-1)} - \frac{1}{2(4+1)} = \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}$

$r=3$: $\frac{1}{2(6-1)} - \frac{1}{2(6+1)} = \cancel{\frac{1}{10}} - \cancel{\frac{1}{14}}$

$\vdots$

$r=n$: $\frac{1}{2(2n-1)} - \frac{1}{2(2n+1)} = \cancel{\frac{1}{4n-2}} - \frac{1}{4n+2}$

So $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{1}{2} - \frac{1}{4n+2} \geqslant 0.49$

$$0.01 \geqslant \frac{1}{4n+2}$$

$$0.01(4n+2) \geqslant 1$$

$$4n+2 \geqslant 100$$

$$4n \geqslant 98$$

$$n \geqslant 24.5$$

So $\boxed{n = 25}$

n is positive integer. so $4n+2$ which is why I can multiply both sides by $4n+2$ without worrying about changing the inequality.

****

34 **In this question you must show detailed reasoning.**

(a) Show that

$$\sum_{r=1}^{n}\frac{5r+6}{r^3+r^2}=\frac{a}{n+1}+b+c\sum_{r=1}^{n}\frac{1}{r^2}$$

where a, b and c are integers whose values are to be determined. **[6]**

You are given that $\sum_{r=1}^{\infty}\frac{1}{r^2}$ exists and is equal to $\frac{1}{6}\pi^2$.

(b) Show that $\sum_{r=1}^{\infty}\frac{5r+6}{r^3+r^2}$ exists and is equal to $(\pi-1)(\pi+1)$. **[2]**

a) $\frac{5r+6}{r^3+r^2} \equiv \frac{5r+6}{r^2(r+1)} = \frac{A}{r}+\frac{B}{r^2}+\frac{C}{r+1}$

$\Rightarrow \quad 5r+6 = Ar(r+1)+B(r+1)+Cr^2$

$r=0: \quad 6 = B$

$r=-1: \quad 1 = C$

$r=1: \quad 11 = 2A+2B+C$

$\Rightarrow 11 = 2A+13$ so $\underline{A=-1}$.

$\sum_{r=1}^{n}\frac{5r+6}{r^3+r^2} = \sum_{r=1}^{n}\left(\frac{-1}{r}+\frac{6}{r^2}+\frac{1}{r+1}\right)$

$= \sum_{r=1}^{n}\left(\frac{-1}{r}+\frac{1}{r+1}\right) + 6\sum_{r=1}^{n}\frac{1}{r^2}$ by inspecting required solution above.

$= -1+\frac{1}{n+1}+6\sum_{r=1}^{n}\frac{1}{r^2}$

$\sum_{r=1}^{n}\left(\frac{-1}{r}+\frac{1}{r+1}\right)$

$r=1: \quad \frac{-1}{1}+\cancel{\frac{1}{2}}$

$r=2: \quad \cancel{-\frac{1}{2}}+\cancel{\frac{1}{3}}$

$r=3: \quad \cancel{-\frac{1}{3}}+\cancel{\frac{1}{4}}$

$\cdots \; r=n \quad \cancel{-\frac{1}{n}}+\frac{1}{n+1}$

$= \frac{1}{n+1}-1+6\sum_{r=1}^{n}\frac{1}{r^2}$

$a=1, \quad b=-1, \quad c=6$

****

b) as $n \to \infty$, $\frac{1}{n+1} \to 0$

so as $n \to \infty$, $\frac{1}{n+1} - 1 + 6 \sum_{r=1}^{n} \frac{1}{r^2}$

$$\to -1 + 6 \sum_{r=1}^{\infty} \frac{1}{r^2} = -1 + 6 \frac{\pi^2}{6}$$

$$= \pi^2 - 1$$

$$= (\pi+1)(\pi-1)$$

35 **(i)** Show that $\dfrac{1}{r} - \dfrac{2}{r+1} + \dfrac{1}{r+2} \equiv \dfrac{2}{r(r+1)(r+2)}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}.$$ **[6]**

(iii) Show that $\displaystyle\sum_{r=n+1}^{\infty} \frac{2}{r(r+1)(r+2)} = \frac{1}{(n+1)(n+2)}$. **[3]**

i) $$\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2} = \frac{(r+1)(r+2) - 2r(r+2) + r(r+1)}{r(r+1)(r+2)}$$

$$= \frac{r^2 + 3r + 2 - 2r^2 - 4r + r^2 + r}{r(r+1)(r+2)}$$

$$= \frac{2}{r(r+1)(r+2)}$$

ii) $$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2} \right)$$

$r=1$: $\dfrac{1}{1} - \dfrac{2}{2} + \cancel{\dfrac{1}{3}}$

$r=2$: $\dfrac{1}{2} - \cancel{\dfrac{2}{3}} + \cancel{\dfrac{1}{4}}$

$r=3$: $\cancel{\dfrac{1}{3}} - \cancel{\dfrac{2}{4}} + \cancel{\dfrac{1}{5}}$

$r=4$: $\cancel{\dfrac{1}{4}} - \cancel{\dfrac{2}{5}} + \cancel{\dfrac{1}{6}}$

$\vdots$

$r=n-1$: $\cancel{\dfrac{1}{n-1}} - \cancel{\dfrac{2}{n}} + \dfrac{1}{n+1}$

$r=n$: $\cancel{\dfrac{1}{n}} - \dfrac{2}{n+1} + \dfrac{1}{n+2}$

$$= \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

****

(ii) $$\sum_{r=n+1}^{\infty} \frac{2}{r(r+1)(r+2)} = \sum_{r=1}^{\infty} \frac{2}{r(r+1)(r+2)} - \sum_{r=1}^{n} \frac{2}{r(r+2)(r+2)}$$

$$= \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2} \right)$$

Since $-\frac{1}{n+1} \to 0$ and $\frac{1}{n+2} \to 0$ as $n \to \infty$.

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{n+2-(n+1)}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}$$

****

36 You are given that $\frac{1}{2r-1}-\frac{1}{2r+3}=\frac{4}{(2r-1)(2r+3)}$ for all integers r.

(i) Use the method of differences to show that

$$\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+3)}=k-\frac{n+1}{(2n+1)(2n+3)},$$

stating the value of k. **[6]**

(ii) The sum of the infinite series

$$\frac{1}{\big(2(n+1)-1\big)\big(2(n+1)+3\big)}+\frac{1}{\big(2(n+2)-1\big)\big(2(n+2)+3\big)}+\frac{1}{\big(2(n+3)-1\big)\big(2(n+3)+3\big)}+\dots$$

is $\frac{7}{195}$. Show that n satisfies $28n^2-139n-174=0$ and hence find the value of n. **[5]**

i) $$\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+3)}=\frac{1}{4}\sum_{r=1}^{n}\frac{4}{(2r-1)(2r+3)}$$

$$=\frac{1}{4}\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+3}\right)$$

$r=1$: $\boxed{\frac{1}{1}}-\cancel{\frac{1}{5}}$

$r=2$: $\boxed{\frac{1}{3}}-\cancel{\frac{1}{7}}$

$r=3$: $\cancel{\frac{1}{5}}-\cancel{\frac{1}{9}}$

$r=4$: $\cancel{\frac{1}{7}}-\cancel{\frac{1}{11}}$

$\vdots$

$r=n-1$: $\frac{1}{2n-3}\boxed{-\frac{1}{2n+1}}$

$r=n$: $\frac{1}{2n-1}\boxed{-\frac{1}{2n+3}}$

$$=\frac{1}{4}\left(1+\frac{1}{3}-\frac{1}{2n+1}-\frac{1}{2n+3}\right)$$

$$=\frac{1}{4}\left(\frac{4}{3}-\frac{1}{2n+1}-\frac{1}{2n+3}\right)$$

$$=\frac{1}{4}\left(\frac{4}{3}-\left(\frac{2n+3+2n+1}{(2n+1)(2n+3)}\right)\right)$$

$$=\frac{1}{4}\left(\frac{4}{3}-\frac{4n+4}{(2n+1)(2n+3)}\right)$$

$$=\boxed{\frac{1}{3}-\frac{n+1}{(2n+1)(2n+3)},\quad k=\frac{1}{3}}$$

****

ii) $\frac{1}{(2(n+1)-1)(2(n+1)+3)}+\frac{1}{(2(n+2)-1)(2(n+2)+3)}+\frac{1}{(2(n+3)-1)(2(n+3)+3)}+\dots = \sum_{r=n+1}^{\infty} \frac{1}{(2r-1)(2r+3)}$

$$= \sum_{r=1}^{\infty} \frac{1}{(2r-1)(2r+3)} - \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+3)}$$

$$= \frac{1}{3} - \left(\frac{1}{3} - \frac{n+1}{(2n+1)(2n+3)}\right)$$

since $\frac{n+1}{(2n+1)(2n+3)} \to 0$ as $n \to \infty$.

$$= \frac{n+1}{(2n+1)(2n+3)}$$

$$\frac{n+1}{(2n+1)(2n+3)} = \frac{7}{195}$$

$$195n + 195 = 7(4n^2 + 8n + 3)$$

$$195n + 195 = 28n^2 + 56n + 21$$

$$0 = 28n^2 - 139n - 174$$

In a new spec paper, it might have said "showing detailed reason..". Hee it doesn't so you can use quadratic solver on calc straight away:

$$n = 6 \quad \text{or} \quad n = \frac{-29}{28}$$

$$n \in \mathbb{Z}^+ \quad \text{so} \quad n \neq \frac{-29}{28} \implies \boxed{n = 6}$$

37 You are given that $\dfrac{3}{4(2r-1)} - \dfrac{1}{2r+1} + \dfrac{1}{4(2r+3)} = \dfrac{2r+5}{(2r-1)(2r+1)(2r+3)}$.

(i) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)} = \frac{2}{3} - \frac{3}{4(2n+1)} + \frac{1}{4(2n+3)}.$$ **[6]**

(ii) Write down the limit to which $\displaystyle\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$ converges as n tends to infinity. **[1]**

(iii) Find the sum of the finite series

$$\frac{45}{39\times41\times43} + \frac{47}{41\times43\times45} + \frac{49}{43\times45\times47} + \ldots + \frac{105}{99\times101\times103},$$

giving your answer to 3 significant figures. **[4]**

i) $\displaystyle\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+5)} = \sum_{r=1}^{n} \left(\frac{3}{4(2r-1)} - \frac{1}{2r+1} + \frac{1}{4(2r+3)} \right)$

$r=1: \quad \frac{3}{4(1)} - \frac{1}{3} + \cancel{\frac{1}{4(5)}}$

$r=2: \quad \frac{3}{4(3)} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4(7)}}$

$r=3: \quad \cancel{\frac{3}{4(5)}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{4(9)}}$

$r=4: \quad \cancel{\frac{3}{4(7)}} - \cancel{\frac{1}{9}} + \cancel{\frac{1}{4(11)}}$

$\vdots$

$r=n-1: \quad \cancel{\frac{3}{4(2n-3)}} - \cancel{\frac{1}{2n-1}} + \frac{1}{4(2n+1)}$

$r=n: \quad \cancel{\frac{3}{4(2n-1)}} - \frac{1}{2n+1} + \frac{1}{4(2n+3)}$

$= \frac{3}{4} - \frac{1}{3} + \frac{3}{4(3)} + \frac{1}{4(2n+1)} - \frac{1}{2n+1} + \frac{1}{4(2n+3)}$

$= \frac{2}{3} + \frac{1}{4(2n+1)} - \frac{4}{4(2n+1)} + \frac{1}{4(2n+3)}$

$= \frac{2}{3} - \frac{3}{4(2n+1)} + \frac{1}{4(2n+3)}$

ii) as $n \to \infty$, $\frac{3}{4(2n+1)} \to 0$ and $\frac{1}{4(2n+3)} \to 0$

****

So $\displaystyle \lim_{n \to \infty}\left(\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}\right) = \boxed{\frac{2}{3}}$

(ii)

$$\frac{45}{39\times41\times43}+\frac{47}{41\times43\times45}+\frac{49}{43\times45\times47}+\ldots+\frac{105}{99\times101\times103} = \sum_{r=20}^{50} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$$

$$= \sum_{r=1}^{50} \frac{2r+5}{(2r-1)(2r+1)(2r+3)} - \sum_{r=1}^{19} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$$

$$= \cancel{\frac{2}{3}} - \frac{3}{4(2(50)+1)} + \frac{1}{4(2(50)+3)} - \left(\cancel{\frac{2}{3}} - \frac{3}{4(2(19)+1)} + \frac{1}{4(2(19)+3)}\right)$$

$$= -\frac{3}{404} + \frac{1}{412} + \frac{3}{156} - \frac{1}{164}$$

$= \boxed{0.00813}$ to 3sf.

****

38 **(i)** Show that $\dfrac{1}{\sqrt{r+2}+\sqrt{r}} \equiv \dfrac{\sqrt{r+2}-\sqrt{r}}{2}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r+2}+\sqrt{r}}.$$ **[6]**

(iii) State, giving a brief reason, whether the series $\sum_{r=1}^{\infty} \dfrac{1}{\sqrt{r+2}+\sqrt{r}}$ converges. **[1]**

i) $$\frac{1}{\sqrt{r+2}+\sqrt{r}} = \frac{1}{\sqrt{r+2}+\sqrt{r}} \times \frac{\sqrt{r+2}-\sqrt{r}}{\sqrt{r+2}-\sqrt{r}}$$

$$= \frac{\sqrt{r+2}-\sqrt{r}}{r+2-r}$$

$(a-b)(a+b) = a^2-b^2$

$$= \frac{\sqrt{r+2}-\sqrt{r}}{2}$$

ii) $$\sum_{r=1}^{n} \frac{1}{\sqrt{r+2}+\sqrt{r}} = \sum_{r=1}^{n} \frac{\sqrt{r+2}-\sqrt{r}}{2} = \sum_{r=1}^{n}\left(\frac{\sqrt{r+2}}{2} - \frac{\sqrt{r}}{2}\right)$$

$r=1$: $\cancel{\frac{\sqrt{3}}{2}} - \frac{\sqrt{1}}{2}$

$r=2$: $\cancel{\frac{\sqrt{4}}{2}} - \frac{\sqrt{2}}{2}$

$r=3$: $\cancel{\frac{\sqrt{5}}{2}} - \cancel{\frac{\sqrt{3}}{2}}$

$r=4$: $\cancel{\frac{\sqrt{6}}{2}} - \cancel{\frac{\sqrt{4}}{2}}$

$\vdots$

$r=n-1$: $\frac{\sqrt{n+1}}{2} - \cancel{\frac{\sqrt{n-1}}{2}}$

$r=n$: $\frac{\sqrt{n+2}}{2} - \cancel{\frac{\sqrt{n}}{2}}$

$$= -\frac{\sqrt{1}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{n+1}}{2} + \frac{\sqrt{n+2}}{2}$$

$$= \frac{\sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1}{2}$$

iii) as $n \to \infty$ $\sqrt{n+2} \to \infty$ and

$\sqrt{n+1} \to \infty$

so $\sum_{r=1}^{\infty} \dfrac{1}{\sqrt{r+2}+\sqrt{r}}$ $\boxed{\text{doesn't converge}}$

****

39 **(i)** Find $\sum_{r=1}^{n} r(r^2+r-7)$, giving your answer in a fully factorised form. **[5]**

A sequence $u_0, u_1, u_2, \ldots$ is defined by

$$u_0 = 5,\ u_n = u_{n-1} + n^3 + n^2 - 7n \text{ for } n \geqslant 1.$$

(ii) By considering $\sum_{r=1}^{n}(u_r - u_{r-1})$, find a formula for u_n in terms of n. **[3]**

[You do not need to factorise your answer.]

i) $\sum_{r=1}^{n} r(r^2+r-7) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2 - 7\sum_{r=1}^{n} r$

$= \underbrace{\frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)}_{\text{Formula booklet}} - 7\frac{n}{2}(n+1)$

If you aren't sure where this comes from see "Series - Standard Formulae" in resources.

$= \frac{1}{12}n(n+1)\left(3n(n+1) + 2(2n+1) - 42\right)$

$= \frac{1}{12}n(n+1)\left(3n^2+3n+4n+2-42\right)$

$= \frac{1}{12}n(n+1)\left(3n^2+7n-40\right)$

$= \boxed{\frac{1}{12}n(n+1)(3n-8)(n+5)}$

****

(ii) $$\sum_{r=1}^{n}(u_r - u_{r-1}) = u_n - u_0$$

$r=1: \cancel{u_1} - u_0$

$r=2: \cancel{u_2} - \cancel{u_1}$

$r=3: \cancel{u_3} - \cancel{u_2}$

$\vdots$

$r=n-1: \cancel{u_{n-1}} - \cancel{u_{n-2}}$

$r=n: u_n - \cancel{u_{n-1}}$

but $$\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n}(u_{r-1} + r^3 + r^2 - 7r - u_{r-1})$$

$$= \sum_{r=1}^{n}(r^3 + r^2 - 7r)$$

$$= \frac{1}{12}n(n+1)(3n-8)(n+5) \quad \text{by i)}$$

so, $$u_n - u_0 = \frac{1}{12}n(n+1)(3n-8)(n+5)$$

$$\Rightarrow \boxed{u_n = \frac{1}{12}n(n+1)(3n-8)(n+5) + 5}$$

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40 **In this question you must show detailed reasoning.**

It is given that $\displaystyle\sum_{r=k}^{98} \frac{5r+2}{r(r+1)(r+2)} = \frac{20539}{34650}$ for some k.

Determine the value of k. **[7]**

$$\frac{5r+2}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$\Rightarrow 5r+2 \equiv A(r+1)(r+2) + Br(r+2) + Cr(r+1)$$

$r=-1$: $\quad -3 = B(-1)(1) \quad \Rightarrow B = 3$

$r=-2$: $\quad -8 = C(-2)(-1) \quad \Rightarrow -8 = 2C \quad \Rightarrow C = -4$

$r=0$: $\quad 2 = 2A \quad \Rightarrow A = 1$

$$\sum_{r=k}^{98} \frac{5r+2}{r(r+1)(r+2)} = \sum_{r=k}^{98}\left(\frac{1}{r} + \frac{3}{r+1} - \frac{4}{r+2}\right)$$

$r=k$: $\quad \frac{1}{k} + \frac{3}{k+1} - \cancel{\frac{4}{k+2}}$

$r=k+1$: $\quad \frac{1}{k+1} + \cancel{\frac{3}{k+2}} - \cancel{\frac{4}{k+3}}$

$r=k+2$: $\quad \cancel{\frac{1}{k+2}} + \cancel{\frac{3}{k+3}} - \cancel{\frac{4}{k+4}}$

$r=96$: $\quad \cancel{\frac{1}{96}} + \cancel{\frac{3}{97}} - \cancel{\frac{4}{98}}$

$r=97$: $\quad \cancel{\frac{1}{97}} + \cancel{\frac{3}{98}} - \frac{4}{99}$

$r=98$: $\quad \cancel{\frac{1}{98}} + \frac{3}{99} - \frac{4}{100}$

$$= \frac{1}{k} + \frac{3}{k+1} + \frac{1}{k+1} - \frac{4}{99} + \frac{3}{99} - \frac{4}{100}$$

$$= \frac{1}{k} + \frac{4}{k+1} - \frac{1}{99} - \frac{4}{100}$$

$$\frac{k+1+4k}{k(k+1)} - \frac{124}{2475} = \frac{20539}{34650}$$

$$\frac{5k+1}{k(k+1)} = \frac{9}{14}$$

$$70k + 14 = 9k^2 + 9k$$

$$9k^2 - 61k - 14 = 0$$

$$(9k+2)(k-7) = 0$$

$$k = \frac{-2}{9} \quad \text{or} \quad k = 7$$

but $k \in \mathbb{N}$, so

$$\boxed{k = 7.}$$

$\rightarrow$ Or use Quadratic Formula but need to show steps since "show detailed reasoning"