London Academy of Maths

# Sums of Complex Numbers

Question Paper

**1** Convergent infinite series $C$ and $S$ are defined by

$$C = 1 + \tfrac{1}{2}\cos\theta + \tfrac{1}{4}\cos 2\theta + \tfrac{1}{8}\cos 3\theta + \ \dots\ ,$$
$$S = \tfrac{1}{2}\sin\theta + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{8}\sin 3\theta + \ \dots\ .$$

**(i)** Show that $C + \mathrm{i}S = \dfrac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}$. **[4]**

**(ii)** Hence show that $C = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$, and find a similar expression for $S$. **[4]**

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**2** **(a)** The infinite series $C$ and $S$ are defined as follows.

$$C = -\frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta - \frac{1}{8}\cos 3\theta + \dots$$
$$S = -\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta - \frac{1}{8}\sin 3\theta + \dots$$

By considering $C + \mathrm{i}S$, show that

$$S = \frac{-2\sin\theta}{5 + 4\cos\theta}\,.$$

Find a corresponding expression for $C$. **[9]**

**3** **(a)** The infinite series $C$ and $S$ are defined as follows.

$$C = 1 + a\cos\theta + a^2\cos 2\theta + \ldots,$$

$$S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \ldots,$$

where $a$ is a real number and $|a| < 1$.

By considering $C + \mathrm{i}S$, show that $C = \dfrac{1 - a\cos\theta}{1 + a^2 - 2a\cos\theta}$ and find a corresponding expression for $S$. **[8]**

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**4.** The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

**(4)**

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

**(4)**

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**5** The integrals $C$ and $S$ are defined by

$$C = \int_0^{\frac{1}{2}\pi} e^{2x} \cos 3x \, dx \qquad \text{and} \qquad S = \int_0^{\frac{1}{2}\pi} e^{2x} \sin 3x \, dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\tfrac{1}{13}\left(2 + 3e^{\pi}\right),$$

and obtain a similar expression for $S$. [8]

(You may assume that the standard result for $\int e^{kx} \, dx$ remains true when $k$ is a complex constant, so that $\int e^{(a+ib)x} \, dx = \dfrac{1}{a + ib} e^{(a+ib)x}$.)

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**6** **(b)** **(i)** Show that $(1 - 2e^{i\theta})(1 - 2e^{-i\theta}) = 5 - 4\cos\theta$. [3]

Series $C$ and $S$ are defined by

$$C = 2\cos\theta + 4\cos 2\theta + 8\cos 3\theta + \ \dots \ + 2^n \cos n\theta,$$
$$S = 2\sin\theta + 4\sin 2\theta + 8\sin 3\theta + \ \dots \ + 2^n \sin n\theta.$$

**(ii)** Show that $C = \dfrac{2\cos\theta - 4 - 2^{n+1}\cos(n+1)\theta + 2^{n+2}\cos n\theta}{5 - 4\cos\theta}$, and find a similar expression for $S$. [9]

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**7** **(b)** Let

$$C = \cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \ldots + \cos\left(\theta + \frac{(2n-2)\pi}{n}\right),$$

and $$S = \sin\theta + \sin\left(\theta + \frac{2\pi}{n}\right) + \sin\left(\theta + \frac{4\pi}{n}\right) + \ldots + \sin\left(\theta + \frac{(2n-2)\pi}{n}\right),$$

where $n$ is an integer greater than 1.

By considering $C + \mathrm{i}S$, show that $C = 0$ and $S = 0$. **[7]**

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**8** Infinite series $C$ and $S$ are defined by

$$C = \cos 2\theta - \tfrac{1}{2}\cos 5\theta + \tfrac{1}{4}\cos 8\theta - \tfrac{1}{8}\cos 11\theta + \dots,$$

$$S = \sin 2\theta - \tfrac{1}{2}\sin 5\theta + \tfrac{1}{4}\sin 8\theta - \tfrac{1}{8}\sin 11\theta + \dots.$$

**(iii)** Show that $C = \dfrac{4\cos 2\theta + 2\cos\theta}{5 + 4\cos 3\theta}$, and find a similar expression for $S$. [8]

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**9.** (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \ldots$$

**(1)**

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots \quad = \frac{2\sin\theta}{5 - 4\cos\theta}$$

**(5)**

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

**(2)**

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**10** **(a)** Find $(3-e^{2i\theta})(3-e^{-2i\theta})$ in terms of $\cos 2\theta$. **[2]**

**(b)** Hence show that the sum of the infinite series

$$\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \frac{1}{27}\sin 7\theta + \ldots$$

can be expressed as $\dfrac{6\sin\theta}{5-3\cos 2\theta}$. **[6]**

**11** The complex number $z$ is defined as $z = \frac{1}{3}e^{i\theta}$ where $0 < \theta < \frac{1}{2}\pi$.

On an Argand diagram, the point O represents the complex number 0, and the points $P_1$, $P_2$, $P_3$, … represent the complex numbers $z$, $z^2$, $z^3$, … respectively.

**(a)** Write down each of the following.

**(i)** The ratio of the lengths $OP_{n+1} : OP_n$ **[1]**

**(ii)** The angle $P_{n+1}OP_n$ **[1]**

**(b)** **(i)** Show that $(3 - e^{i\theta})(3 - e^{-i\theta}) = a + b\cos\theta$, where $a$ and $b$ are integers to be determined. **[2]**

**(ii)** By considering the sum to infinity of the series $z + z^2 + z^3 + \ldots$, show that

$$\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \frac{1}{27}\sin 3\theta + \ldots = \frac{3\sin\theta}{10 - 6\cos\theta}.$$ **[6]**

**12** **(a)** Show that $(2-\mathrm{e}^{\mathrm{i}\theta})(2-\mathrm{e}^{-\mathrm{i}\theta}) = 5-4\cos\theta$. **[3]**

Series $C$ and $S$ are defined by

$$C = \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \ldots + \frac{1}{2^n}\cos n\theta,$$
$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots + \frac{1}{2^n}\sin n\theta.$$

**(b)** Show that $C = \dfrac{2^n(2\cos\theta - 1) - 2\cos(n+1)\theta + \cos n\theta}{2^n(5-4\cos\theta)}$. **[9]**

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**13** **(a)** **(i)** Show that

$$1 + e^{i2\theta} = 2\cos\theta(\cos\theta + i\sin\theta).$$ **[2]**

**(ii)** The series $C$ and $S$ are defined as follows.

$$C = 1 + \binom{n}{1}\cos 2\theta + \binom{n}{2}\cos 4\theta + \ldots + \cos 2n\theta$$

$$S = \binom{n}{1}\sin 2\theta + \binom{n}{2}\sin 4\theta + \ldots + \sin 2n\theta$$

By considering $C + iS$, show that

$$C = 2^n \cos^n\theta \cos n\theta,$$

and find a corresponding expression for $S$. **[7]**

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**14** **(a)** **(i)** Express $2\sin\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta - \mathrm{i}\cos\frac{1}{2}\theta\right)$ in terms of $z$ where $z = \cos\theta + \mathrm{i}\sin\theta$. **[3]**

**(ii)** The series $C$ and $S$ are defined as follows.

$$C = 1 - \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta - \ldots + (-1)^n\binom{n}{n}\cos n\theta$$

$$S = -\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta - \ldots + (-1)^n\binom{n}{n}\sin n\theta$$

Show that

$$C + \mathrm{i}S = \left\{-2\mathrm{i}\sin\tfrac{1}{2}\theta\left(\cos\tfrac{1}{2}\theta + \mathrm{i}\sin\tfrac{1}{2}\theta\right)\right\}^n.$$

Hence show that, for even values of $n$,

$$\frac{C}{S} = \cot\left(\tfrac{1}{2}n\theta\right).$$ **[8]**

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**15** Let $S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{10i\theta}$.

**(i) (a)** Show that, for $\theta \neq 2n\pi$, where $n$ is an integer,

$$S = \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}.$$ **[4]**

**(b)** State the value of $S$ for $\theta = 2n\pi$, where $n$ is an integer. **[1]**

**(ii)** Hence show that, for $\theta \neq 2n\pi$, where $n$ is an integer,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}.$$ **[3]**

**(iii)** Hence show that $\theta = \frac{1}{11}\pi$ is a root of $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = 0$ and find another root in the interval $0 < \theta < \frac{1}{4}\pi$. **[4]**

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**16** Let $C = \sum_{r=0}^{20} \binom{20}{r} \cos r\theta$. Show that $C = 2^{20} \cos^{20}\left(\frac{1}{2}\theta\right) \cos 10\theta$. **[8]**

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**17** **(i)** Show that, if $z \neq \pm 1$ and $z \neq 0$,

$$\sum_{r=1}^{n} z^{2r-1} = \frac{1-z^{2n}}{z^{-1}-z}.$$ **[2]**

**(ii)** Hence show that, if $\sin\theta \neq 0$,

$$\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}.$$ **[6]**

**(iii)** Hence find the exact value of

$$\int_0^{\frac{1}{6}\pi} \frac{\sin^2 3\theta}{\sin\theta}\,\mathrm{d}\theta.$$ **[3]**

**18** Series $C$ and $S$ are defined by

$$C = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \binom{n}{3}\cos 3 \quad + \ldots + \binom{n}{n}\cos n\theta,$$

$$S = \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \binom{n}{3}\sin 3 \quad + \ldots + \binom{n}{n}\sin n\theta.$$

**(ii)** Find $C$ and $S$, and show that $\dfrac{S}{C} = \tan \tfrac{1}{2}n\theta$. [7]