London Academy of Maths

Sums of Complex Numbers

Worked Solutions

1 Convergent infinite series C and S are defined by

$$C = 1 + \tfrac{1}{2}\cos\theta + \tfrac{1}{4}\cos 2\theta + \tfrac{1}{8}\cos 3\theta + \ldots ,$$
$$S = \tfrac{1}{2}\sin\theta + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{8}\sin 3\theta + \ldots .$$

(i) Show that $C + \mathrm{i}S = \dfrac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}$. **[4]**

(ii) Hence show that $C = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$, and find a similar expression for S. **[4]**

a) $C + iS = 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \ldots$

$= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{8}e^{3i\theta} + \ldots$

$= \dfrac{1}{1 - \frac{1}{2}e^{i\theta}} \times \dfrac{2}{2}$ $\quad \left(S_\infty = \dfrac{a}{1-r} \right)$

$= \dfrac{2}{2 - e^{i\theta}}$

b) $C + iS = \dfrac{2}{2 - e^{i\theta}} \times \dfrac{2 - e^{-i\theta}}{2 - e^{-i\theta}}$

$= \dfrac{4 - 2e^{-i\theta}}{4 - 2e^{i\theta} - 2e^{-i\theta} + 1}$

$= \dfrac{4 - 2\cos(-\theta) - 2i\sin(-\theta)}{5 - 2(e^{i\theta} + e^{-i\theta})}$

$= \dfrac{4 - 2\cos\theta + 2i\sin(\theta)}{5 - 2(2\cos\theta)}$ $\leftarrow \begin{cases} \cos(-\theta) = \cos(\theta) \\ \sin(-\theta) = -\sin(\theta) \end{cases}$

$\leftarrow$ see derivation of this on next page.

P-1.0

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$$\therefore C = \mathrm{Re}\left(C+iS\right) = \frac{4-2\cos\theta}{5-4\cos\theta}$$

$$\& \quad S = \mathrm{Im}\left(C+iS\right) = \frac{2\sin\theta}{5-4\cos\theta}$$

$$e^{i\theta}+e^{-i\theta} = \cos\theta + i\sin\theta \quad + \cos -\theta \quad + i\sin -\theta$$

$$= \cos\theta + i\sin\theta \quad + \cos\theta \quad - i\sin\theta$$

since $\cos(-\theta)=\cos\theta$
and $\sin(-\theta) = -\sin\theta$

$$= 2\cos\theta$$

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2 **(a)** The infinite series C and S are defined as follows.

$$C = -\frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta - \frac{1}{8}\cos 3\theta + \ldots$$
$$S = -\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta - \frac{1}{8}\sin 3\theta + \ldots$$

By considering $C + \mathrm{i}S$, show that

$$S = \frac{-2\sin\theta}{5 + 4\cos\theta}.$$

Find a corresponding expression for C. **[9]**

$$C + iS = -\frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \ldots$$

$$= -\frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} - \frac{1}{8}e^{3i\theta} + \ldots$$

$$= \frac{-\frac{1}{2}e^{i\theta}}{1 + \frac{1}{2}e^{i\theta}} \times \frac{2}{2}$$

$a = r = -\frac{1}{2}e^{i\theta}$

$S_\infty = \frac{a}{1-r}$

$$= \frac{-e^{i\theta}}{2 + e^{i\theta}} \times \frac{2 + e^{-i\theta}}{2 + e^{-i\theta}}$$

$$= \frac{-2e^{i\theta} - 1}{5 + 2(e^{i\theta} + e^{-i\theta})}$$

$e^{i\theta} + e^{-i\theta}$

$= \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}$

$= 2\cos\theta$

$$= \frac{-2\cos\theta - 2i\sin\theta - 1}{5 + 4\cos\theta}$$

$$C = \mathrm{Re}(C + iS) = \frac{-2\cos\theta - 1}{5 + 4\cos\theta}$$

$$S = \mathrm{Im}(C + iS) = \frac{-2\sin\theta}{5 + 4\cos\theta}$$

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3 **(a)** The infinite series C and S are defined as follows.

$$C = 1 + a\cos\theta + a^2\cos 2\theta + \ldots,$$

$$S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \ldots,$$

where a is a real number and $|a| < 1$.

By considering $C + \mathrm{i}S$, show that $C = \dfrac{1 - a\cos\theta}{1 + a^2 - 2a\cos\theta}$ and find a corresponding expression for S. **[8]**

$$C + iS = 1 + a(\cos\theta + i\sin\theta) + a^2(\cos 2\theta + i\sin 2\theta) + \ldots$$

$$= 1 + ae^{i\theta} + a^2e^{2i\theta} + \ldots$$

$$= \frac{1}{1 - ae^{i\theta}}$$

$$\left(S_\infty = \frac{a}{1-r} \right)$$

$$= \frac{1}{1 - ae^{i\theta}} \times \frac{1 - ae^{-i\theta}}{1 - ae^{-i\theta}}$$

$$= \frac{1 - ae^{-i\theta}}{1 + a^2 - a(e^{i\theta} + e^{-i\theta})}$$

$$= \frac{1 - a(\cos(-\theta) + i\sin(-\theta))}{1 + a^2 - a(2\cos\theta)}$$

$$= \frac{1 - a\cos\theta + ai\sin\theta}{1 + a^2 - 2a\cos\theta}$$

$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

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$$C = \operatorname{Re}(C+iS) = \frac{1 - a\cos\theta}{1 + a^2 - 2a\cos\theta}$$

$$S = \operatorname{Im}(C+iS) = \frac{a\sin\theta}{1 + a^2 - 2a\cos\theta}$$

4. The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

a) $C+iS = \cos\theta + i\sin\theta + \frac{1}{2}(\cos 5\theta + i\sin 5\theta) + \frac{1}{4}(\cos 9\theta + i\sin 9\theta) + \ldots$

$= e^{i\theta} + \frac{1}{2}e^{5i\theta} + \frac{1}{4}e^{9i\theta} + \frac{1}{8}e^{13i\theta} + \ldots$

$= \dfrac{e^{i\theta}}{1 - \frac{1}{2}e^{4i\theta}} \times \dfrac{2}{2}$ $\quad \left(S_\infty = \dfrac{a}{1-r}\right)$

$= \dfrac{2e^{i\theta}}{2 - e^{4i\theta}}$

b) $C+iS = \dfrac{2e^{i\theta}}{2-e^{4i\theta}} \times \dfrac{2-e^{-4i\theta}}{2-e^{-4i\theta}}$

$= \dfrac{4e^{i\theta} - 2e^{-3i\theta}}{5 - 2(e^{4i\theta} + e^{-4i\theta})}$

$$= \frac{4\cos\theta + 4i\sin\theta \quad - 2\cos(-3\theta) - 2i\sin(-3\theta)}{5 \quad - 2(2\cos 4\theta)}$$

$$= \frac{4\cos\theta + 4i\sin\theta \quad - 2\cos 3\theta \quad + 2i\sin 3\theta}{5 - 4\cos 4\theta}$$

since $\sin(-3\theta)$
$= -\sin(3\theta)$
& $\cos(-3\theta) = \cos(3\theta)$

$$S = \operatorname{Im}(C + iS)$$

$$= \frac{4\sin\theta \ + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

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5 The integrals C and S are defined by

$$C = \int_0^{\frac{1}{2}\pi} e^{2x} \cos 3x \, dx \qquad \text{and} \qquad S = \int_0^{\frac{1}{2}\pi} e^{2x} \sin 3x \, dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\tfrac{1}{13}(2 + 3e^{\pi}),$$

and obtain a similar expression for S. [8]

(You may assume that the standard result for $\int e^{kx}\,dx$ remains true when k is a complex constant, so that $\int e^{(a+ib)x}\,dx = \frac{1}{a+ib}e^{(a+ib)x}$.) (*)

$$C+iS = \int_0^{\frac{1}{2}\pi} e^{2x}\cos(3x)\,dx + i\int_0^{\frac{1}{2}\pi} e^{2x}\sin(3x)\,dx$$

$$= \int_0^{\frac{1}{2}\pi} \left(e^{2x}\cos(3x) + ie^{2x}\sin(3x)\right) dx$$

$$= \int_0^{\frac{1}{2}\pi} e^{2x}\left(\cos(3x) + i\sin(3x)\right) dx$$

$$= \int_0^{\frac{1}{2}\pi} e^{2x} e^{3ix}\, dx$$

$$= \int_0^{\frac{1}{2}\pi} e^{(2+3i)x}\, dx$$

$$= \left[\frac{1}{2+3i} e^{(2+3i)x}\right]_0^{\frac{1}{2}\pi} \qquad \text{(by *)}$$

$$= \frac{1}{2+3i}\left[e^{(2+3i)x}\right]_0^{\frac{1}{2}\pi}$$

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$$= \frac{1}{2+3i}\left(e^{(2+3i)\frac{1}{2}\pi} - e^{(2+3i)0}\right)$$

$$= \frac{1}{2+3i}\left(e^{\pi+\frac{3}{2}\pi i} - 1\right)$$

$$= \frac{1}{2+3i}\left(e^{\pi}e^{3/2\pi i} - 1\right)$$

$$= \frac{1}{2+3i}\left(e^{\pi}\left(\cos\left(\tfrac{3}{2}\pi\right)+i\sin\left(\tfrac{3}{2}\pi\right)\right) - 1\right)$$

$$= \frac{1}{2+3i}\left(e^{\pi}(-i) - 1\right)$$

$$= \frac{2-3i}{13}\left(-e^{\pi}i - 1\right) \qquad \text{by rationalising } \frac{1}{2+3i}$$

$$= \frac{-2-3e^{\pi}+i(-2e^{\pi}+3)}{13}$$

$$C = \mathrm{Re}(C+iS) = \frac{-2-3e^{\pi}}{13} = \frac{-1}{13}\left(3e^{\pi}+2\right)$$

$$S = \mathrm{Im}(C+iS) = \frac{3-2e^{\pi}}{13}$$

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6 **(b)** **(i)** Show that $(1-2e^{i\theta})(1-2e^{-i\theta}) = 5 - 4\cos\theta$. [3]

Series C and S are defined by

$$C = 2\cos\theta + 4\cos 2\theta + 8\cos 3\theta + \ldots + 2^n \cos n\theta,$$
$$S = 2\sin\theta + 4\sin 2\theta + 8\sin 3\theta + \ldots + 2^n \sin n\theta.$$

(ii) Show that $C = \dfrac{2\cos\theta - 4 - 2^{n+1}\cos(n+1)\theta + 2^{n+2}\cos n\theta}{5 - 4\cos\theta}$, and find a similar expression for S. [9]

$$2e^{i\theta} + 4e^{2i\theta} + 8e^{3i\theta} + \ldots + 2^n e^{in\theta}$$

i) $(1-2e^{i\theta})(1-2e^{-i\theta}) = 1 + 4 - 2(e^{i\theta} + e^{-i\theta})$

$= 5 - 2(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta))$

$= 5 - 2(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)$

$= 5 - 2(2\cos\theta)$

$= 5 - 4\cos\theta$

ii) $C + iS = 2(\cos\theta + i\sin\theta) + 4(\cos 2\theta + i\sin 2\theta) + \ldots +$

$2^n(\cos n\theta + i\sin n\theta)$

$= 2e^{i\theta} + 4e^{2i\theta} + \ldots + 2^n e^{in\theta}$

$= \dfrac{2e^{i\theta}(1 - (2e^{i\theta})^n)}{1 - 2e^{i\theta}}$ $\left(S_n = \dfrac{a(1-r^n)}{1-r}\right)$

**

$$= \frac{2e^{i\theta} - 2^{n+1} e^{i(n+1)\theta}}{1-2e^{i\theta}} \times \frac{1-2e^{-i\theta}}{1-2e^{-i\theta}}$$

$$= \frac{2e^{i\theta} - 2^{n+1} e^{i(n+1)\theta} - 4 + 2^{n+2} e^{in\theta}}{5 - 4\cos\theta} \longleftarrow \text{from (i)}$$

$$= \frac{2(\cos\theta + i\sin\theta) - 2^{n+1}(\cos(n+1)\theta + i\sin(n+1)\theta) - 4 + 2^{n+2}(\cos n\theta + i\sin n\theta)}{5 - 4\cos\theta}$$

$$C = \mathrm{Re}(C+iS) = \frac{2\cos\theta - 2^{n+1}\cos(n+1)\theta - 4 - 2^{n+2}\cos n\theta}{5-4\cos\theta}$$

$$S = \mathrm{Im}(C+iS) = \frac{2\sin\theta - 2^{n+1}\sin(n+1)\theta + 2^{n+2}\sin n\theta}{5-4\cos\theta}$$

**

7 **(b)** Let

$$C = \cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \ldots + \cos\left(\theta + \frac{(2n-2)\pi}{n}\right),$$

and $S = \sin\theta + \sin\left(\theta + \frac{2\pi}{n}\right) + \sin\left(\theta + \frac{4\pi}{n}\right) + \ldots + \sin\left(\theta + \frac{(2n-2)\pi}{n}\right),$

where n is an integer greater than 1.

By considering $C + \mathrm{i}S$, show that $C = 0$ and $S = 0$. **[7]**

b) $C + iS = \cos\theta + i\sin\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + i\sin\left(\theta + \frac{2\pi}{n}\right) + \ldots + \cos\left(\theta + \frac{(2n-2)\pi}{n}\right) + i\sin\left(\theta + \frac{(2n-2)\pi}{n}\right)$

$$= e^{i\theta} + e^{i\left(\theta + \frac{2\pi}{n}\right)} + \ldots + e^{i\left(\theta + \frac{(2n-2)\pi}{n}\right)}$$

$$= \frac{e^{i\theta}\left(1 - \left(e^{i\frac{2\pi}{n}}\right)^{n}\right)}{1 - e^{i\frac{2\pi}{n}}}$$

Need to count how many terms from $0, 2, 4, \ldots, (2n-2)$

Same as $0, 1, 2, \ldots, n-1$

So n terms in total.

$$= \frac{e^{i\theta}\left(1 - e^{2\pi i}\right)}{1 - e^{i\frac{2\pi}{n}}}$$

$$= \frac{e^{i\theta}\left(1 - \left(\cos(2\pi) + i\sin(2\pi)\right)\right)}{1 - e^{i\frac{2\pi}{n}}}$$

$$= \frac{e^{i\theta}(1-1)}{1 - e^{i\frac{2\pi}{n}}} = \frac{0}{1 - e^{i\frac{2\pi}{n}}} = 0$$

$C = \mathrm{Re}(C + iS) = 0$

& $S = \mathrm{Im}(C + iS) = 0$

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8 Infinite series C and S are defined by

$$C = \cos 2\theta - \tfrac{1}{2}\cos 5\theta + \tfrac{1}{4}\cos 8\theta - \tfrac{1}{8}\cos 11\theta + \ldots,$$

$$S = \sin 2\theta - \tfrac{1}{2}\sin 5\theta + \tfrac{1}{4}\sin 8\theta - \tfrac{1}{8}\sin 11\theta + \ldots.$$

(iii) Show that $C = \dfrac{4\cos 2\theta + 2\cos\theta}{5 + 4\cos 3\theta}$, and find a similar expression for S. [8]

$$C + iS = \cos 2\theta + i\sin 2\theta - \tfrac{1}{2}(\cos 5\theta + i\sin 5\theta) + \tfrac{1}{4}(\cos 8\theta + i\sin 8\theta) + \ldots$$

$$= e^{2i\theta} - \tfrac{1}{2}e^{5i\theta} + \tfrac{1}{4}e^{8i\theta} - \ldots$$

$$= \frac{e^{2i\theta}}{1 + \frac{1}{2}e^{3i\theta}}$$

$a = e^{2i\theta}$

$r = -\tfrac{1}{2}e^{3i\theta}$

$S_\infty = \dfrac{a}{1-r}$

$$= \frac{2e^{2i\theta}}{2 + e^{3i\theta}} \times \frac{2 + e^{-3i\theta}}{2 + e^{-3i\theta}}$$

$$= \frac{4e^{2i\theta} + 2e^{-i\theta}}{5 + 2(e^{3i\theta} + e^{-3i\theta})}$$

$$= \frac{4\cos 2\theta + 4i\sin 2\theta + 2\cos(-\theta) + 2i\sin(-\theta)}{5 + 4\cos 3\theta}$$

$$= \frac{4\cos 2\theta + 4i\sin 2\theta + 2\cos\theta - 2i\sin\theta}{5 + 4\cos 3\theta}$$

$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

**

$$C = \mathrm{Re}(C+iS) = \frac{4\cos 2\theta + 2\cos\theta}{5+4\cos\theta}$$

$$S = \mathrm{Im}(C+iS) = \frac{4\sin 2\theta - 2\sin\theta}{5+4\cos\theta}$$

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9. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \ldots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

(2)

a) $1 + z + z^2 + z^3 + \ldots = \frac{1}{1-z}$ $\left(S_\infty = \frac{a}{1-r} \right)$

b) let $C + iS = 1 + z + z^2 + z^3 + \ldots$

$= 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \ldots$

$= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \ldots$

$= \frac{1}{1 - \frac{1}{2}e^{i\theta}}$ $\times\frac{2}{2}$ by (a)

$= \frac{2}{2 - e^{i\theta}} \times \frac{2 - e^{-i\theta}}{2 - e^{i\theta}}$

$= \frac{4 - 2e^{-i\theta}}{5 - 2(e^{i\theta} + e^{-i\theta})}$

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$$= \frac{4 - 2\cos(-\theta) - 2i\sin(-\theta)}{5 - 4\cos\theta}$$

$$= \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}$$

$\cos(-\theta) = \cos\theta$

& $\sin(-\theta) = -\sin\theta$

$$\therefore \operatorname{Im}(C+iS) = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta$$

$$= \frac{2\sin\theta}{5 - 4\cos\theta}$$

c) $\operatorname{Re}(C+iS) = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$

$4 - 2\cos\theta \geqslant 2$ for all values of θ since $-1 \leqslant \cos\theta \leqslant 1$

so $4 - 2\cos\theta \neq 0$ $\Rightarrow \operatorname{Re}(C+iS) \neq 0$

$\Rightarrow 1 + z + z^2 + \dots$ is not purely imaginary.

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10 **(a)** Find $(3-e^{2i\theta})(3-e^{-2i\theta})$ in terms of $\cos 2\theta$. **[2]**

(b) Hence show that the sum of the infinite series

$$\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{9}\sin 5\theta+\frac{1}{27}\sin 7\theta+\ldots$$

can be expressed as $\dfrac{6\sin\theta}{5-3\cos 2\theta}$. **[6]**

a) $(3-e^{2i\theta})(3-e^{-2i\theta}) = 10-3(e^{2i\theta}+e^{-2i\theta})$

$= 10-3(2\cos 2\theta)$

$= 10-6\cos 2\theta$

b) let $C = \cos\theta+\frac{1}{3}\cos 3\theta+\frac{1}{9}\cos 5\theta+\ldots$

& $S = \sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{9}\sin 5\theta+\ldots$

$C+iS = \cos\theta+i\sin\theta+\frac{1}{3}(\cos 3\theta+i\sin 3\theta)+\ldots$

$= e^{i\theta}+\frac{1}{3}e^{3i\theta}+\frac{1}{9}e^{5i\theta}+\ldots$

$= \dfrac{e^{i\theta}}{1-\frac{1}{3}e^{2i\theta}} \times \dfrac{3}{3}$ $\left(S_\infty = \dfrac{a}{1-r}\right)$

$= \dfrac{3e^{i\theta}}{3-e^{2i\theta}} \times \dfrac{3-e^{-2i\theta}}{3-e^{-2i\theta}}$

$= \dfrac{9e^{i\theta}-3e^{-i\theta}}{10-6\cos 2\theta}$ from (a)

**

$$= \frac{9\cos\theta + 9i\sin\theta - 3\cos(-\theta) - 3i\sin(-\theta)}{10 - 6\cos 2\theta}$$

$$= \frac{9\cos\theta - 3\cos\theta + 9i\sin\theta + 3i\sin\theta}{10 - 6\cos 2\theta}$$

$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

$$= \frac{6\cos\theta + 12i\sin\theta}{10 - 6\cos 2\theta}$$

$$= \frac{3\cos\theta + 6i\sin\theta}{5 - 3\cos 2\theta}$$

$$S = \operatorname{Im}(C + iS) = \frac{6\sin\theta}{5 - 3\cos 2\theta} \qquad \square$$

11 The complex number z is defined as $z = \frac{1}{3}e^{i\theta}$ where $0 < \theta < \frac{1}{2}\pi$.

On an Argand diagram, the point O represents the complex number 0, and the points P_1, P_2, P_3, ... represent the complex numbers z, z^2, z^3, ... respectively.

(a) Write down each of the following.

(i) The ratio of the lengths $OP_{n+1} : OP_n$ **[1]**

(ii) The angle $P_{n+1}OP_n$ **[1]**

(b) **(i)** Show that $(3-e^{i\theta})(3-e^{-i\theta}) = a + b\cos\theta$, where a and b are integers to be determined. **[2]**

(ii) By considering the sum to infinity of the series $z + z^2 + z^3 + \dots$, show that

$$\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \frac{1}{27}\sin 3\theta + \dots = \frac{3\sin\theta}{10 - 6\cos\theta}.$$ **[6]**

i) $|OP_1| = \frac{1}{3}$ since magnitude of $re^{i\theta}$ is r

$|OP_{n+1}| = \left(\frac{1}{3}\right)^{n+1}$, $|OP_n| = \left(\frac{1}{3}\right)^n$

So $|OP_{n+1}| : |OP_n| = \left(\frac{1}{3}\right)^{n+1} : \left(\frac{1}{3}\right)^n$

$= \frac{1}{3} : 1 = 1 : 3$

ii) $\angle P_{n+1}OP_n = \arg(P_{n+1}) - \arg(P_n)$

$= (n+1)\theta - n\theta$

$= \theta$

$\arg(re^{i\theta}) = \theta$ by definition.

bi) $(3-e^{i\theta})(3-e^{-i\theta}) = 9 + 1 - 3(e^{i\theta} + e^{-i\theta})$

$= 10 - 3(2\cos\theta)$

$= 10 - 6\cos\theta$ $a = 10$, $b = -6$

**

ii) $z + z^2 + \ldots = \dfrac{z}{1-z}$ $(*)$ $\left(S_\infty = \dfrac{a}{1-r}\right)$

Now let $C = \frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta + \frac{1}{27}\cos 3\theta + \ldots$

& $S = \frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \frac{1}{27}\sin 3\theta + \ldots$

$C + iS = \frac{1}{3}(\cos\theta + i\sin\theta) + \frac{1}{9}(\cos 2\theta + i\sin 2\theta) + \ldots$

$= z + z^2 + z^3 + \ldots$ where $z = \frac{1}{3}(\cos\theta + i\sin\theta)$

$= \dfrac{\frac{1}{3}e^{i\theta}}{1 - \frac{1}{3}e^{i\theta}} \times \dfrac{3}{3}$ by $(*)$

$= \dfrac{e^{i\theta}}{3 - e^{i\theta}} \times \dfrac{3 - e^{-i\theta}}{3 - e^{-i\theta}}$

$= \dfrac{3e^{i\theta} - 1}{10 - 6\cos\theta}$ ← by bi)

$= \dfrac{3\cos\theta + 3i\sin\theta - 1}{10 - 6\cos\theta}$

So $\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \ldots = \mathrm{Im}(C + iS)$

$= \dfrac{3\sin\theta}{10 - 6\cos\theta}$ $\square$

**

12 **(a)** Show that $(2-e^{i\theta})(2-e^{-i\theta}) = 5-4\cos\theta$. **[3]**

Series C and S are defined by

$$C = \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \ldots + \frac{1}{2^n}\cos n\theta,$$

$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots + \frac{1}{2^n}\sin n\theta.$$

(b) Show that $C = \dfrac{2^n(2\cos\theta - 1) - 2\cos(n+1)\theta + \cos n\theta}{2^n(5-4\cos\theta)}$. **[9]**

a) $(2-e^{i\theta})(2-e^{-i\theta}) = 4+1-2(e^{i\theta}+e^{-i\theta})$

$= 5 - 2(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta))$

$= 5 - 2(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)$

$= 5 - 2(2\cos\theta)$

$= 5 - 4\cos\theta$

b) $C + iS = \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \ldots$

$+ \frac{1}{2^n}(\cos n\theta + i\sin n\theta)$

$= \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \ldots + \frac{1}{2^n}e^{in\theta}$

$= \dfrac{\frac{1}{2}e^{i\theta}\left(1-\left(\frac{1}{2}e^{i\theta}\right)^n\right)}{1-\frac{1}{2}e^{i\theta}}$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$$= \frac{\frac{1}{2} e^{i\theta}\left(1 - \frac{1}{2^n} e^{in\theta}\right)}{1 - \frac{1}{2} e^{i\theta}} \times \frac{2}{2}$$

$$= \frac{e^{i\theta}\left(1 - \frac{1}{2^n} e^{in\theta}\right)}{2 - e^{i\theta}} \times \frac{2 - e^{-i\theta}}{2 - e^{-i\theta}}$$

$$= \frac{\frac{1}{2^n} e^{i\theta}\left(2^n - e^{in\theta}\right)\left(2 - e^{-i\theta}\right)}{5 - 4\cos\theta}$$ by factoring out $\frac{1}{2^n}$

$$= \frac{e^{i\theta}\left(2^{n+1} - 2e^{in\theta} - 2^n e^{-i\theta} + e^{i(n-1)\theta}\right)}{2^n(5 - 4\cos\theta)}$$

$$= \frac{2^{n+1} e^{i\theta} - 2e^{i(n+1)\theta} - 2^n + e^{in\theta}}{2^n(5-4\cos\theta)}$$

$$= \frac{2^{n+1}(\cos\theta + i\sin\theta) - 2\left(\cos(n+1)\theta + i\sin(n+1)\theta\right) - 2^n + \cos n\theta + i\sin n\theta}{2^n(5-4\cos\theta)}$$

$$C = \mathrm{Re}\,(C + iS)$$

$$= \frac{2^{n+1}\cos\theta - 2\cos(n+1)\theta - 2^n + \cos n\theta}{2^n(5-4\cos\theta)}$$

$$= \frac{2^n(2\cos\theta - 1) - 2\cos(n+1) + \cos n\theta}{2^n(5-4\cos\theta)}$$

$\square$

***

13 **(a)** **(i)** Show that

$$1 + e^{i2\theta} = 2\cos\theta(\cos\theta + i\sin\theta).$$ **[2]**

(ii) The series C and S are defined as follows.

$$C = 1 + \binom{n}{1}\cos 2\theta + \binom{n}{2}\cos 4\theta + \ldots + \cos 2n\theta$$

$$S = \binom{n}{1}\sin 2\theta + \binom{n}{2}\sin 4\theta + \ldots + \sin 2n\theta$$

By considering $C + iS$, show that

$$C = 2^n \cos^n\theta \cos n\theta,$$

and find a corresponding expression for S. **[7]**

i) $1 + e^{i2\theta} = 1 + \cos 2\theta + i\sin 2\theta$

$= 1 + 2\cos^2\theta - 1 + 2i\sin\theta\cos\theta$

$= 2\cos^2\theta + 2i\sin\theta\cos\theta$

$= 2\cos\theta(\cos\theta + i\sin\theta)$

ii) $C + iS = 1 + \binom{n}{1}(\cos 2\theta + i\sin 2\theta) + \binom{n}{2}(\cos 4\theta + i\sin 4\theta) + \ldots$

$+ \binom{n}{n}(\cos 2n\theta + i\sin 2n\theta)$

$= 1 + \binom{n}{1}e^{2i\theta} + \binom{n}{2}e^{4i\theta} + \ldots + \binom{n}{n}e^{2ni\theta}$

$= (1 + e^{2i\theta})^n$ by Binomial Theorem

$= (2\cos\theta(\cos\theta + i\sin\theta))^n$

$= 2^n\cos^n\theta(\cos n\theta + i\sin n\theta)$ by De Moivre's Theorem

$= 2^n\cos^n\theta\cos n\theta + 2^n i\cos^n\theta\sin n\theta$

$$C = \mathrm{Re}(C+iS) = 2^n \cos^n\theta \cos n\theta$$

$$S = \mathrm{Im}(C+iS) = 2^n \cos^n\theta \sin n\theta$$

 ****

14 **(a)** **(i)** Express $2\sin\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta - \mathrm{i}\cos\frac{1}{2}\theta\right)$ in terms of z where $z = \cos\theta + \mathrm{i}\sin\theta$. **[3]**

(ii) The series C and S are defined as follows.

$$C = 1 - \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta - \ldots + (-1)^n\binom{n}{n}\cos n\theta$$

$$S = -\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta - \ldots + (-1)^n\binom{n}{n}\sin n\theta$$

Show that

$$C + \mathrm{i}S = \left\{-2\mathrm{i}\sin\tfrac{1}{2}\theta\left(\cos\tfrac{1}{2}\theta + \mathrm{i}\sin\tfrac{1}{2}\theta\right)\right\}^n.$$

Hence show that, for even values of n,

$$\frac{C}{S} = \cot\left(\tfrac{1}{2}n\theta\right).$$ **[8]**

ai) $2\sin\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta - i\cos\frac{1}{2}\theta\right)$

$= 2\sin^2\frac{1}{2}\theta - 2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$

$\Rightarrow 1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$

$= 1 - \cos\theta - i\sin\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 1 - (\cos\theta + i\sin\theta)$

$= 1 - z$

ii) $C + iS = 1 - \binom{n}{1}(\cos\theta + i\sin\theta) + \binom{n}{2}(\cos 2\theta + i\sin 2\theta) + \ldots$

$+ (-1)^n\binom{n}{n}(\cos n\theta + i\sin n\theta)$

$= 1 - \binom{n}{1}e^{i\theta} + \binom{n}{2}e^{2i\theta} + \ldots + (-1)^n e^{in\theta}$

$= (1 - e^{i\theta})^n$

$= (1 - z)^n$

$$= \left(2\sin\tfrac{1}{2}\theta\left(\sin\tfrac{1}{2}\theta - i\cos\tfrac{1}{2}\theta\right)\right)^n$$ by (i)

$$= \left(-2i^2\sin\tfrac{1}{2}\theta\left(\sin\tfrac{1}{2}\theta - i\cos\tfrac{1}{2}\theta\right)\right)^n$$ $-i^2 = 1$

$$= \left(-2i\sin\left(\tfrac{1}{2}\theta\right)\left(i\sin\tfrac{1}{2}\theta - i\times i\cos\tfrac{1}{2}\theta\right)\right)^n$$

I split i^2 up as $i \times i$ and multiply one of the i's in to the brackets.

$$= \left(-2i\sin\left(\tfrac{1}{2}\theta\right)\left(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta\right)\right)^n$$

$$= (-2i)^n\sin^n\left(\tfrac{1}{2}\theta\right)\left(\cos\tfrac{1}{2}n\theta + i\sin\tfrac{1}{2}n\theta\right)$$

if n is even,

note $i^n = \pm 1$ if n is even

$$C + iS = 2^n i^n \sin^n\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}n\theta\right) + 2^n i^n i \sin^n\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}n\theta\right)$$

$$C = \operatorname{Re}(C + iS) = 2^n i^n \sin^n\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}n\theta\right)$$

$$S = \operatorname{Im}(C + iS) = 2^n i^n \sin^n\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}n\theta\right)$$

$$\frac{C}{S} = \frac{\cancel{2^n}\cancel{i^n}\cancel{\sin^n\left(\tfrac{1}{2}\theta\right)}\cos\left(\tfrac{1}{2}n\theta\right)}{\cancel{2^n}\cancel{i^n}\cancel{\sin^n\left(\tfrac{1}{2}\theta\right)}\sin\left(\tfrac{1}{2}n\theta\right)} = \cot\left(\tfrac{1}{2}n\theta\right)$$

****

15 Let $S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{10i\theta}$.

(i) (a) Show that, for $\theta \neq 2n\pi$, where n is an integer,

$$S = \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}.$$ **[4]**

(b) State the value of S for $\theta = 2n\pi$, where n is an integer. **[1]**

(ii) Hence show that, for $\theta \neq 2n\pi$, where n is an integer,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}.$$ **[3]**

(iii) Hence show that $\theta = \frac{1}{11}\pi$ is a root of $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = 0$ and find another root in the interval $0 < \theta < \frac{1}{4}\pi$. **[4]**

i)a

$$S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{10i\theta}$$

$$= \frac{e^{i\theta}\left(1 - \left(e^{i\theta}\right)^{10}\right)}{1 - e^{i\theta}}$$

$$= \frac{e^{i\theta}\left(1 - e^{10i\theta}\right)}{1 - e^{i\theta}} \times \frac{-e^{-\frac{1}{2}i\theta}}{-e^{-\frac{1}{2}i\theta}}$$

$$= \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}\theta}}$$

$$= \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{\cos\left(\frac{1}{2}\theta\right) + i\sin\left(\frac{1}{2}\theta\right) - \cos\left(-\frac{1}{2}\theta\right) + i\sin\left(-\frac{1}{2}\theta\right)}$$

$$= \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}$$

since $i\sin\left(-\frac{1}{2}\theta\right) = -i\sin\left(\frac{1}{2}\theta\right)$

& $\cos\left(-\frac{1}{2}\theta\right) = \cos\left(\frac{1}{2}\theta\right)$

i b) if $\theta = 2n\pi$

$$S = e^{2n\pi i} + e^{4n\pi i} + \dots + e^{20n\pi i}$$

but $\cos(2n\pi) = 1$ & $\sin(2n\pi) = 0$ for all $n \in \mathbb{Z}$.

$$\therefore S = 1 + 1 + \dots + 1 = 10$$

ii) where $\theta \neq 2n\pi$, $S = \dfrac{e^{\frac{1}{2}i\theta}(e^{10i\theta} - 1)}{2i\sin(\frac{1}{2}\theta)} \times \dfrac{-i}{-i}$

$$= \frac{-ie^{\frac{21}{2}i\theta} + ie^{\frac{1}{2}i\theta}}{2\sin(\frac{1}{2}\theta)}$$

$$= \frac{-i\cos(\frac{21}{2}\theta) + \sin(\frac{21}{2}\theta) + i\cos(\frac{1}{2}\theta) - \sin(\frac{1}{2}\theta)}{2\sin(\frac{1}{2}\theta)}$$

$\cos\theta + \cos(2\theta) + \dots + \cos(10\theta) = \mathrm{Re}(S)$

$$= \frac{\sin(\frac{21}{2}\theta) - \sin(\frac{1}{2}\theta)}{2\sin(\frac{1}{2}\theta)}$$

$$\boxed{= \frac{\sin(\frac{21}{2}\theta)}{2\sin(\frac{1}{2}\theta)} - \frac{1}{2}}$$

iii) $\dfrac{\sin(\frac{21}{2} \times \frac{1}{11}\pi)}{2\sin(\frac{1}{2} \times \frac{1}{11}\pi)} - \dfrac{1}{2} = \dfrac{\sin(\frac{21}{22}\pi)}{2\sin(\frac{1}{22}\pi)} - \dfrac{1}{2}$

$$= \frac{\cancel{\sin(\frac{1}{22}\pi)}}{2\cancel{\sin(\frac{1}{22}\pi)}} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

$\sin(\pi - \theta) \equiv \sin(\theta)$

to find other root

$$\frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2} = 0$$

$$\Longleftrightarrow \quad \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} = \frac{1}{2}$$

$$\sin\left(\frac{21}{2}\theta\right) = \sin\left(\tfrac{1}{2}\theta\right)$$

$\frac{21}{2}\theta = \frac{1}{2}\theta$ $\quad$ OR $\quad$ $\frac{21}{2}\theta = \pi - \frac{1}{2}\theta$ $\quad$ OR $\quad$ $\frac{21}{2}\theta = \frac{1}{2}\theta + 2\pi$

$\theta = 0$

not valid

$11\theta = \pi$

$\theta = \frac{1}{11}\pi$

used already

$10\theta = 2\pi$

$$\theta = \frac{1}{5}\pi$$

 *****

16 Let $C = \sum_{r=0}^{20} \binom{20}{r} \cos r\theta$. Show that $C = 2^{20}\cos^{20}\left(\frac{1}{2}\theta\right)\cos 10\theta$. **[8]**

let $S = \sum_{r=0}^{20} \binom{20}{r} \sin r\theta$

$$C + iS = \sum_{r=0}^{20} \binom{20}{r} \cos r\theta + i \sum_{r=0}^{20} \binom{20}{r} \sin r\theta$$

$$= \sum_{r=0}^{20} \binom{20}{r} (\cos r\theta + i \sin r\theta)$$

$$= \sum_{r=0}^{20} \binom{20}{r} e^{ir\theta}$$

$$= (1 + e^{i\theta})^{20}$$

by the Binomial Theorem

It might help to write the terms out if you're not sure where this has come from.

This is where the problem becomes tricky again.

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow \cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

so $(1 + e^{i\theta})^{20} = \left(1 + 2\cos^2\left(\frac{\theta}{2}\right) - 1 + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{20}$

$$= \left(2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right)^{20}$$

$$= 2^{20}\cos^{20}\left(\frac{\theta}{2}\right)\left(\cos 10\theta + i\sin 10\theta\right)$$ by De Moivre's Theorem.

$$= 2^{20}\cos^{20}\left(\frac{\theta}{2}\right)\cos 10\theta + 2^{20} i\cos^{20}\left(\frac{\theta}{2}\right)\sin 10\theta$$

$$C = \mathrm{Re}(C + iS)$$

$$= 2^{20}\cos^{20}\left(\tfrac{1}{2}\theta\right)\cos 10\theta$$

$\square$

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17 **(i)** Show that, if $z \neq \pm 1$ and $z \neq 0$,

$$\sum_{r=1}^{n} z^{2r-1} = \frac{1-z^{2n}}{z^{-1}-z}. \quad \textbf{[2]}$$

(ii) Hence show that, if $\sin\theta \neq 0$,

$$\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}. \quad \textbf{[6]}$$

(iii) Hence find the exact value of

$$\int_0^{\frac{1}{6}\pi} \frac{\sin^2 3\theta}{\sin\theta}\,d\theta. \quad \textbf{[3]}$$

i) $\sum_{r=1}^{n} z^{2r-1} = z + z^3 + z^5 + \dots + z^{2n-1}$

$$= \frac{z(1-(z^2)^n)}{1-z^2}$$

$$= \frac{z(1-z^{2n})}{1-z^2} \times \frac{1/z}{1/z}$$

$$= \frac{1-z^{2n}}{z^{-1}-z}$$

ii) let $z = \cos\theta + i\sin\theta$

$$C+iS = \sum_{r=1}^{n} z^{2r-1} = \frac{1-z^{2n}}{z^{-1}-z} = \frac{1-e^{2in\theta}}{\cos(-\theta)+i\sin(-\theta)-\cos(\theta)-i\sin(\theta)}$$

$$= \frac{1-e^{2in\theta}}{-2i\sin(\theta)} \times \frac{i}{i}$$ seen similar

$$= \frac{i-ie^{2in\theta}}{2\sin\theta}$$

$$= \frac{i-i\cos(2n\theta)+\sin(2n\theta)}{2\sin(\theta)}$$

P.T.O

but $S := \sum_{r=1}^{n} \sin(2r-1)\theta$ is just $\text{Im}(C+iS)$

$$\therefore\ S = \frac{1-\cos(2n\theta)}{2\sin(\theta)}$$

$$= \frac{1-(1-2\sin^2(n\theta))}{2\sin(\theta)} \qquad \left(\text{since } \cos(2A) = 1-2\sin^2(A)\right)$$

$$= \frac{2\sin^2(n\theta)}{2\sin(\theta)}$$

$$= \frac{\sin^2(n\theta)}{\sin(\theta)} \quad \text{as required.}$$

iii) $$\int_0^{\frac{1}{6}\pi} \frac{\sin^2(3\theta)}{\sin(\theta)} = \int_0^{\frac{1}{6}\pi} \sum_{r=1}^{3} \sin((2r-1)\theta)\ d\theta$$

$$= \int_0^{\frac{1}{6}\pi} (\sin\theta + \sin(3\theta) + \sin(5\theta))\ d\theta$$

$$= \left[-\cos\theta - \frac{1}{3}\cos(3\theta) - \frac{1}{5}\cos(5\theta)\right]_0^{\frac{1}{6}\pi}$$

$= \ldots$

$$= \left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{10}\right) - \left(-1 - \frac{1}{3} - \frac{1}{5}\right)$$

$$= \boxed{\frac{1}{15}(23-6\sqrt{3})} \quad \text{or} \quad \boxed{\frac{23}{15} - \frac{2\sqrt{3}}{5}}$$

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18 Series C and S are defined by

$$C = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \binom{n}{3}\cos 3 \ + \ldots + \binom{n}{n}\cos n\theta,$$

$$S = \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \binom{n}{3}\sin 3 \ + \ldots + \binom{n}{n}\sin n\theta.$$

(ii) Find C and S, and show that $\frac{S}{C} = \tan\frac{1}{2}n\theta$. [7]

$$C + iS = 1 + \binom{n}{1}(\cos\theta + i\sin\theta) + \binom{n}{2}(\cos 2\theta + i\sin 2\theta) + \cdots + \binom{n}{n}(\cos n\theta + i\sin n\theta)$$

$$= 1 + \binom{n}{1}e^{i\theta} + \binom{n}{2}e^{2i\theta} + \cdots + \binom{n}{n}e^{in\theta}$$

$$= (1 + e^{i\theta})^n$$ by the Binomial Theorem

This is where it gets tricky. I need to write $1 + e^{i\theta}$ as a product of terms so I can distribute the n easily. Idea (Similar to Jan 13 OCR MEI question):

$$\cos 2\theta \equiv 2\cos^2\theta - 1$$

$$\Rightarrow \cos\theta \equiv 2\cos^2\frac{\theta}{2} - 1$$

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

$$\Rightarrow \sin\theta \equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

So $$(1 + e^{i\theta})^n = \left(1 + 2\cos^2\frac{\theta}{2} - 1 + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$= \left(2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right)^n$$

So $C + iS = \left(\cos\frac{\theta}{2} \times e^{i\frac{\theta}{2}}\right)^n$

$$= \cos^n\left(\frac{\theta}{2}\right) e^{i\frac{n\theta}{2}}$$

$$= \cos^n\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{n}{2}\theta\right) + i\sin\left(\frac{n}{2}\theta\right)\right)$$

$$= \cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n}{2}\theta\right) + i\cos^n\left(\frac{\theta}{2}\right)\sin\left(\frac{n}{2}\theta\right)$$

so $C = \operatorname{Re}(C+iS) = \cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n}{2}\theta\right)$

$S = \operatorname{Im}(C+iS) = \cos^n\left(\frac{\theta}{2}\right)\sin\left(\frac{n}{2}\theta\right)$

$$\frac{S}{C} = \frac{\cancel{\cos^n}\left(\frac{\theta}{2}\right)\sin\left(\frac{n}{2}\theta\right)}{\cancel{\cos^n}\left(\frac{\theta}{2}\right)\cos\left(\frac{n}{2}\theta\right)} = \frac{\sin\left(\frac{n}{2}\theta\right)}{\cos\left(\frac{n}{2}\theta\right)}$$

$$= \tan\left(\frac{1}{2}n\theta\right)$$

$\square$