London Academy of Maths

Consistency of Solutions

Question Paper

1 The equations of three planes are

$$\begin{aligned} 2x + y + 3z &= 3, \\ 3x - y - 2z &= 2, \\ -4x + 3y + 7z &= k, \end{aligned}$$

where k is a constant.

(a) By considering a suitable determinant, show that the planes do **not** meet at a single point. **[2]**

(b) Given that the planes form a sheaf, determine the value of k. **[4]**

2 Three planes have equations

$$x + 2y - 3z = 0,$$
$$-x + 3y - 2z = 0,$$
$$x - 2y + kz = k,$$

where k is a constant.

(a) For the case $k = 0$, the origin lies on all three planes.

Use a determinant to explain whether there are any other points that lie on all three planes in this case. **[2]**

(b) You are now given that $k = 1$.

(i) Show that there are no points that lie on all three planes. **[3]**

(ii) Describe the geometrical arrangement of the three planes. **[1]**

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3 You are given the matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} a & 2 & 0 \\ 0 & a & 2 \\ 4 & 5 & 1 \end{pmatrix}$.

(a) Find, in terms of a, the determinant of $\mathbf{A}$, simplifying your answer. **[2]**

(b) Hence find the values of a for which $\mathbf{A}$ is singular. **[2]**

You are given the following equations which are to be solved simultaneously.

$$\begin{aligned} ax + 2y \phantom{{}+2z} &= 6 \\ ay + 2z &= 8 \\ 4x + 5y + z &= 16 \end{aligned}$$

(c) For each of the values of a found in part **(b)** determine whether the equations have

- a unique solution, which should be found, or
- an infinite set of solutions or
- no solution. **[7]**

4 **(i)** Find the solution to the following simultaneous equations.

$$\begin{aligned} x + \quad y + \quad z &= 3 \\ 2x + 4y + 5z &= 9 \\ 7x + 11y + 12z &= 20 \end{aligned}$$

[2]

(ii) Determine the values of p and k for which there are an infinity of solutions to the following simultaneous equations.

$$\begin{aligned} x + \quad y + \quad z &= 3 \\ 2x + 4y + 5z &= 9 \\ 7x + 11y + pz &= k \end{aligned}$$

[6]

5 Three planes have equations

$$kx + y - 2z = 0$$
$$2x + 3y - 6z = -5$$
$$3x - 2y + 5z = 1$$

where k is a constant.

Investigate the arrangement of the planes for each of the following cases. If in either case the planes meet at a unique point, find the coordinates of that point.

(a) $k = -1$ **[3]**

(b) $k = \frac{2}{3}$ **[4]**

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6.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix **M** has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$\begin{aligned} 2x - y + z &= p \\ 3x - 6y + 4z &= 1 \\ 3x + 2y - z &= 0 \end{aligned}$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$\begin{aligned} 2x - y + z &= 1 \\ 3x - 5y + 4z &= q \\ 3x + 2y - z &= 0 \end{aligned}$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

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7 The equations of three planes are

$$\begin{aligned} -4x + ky + 7z &= 4, \\ x - 2y + 5z &= l, \\ 2x + 3y + z &= 2. \end{aligned}$$

Given that the planes form a sheaf, determine the values of k and l. **[6]**

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8 Three planes have equations

$$\begin{aligned} -x + ay \quad\; &= 2 \\ 2x + 3y + z &= -3 \\ x + by + z &= c \end{aligned}$$

where a, b and c are constants.

(a) In the case where the planes **do not** intersect at a unique point,

(i) find b in terms of a, **[4]**

(ii) find the value of c for which the planes form a sheaf. **[3]**

(b) In the case where $b = a$ and $c = 1$, find the coordinates of the point of intersection of the planes in terms of a. **[6]**

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9 Matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} k & 1 & -5 \\ 2 & 3 & -3 \\ -1 & 2 & 2 \end{pmatrix}$, where k is a constant.

(i) Show that $\det \mathbf{M} = 12(k-3)$. **[2]**

(ii) Find a solution of the following simultaneous equations for which $x \neq z$.

$$\begin{aligned} 4x^2 + y^2 - 5z^2 &= 6 \\ 2x^2 + 3y^2 - 3z^2 &= 6 \\ -x^2 + 2y^2 + 2z^2 &= -6 \end{aligned}$$

[3]

(iii) **(*A*)** Verify that the point $(2, 0, 1)$ lies on each of the following three planes.

$$\begin{aligned} 3x + y - 5z &= 1 \\ 2x + 3y - 3z &= 1 \\ -x + 2y + 2z &= 0 \end{aligned}$$

[1]

(*B*) Describe how the three planes in part **(iii)** (*A*) are arranged in 3-D space. Give reasons for your answer. **[4]**

(iv) Find the values of k for which the transformation represented by $\mathbf{M}$ has a volume scale factor of 6. **[3]**

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10 Three planes have equations

$$\begin{aligned} -x + 2y + z &= 0 \\ 2x - \ y - z &= 0 \\ x + \ y \quad &= a \end{aligned}$$

where a is a constant.

(i) Investigate the arrangement of the planes:

- when $a = 0$;
- when $a \neq 0$. **[6]**

(ii) Chris claims that the position vectors $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $2\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ lie in a plane. Determine whether or not Chris is correct. **[2]**

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11 You are given that a is a parameter which can take only real values.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 2 & 4 & -6 \\ -3 & 10-4a & 9 \\ 7 & 4 & 4 \end{pmatrix}$.

(a) Find an expression for the determinant of **A** in terms of a. **[2]**

You are given the following system of equations in x, y and z.

$$\begin{aligned} 2x + \quad\quad 4y - 6z &= 6 \\ -3x + (10-4a)y + 9z &= -9 \\ 7x + \quad\quad 4y + 4z &= 11 \end{aligned}$$

The system can be written in the form $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 11 \end{pmatrix}$.

(b) (i) In the case where **A** is **not** singular, solve the given system of equations by using $\mathbf{A}^{-1}$. **[5]**

(ii) In the case where **A is** singular describe the configuration of the planes whose equations are the three equations of the system. **[3]**

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12.

$$\mathbf{M} = \begin{pmatrix} k & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Given that $k \neq 4$, find, in terms of k, the inverse of the matrix $\mathbf{M}$.

(4)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect.

$$\begin{aligned} 2x + 5y + 7z &= 1 \\ x + y + z &= p \\ 2x + y - z &= 2 \end{aligned}$$

(3)

(c) (i) Find the value of q for which the following planes intersect in a straight line.

$$\begin{aligned} 4x + 5y + 7z &= 1 \\ x + y + z &= q \\ 2x + y - z &= 2 \end{aligned}$$

(ii) For this value of q, determine a vector equation for the line of intersection.

(7)