London Academy of Maths

Consistency of Solutions

Worked Solutions

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1 The equations of three planes are

$$2x + y + 3z = 3,$$
$$3x - y - 2z = 2,$$
$$-4x + 3y + 7z = k,$$

where k is a constant.

(a) By considering a suitable determinant, show that the planes do **not** meet at a single point. **[2]**

(b) Given that the planes form a sheaf, determine the value of k. **[4]**

a)

$$M = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ -4 & 3 & 7 \end{pmatrix} \qquad \det M = 2(-7+6) - 1(21-8) + 3(9-4)$$
$$= -2 - 13 + 15 = 0$$

so $\det M = 0$

$\Rightarrow$ the planes don't intersect at a unique point

b) Easiest variable to eliminate is y:

A $\quad 2x + y + 3z = 3,$

B $\quad 3x - y - 2z = 2,$

C $\quad -4x + 3y + 7z = k,$

A + B : $5x + z = 5$

3B + C : $5x + z = 6 + k$

$\Rightarrow 5 = 6 + k$

So $\boxed{k = -1}$

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2 Three planes have equations

$$x+2y-3z=0,$$
$$-x+3y-2z=0,$$
$$x-2y+kz=k,$$

where k is a constant.

(a) For the case $k=0$, the origin lies on all three planes.

Use a determinant to explain whether there are any other points that lie on all three planes in this case. **[2]**

(b) You are now given that $k=1$.

(i) Show that there are no points that lie on all three planes. **[3]**

(ii) Describe the geometrical arrangement of the three planes. **[1]**

a) $M=\begin{pmatrix} 1 & 2 & -3 \\ -1 & 3 & -2 \\ 1 & -2 & k \end{pmatrix}$

$\det M = 1(3k-4)-2(-k+2)-3(2-3)$

$= 5k-5$

if $k=0$ $\det M = -5 \neq 0$

so planes intersect at a unique point,

so no other point lies on all three planes.

bi) $\det M = 5(1)-5 = 0$ (when $k=1$)

(A) $x+2y-3z=0$

(B) $-x+3y-2z=0$

(C) $x-2y+z=1$

eliminate either x, y or z from two pairs of equations. Here, x is the easiest to eliminate. P.T.O

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(A) + (B) : $5y - 5z = 0$
$\Rightarrow y = z$

(B) + (C) : $y - z = 1$
$\Rightarrow y = z + 1$

$y = z$ and $y = z + 1$ are not scalar multiples of each other so the equations are inconsistent, so the planes have no common point

b) ii) The planes form a prism
(prismatic intersection)

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3 You are given the matrix **A** where $\mathbf{A} = \begin{pmatrix} a & 2 & 0 \\ 0 & a & 2 \\ 4 & 5 & 1 \end{pmatrix}$.

(a) Find, in terms of a, the determinant of **A**, simplifying your answer. **[2]**

(b) Hence find the values of a for which **A** is singular. **[2]**

You are given the following equations which are to be solved simultaneously.

$$\begin{aligned} ax + 2y \quad\quad &= 6 \\ ay + 2z &= 8 \\ 4x + 5y + z &= 16 \end{aligned}$$

(c) For each of the values of a found in part **(b)** determine whether the equations have

- a unique solution, which should be found, or
- an infinite set of solutions or
- no solution. **[7]**

a) $\det A = a(a-10) - 2(-8)$

$= a^2 - 10a + 16$

b) A is singular $\Leftrightarrow \det A = 0$

$\Leftrightarrow a^2 - 10a + 16 = 0$

$(a-8)(a-2) = 0$

$a = 8$ or 2

c) Since $\det A = 0$ for $a = 8$ and $a = 2$, we know there is no unique solution for either set of equations

P.T.O.

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$a=8$:

Ⓐ $8x+2y = 6$

Ⓑ $8y+2z=8$

Ⓒ $4x+5y+z=16$

eliminate either x or z (as it's easier than eliminating y).

$2Ⓒ - Ⓐ$: $8y+2z=26$

Ⓑ : $8y+2z=8$

These are not scalar multiples of eachother. The equations are inconsistent so there is no solution

(the planes form a prism).

$a=2$:

$$\begin{aligned} ax + 2y \quad &= 6 \\ ay + 2z &= 8 \\ 4x + 5y + z &= 16 \end{aligned}$$

Ⓐ $2x+2y = 6$

Ⓑ $2y+2z=8$

Ⓒ $4x+5y+z=16$

eliminate either x or z (as it's easier than eliminating y).

$Ⓒ - 2Ⓐ$: $y+z=4 \Rightarrow 2y+2z=8$

Ⓑ : $2y+2z=8$

So the equations are consistent

$\Rightarrow$ there is an infinite set of solutions

(the planes form a sheaf).

4 **(i)** Find the solution to the following simultaneous equations.

$$\begin{aligned} x + \quad y + \quad z &= 3 \\ 2x + 4y + 5z &= 9 \\ 7x + 11y + 12z &= 20 \end{aligned}$$

[2]

(ii) Determine the values of p and k for which there are an infinity of solutions to the following simultaneous equations.

$$\begin{aligned} x + \quad y + \quad z &= 3 \\ 2x + 4y + 5z &= 9 \\ 7x + 11y + pz &= k \end{aligned}$$

[6]

i) $$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 7 & 11 & 12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 7 & 11 & 12 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 9 \\ 20 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ -9 \\ 7 \end{pmatrix}}$$

ii) infinity of solutions

$\Rightarrow \det M = 0$ and planes form sheaf.

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 7 & 11 & p \end{pmatrix} \qquad \det M = 1(4p-55) - 1(2p-35) + 1(22-28)$$

$$= 2p - 26$$

so $\det M = 0 \Rightarrow \boxed{p = 13}$

P.T.O.

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Ⓐ $x + y + z = 3$

Ⓑ $2x + 4y + 5z = 9$

Ⓒ $7x + 11y + 13z = k$

As usual I need to eliminate either x, y or z from two pairs of equations. I'll pick x here.

Ⓑ $- 2$Ⓐ: $2y + 3z = 3$ $\times 2$ $4y + 6z = 6$

Ⓒ $- 7$Ⓐ: $4y + 6z = k - 21$

for consistency of equations,

$$6 = k - 21 \implies \boxed{k = 27}$$

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5 Three planes have equations

$$kx + y - 2z = 0$$
$$2x + 3y - 6z = -5$$
$$3x - 2y + 5z = 1$$

where k is a constant.

Investigate the arrangement of the planes for each of the following cases. If in either case the planes meet at a unique point, find the coordinates of that point.

(a) $k = -1$ **[3]**

(b) $k = \frac{2}{3}$ **[4]**

$$M = \begin{pmatrix} k & 1 & -2 \\ 2 & 3 & -6 \\ 3 & -2 & 5 \end{pmatrix}$$

$$\det M = k(15-12) - 1(10+18) - 2(-4-9)$$
$$= 3k - 2$$

a) $k=-1$, $\det M = -5 \neq 0$

so the planes meet at a unique point.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$ by calculator

So point of intersection is $(-1, 3, 2)$

b) $k = 2/3$, $\det M = 0$

So planes don't meet at a unique point

P-T-O.

Ⓐ $\frac{2}{3}x + y - 2z = 0$

Ⓑ $2x + 3y - 6z = -5$

Ⓒ $3x - 2y + 5z = 1$

Notice that Ⓐ $\times 3$ is $2x + 3y - 6z = 0$

so Ⓐ and Ⓑ are parallel (but not identical)

so the first two planes are parallel and intersected by the third plane.

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6.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix **M** has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$\begin{aligned} 2x - y + z &= p \\ 3x - 6y + 4z &= 1 \\ 3x + 2y - z &= 0 \end{aligned}$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$\begin{aligned} 2x - y + z &= 1 \\ 3x - 5y + 4z &= q \\ 3x + 2y - z &= 0 \end{aligned}$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

a) $\det M = 0$

$\Rightarrow 2(-k-8) + 1(-3-12) + 1(6-3k) = 0$

$-2k - 16 - 15 + 6 - 3k = 0$

$-5k - 25 = 0$

$5k = -25$

$k = -5$

$\boxed{\text{so for } k \neq -5 \quad M \text{ has an inverse.}}$

b) we are solving $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$

where $k = -6$ for M.

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by multiplicating both sides (on the left) by M^{-1}, we get

$$M^{-1}M\begin{pmatrix}x\\y\\z\end{pmatrix}=M^{-1}\begin{pmatrix}p\\1\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=M^{-1}\begin{pmatrix}p\\1\\0\end{pmatrix}$$

Note, the question doesn't mention anything about detailed reasoning so we can use a calculator to find M^{-1}.

$$M^{-1}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-12 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}\begin{pmatrix}p\\1\\0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-12p+1\\ 15p-5\\ 24p-7\end{pmatrix}$$

Question wants the coordinates of intersection:

$$(x,y,z)=\left(\frac{-12p+1}{5},\ 3p-1,\ \frac{24p-7}{5}\right)$$

c i) Determinant $=0$, by (a) so unique solution can't exist.

Either there are no solutions, or infinitely many solutions.

we want the latter. (infinitely many).

Ⓐ $2x-y+z=1$

Ⓑ $3x-5y+4z=q$

Ⓒ $3x+2y-z=0$

$4\times$Ⓒ : $12x+8y-4z=0$

Ⓐ $+$ Ⓒ : $5x+y=1$

I need to find another equation with only x & y.

Ⓑ $+\ 4\times$Ⓒ : $15x+3y=q$

$\Rightarrow 5x+y=\frac{q}{3}$

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So $1 = \frac{a}{3} \Rightarrow \boxed{a = 3}$

ii) The planes form a sheaf

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7 The equations of three planes are

$$-4x+ky+7z=4,$$
$$x-2y+5z=l,$$
$$2x+3y+z=2.$$

Given that the planes form a sheaf, determine the values of k and l. **[6]**

Planes form a sheaf $\Rightarrow \det M = 0$

where $M = \begin{pmatrix} -4 & k & 7 \\ 1 & -2 & 5 \\ 2 & 3 & 1 \end{pmatrix}$

$\det M = -4(-2-15) - k(1-10) + 7(3+4)$

$= 68 + 9k + 49$

$= 9k + 117 = 0$

$9k = -117$

$\boxed{k = -13}$

Now we eliminate either x, y or z from the equations:

(A) $-4x - 13y + 7z = 4$

(B) $x - 2y + 5z = l$ $\qquad 4\times$(B): $4x - 8y + 20z = 4l$

(C) $2x + 3y + z = 2$ $\qquad 2\times$(C): $4x + 6y + 2z = 4$

(A) $+ 4\times$(B): $-21y + 27z = 4 + 4l$

(A) $+ 2\times$(C): $-7y + 9z = 8$ $\quad \times 3$

$\times 3$ $\quad -21y + 27z = 24$

So $4 + 4l = 24 \Rightarrow 4l = 20 \Rightarrow \boxed{l = 5}$

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8 Three planes have equations

$$-x+ay \quad =2$$
$$2x+3y+z=-3$$
$$x+by+z=c$$

where a, b and c are constants.

(a) In the case where the planes **do not** intersect at a unique point,

(i) find b in terms of a, **[4]**

(ii) find the value of c for which the planes form a sheaf. **[3]**

(b) In the case where $b=a$ and $c=1$, find the coordinates of the point of intersection of the planes in terms of a. **[6]**

a) i) $M = \begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & b & 1 \end{pmatrix}$

$\det M = 0$ since planes don't intersect at unique point.

$\Rightarrow -1(3-b) - a(2-1) = 0$

$\Rightarrow -3+b-a=0$

$\Rightarrow b = a+3$

ii)

(A) $-x+ay \quad =2$

(B) $2x+3y+z=-3$

(C) $x+by+z=c$

eliminate z as it's the easiest.

(B) − (C) : $x+(3-b)y = -3-c$

so $x+(3-b)y=-3-c$ is a scalar multiple of $-x+ay=2$

$\Rightarrow 2 = -1(-3-c)$ since $-x \times -1 = x$

So $2 = 3+c$ $\Rightarrow c = -1$

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b) $M = \begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & a & 1 \end{pmatrix}$

We need the inverse of M.

Matrix of Minors $= \begin{pmatrix} 3-a & 1 & 2a-3 \\ a & -1 & -2a \\ a & -1 & -3-2a \end{pmatrix}$

Matrix of Cofactors $= \begin{pmatrix} 3-a & -1 & 2a-3 \\ -a & -1 & 2a \\ a & 1 & -3-2a \end{pmatrix}$

Matrix of Cofactors^T $= \begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}$

$\det M = -1(3-a) - a(2-1) = -3$

so $M^{-1} = \frac{-1}{3}\begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}$

$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \frac{-1}{3}\begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \frac{-1}{3}\begin{pmatrix} 6+2a \\ 2 \\ -4a-9 \end{pmatrix}$

So coordinates are $\boxed{\left(-\frac{6+2a}{3}, -\frac{2}{3}, \frac{4a+9}{3}\right)}$

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9 Matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} k & 1 & -5 \\ 2 & 3 & -3 \\ -1 & 2 & 2 \end{pmatrix}$, where k is a constant.

(i) Show that $\det \mathbf{M} = 12(k-3)$. **[2]**

(ii) Find a solution of the following simultaneous equations for which $x \neq z$.

$$\begin{aligned} 4x^2 + \ \ y^2 - 5z^2 &= 6 \\ 2x^2 + 3y^2 - 3z^2 &= 6 \\ -x^2 + 2y^2 + 2z^2 &= -6 \end{aligned}$$

[3]

(iii) **(*A*)** Verify that the point $(2, 0, 1)$ lies on each of the following three planes.

$$\begin{aligned} 3x + \ \ y - 5z &= 1 \\ 2x + 3y - 3z &= 1 \\ -x + 2y + 2z &= 0 \end{aligned}$$

[1]

(*B*) Describe how the three planes in part **(iii)** (*A*) are arranged in 3-D space. Give reasons for your answer. **[4]**

(iv) Find the values of k for which the transformation represented by $\mathbf{M}$ has a volume scale factor of 6. **[3]**

i) $\det M = k(3\times 2 + 3\times 2) - 1(2\times 2 - 3\times 1) - 5(2\times 2 + 3\times 1)$

$= 12k - 1 - 35$

$= 12k - 36$

$= 12(k-3)$

ii) $M\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = M^{-1}\begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}$

$$M^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1/12 & 1/4 & 1/6 \\ 7/12 & -3/4 & 5/6 \end{pmatrix}$$

So $$\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = M^{-1} \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix}$$

$\Rightarrow \quad x = \sqrt{6}\,i \quad , y = 0 \quad$ and $\quad z = -\sqrt{6}\,i \qquad$ (since $x \neq z$)

or $\quad x = -\sqrt{6}\,i \quad , y = 0 \quad$ and $z = \sqrt{6}\,i$

iii) A) $\quad$ LHS $= 3(2) + 0 - 5(1) = 6 - 5 = 1 \quad$ = RHS

LHS $= 2(2) + 3(0) - 3(1) = 4 - 3 = 1 \quad$ = RHS

LHS $= -2 + 2(0) + 2(1) = -2 + 2 = 0 \quad$ = RHS

B) $\quad \det M = 0 \quad$ by using (a) and $k = 3$.

so planes don't intersect at unique point.

Ⓐ $\quad 3x + y - 5z = 1$

Ⓑ $\quad 2x + 3y - 3z = 1$

Ⓒ $\quad -x + 2y + 2z = 0$

Ⓐ $+ 3\times$Ⓒ: $\quad 7y + z = 1$

Ⓐ $+ 2\times$Ⓒ: $\quad 7y + z = 1$

so eliminating x gives the same equation. Which means the planes must form a <u>sheaf</u>.

iv) Not related to consistency of solutions but good revision nonetheless.

$\det M = \pm 6 \implies 12(k-3)=6$ or $12(k-3)=-6$

$\implies \boxed{k = 7/2 \quad \text{or} \quad k = 5/2}$

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10 Three planes have equations

$$-x+2y+z=0$$
$$2x-y-z=0$$
$$x+y=a$$

where a is a constant.

(i) Investigate the arrangement of the planes:
- when $a=0$;
- when $a\neq 0$. **[6]**

(ii) Chris claims that the position vectors $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $2\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ lie in a plane. Determine whether or not Chris is correct. **[2]**

i) $M = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$

$\det M = -1(1) - 2(1) + 1(2+1)$

$= -1-2+3 = 0$

so the planes never intersect at a unique point.

(A) $-x+2y+z=0$

(B) $2x-y-z=0$

(C) $x+y=a$

easiest variable to eliminate is z:

(A) + (B) : $x+y=0$

(C) : $x+y=a$

if $a=0$, these equations are consistent and so the planes would form a sheaf.

if $a \neq 0$, these equations are inconsistent (since they're not scalar multiples of eachother) and would form a prism / prismatic intersection.

ii) These are the normals to the three planes. In either case $(a=0, a\neq 0)$ they must lie in the same plane.

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11 You are given that a is a parameter which can take only real values.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 2 & 4 & -6 \\ -3 & 10-4a & 9 \\ 7 & 4 & 4 \end{pmatrix}$.

(a) Find an expression for the determinant of **A** in terms of a. **[2]**

You are given the following system of equations in x, y and z.

$$\begin{aligned} 2x + \quad\quad 4y - 6z &= 6 \\ -3x + (10-4a)y + 9z &= -9 \\ 7x + \quad\quad 4y + 4z &= 11 \end{aligned}$$

The system can be written in the form $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 11 \end{pmatrix}$.

(b) **(i)** In the case where **A** is **not** singular, solve the given system of equations by using $\mathbf{A}^{-1}$. **[5]**

(ii) In the case where **A is** singular describe the configuration of the planes whose equations are the three equations of the system. **[3]**

a) $\det A = 2(40 - 16a - 36) - 4(-12 - 63) - 6(-12 - 70 + 28a)$

$= 2(4 - 16a) - 4(-75) - 6(-82 + 28a)$

$= 8 - 32a + 300 + 492 - 168a$

$= 800 - 200a$

b) $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 11 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 6 \\ -9 \\ 11 \end{pmatrix}$

Need to find A^{-1}:

P.T.O.

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$$\text{Matrix of Minors} = \begin{pmatrix} 4-16a & -75 & 28a-82 \\ 40 & 50 & -20 \\ 96-24a & 0 & 32-8a \end{pmatrix}$$

$$\text{Matrix of Cofactors} = \begin{pmatrix} 4-16a & 75 & 28a-82 \\ -40 & 50 & 20 \\ 96-24a & 0 & 32-8a \end{pmatrix}$$

$$\text{Matrix of Cofactors}^{T} = \begin{pmatrix} 4-16a & -40 & 96-24a \\ 75 & 50 & 0 \\ 28a-82 & 20 & 32-8a \end{pmatrix}$$

$$A^{-1} = \frac{1}{800-200a}\begin{pmatrix} 4-16a & -40 & 96-24a \\ 75 & 50 & 0 \\ 28a-82 & 20 & 32-8a \end{pmatrix}$$

$$\text{so} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 6 \\ -9 \\ 11 \end{pmatrix}$$

$$= \frac{1}{800-200a}\begin{pmatrix} 1440-360a \\ 0 \\ -320+80a \end{pmatrix}$$

$$= \frac{1}{200(4-a)}\begin{pmatrix} 360(4-a) \\ 0 \\ -80(4-a) \end{pmatrix}$$

$$= \frac{1}{200}\begin{pmatrix} 360 \\ 0 \\ -80 \end{pmatrix} = \boxed{\begin{pmatrix} 9/5 \\ 0 \\ -2/5 \end{pmatrix}}$$

can also express as

$x = 9/5, \ y = 0, \ z = -2/5$

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b(ii) A is singular $\Rightarrow$ $\det A = 0$

$\Rightarrow$ $800 - 200a = 0$

$\Rightarrow$ $a = 4$

So the equations become.

(A) $2x + 4y - 6z = 6$

(B) $-3x - 6y + 9z = -9$

(C) $7x + 4y + 4z = 11$

Note that (B) $=$ (A) $\times \frac{-3}{2}$

So the planes (A) and (B) are identical.

They intersect the third plane in a line (since it isn't parallel to (A)/(B)).

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12.

$$\mathbf{M} = \begin{pmatrix} k & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Given that $k \neq 4$, find, in terms of k, the inverse of the matrix $\mathbf{M}$.

(4)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect.

$$\begin{aligned} 2x + 5y + 7z &= 1 \\ x + y + z &= p \\ 2x + y - z &= 2 \end{aligned}$$

(3)

(c) (i) Find the value of q for which the following planes intersect in a straight line.

$$\begin{aligned} 4x + 5y + 7z &= 1 \\ x + y + z &= q \\ 2x + y - z &= 2 \end{aligned}$$

(ii) For this value of q, determine a vector equation for the line of intersection.

(7)

a) $\det M = k(-1-1) - 5(-1-2) + 7(1-2)$

$= -2k + 15 - 7 = 8 - 2k$

Matrix of minors $= \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} k & 7 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} k & 5 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 5 & 7 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} k & 7 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} k & 5 \\ 1 & 1 \end{vmatrix} \end{pmatrix}$

$= \begin{pmatrix} -2 & -3 & -1 \\ -12 & -k-14 & k-10 \\ -2 & k-7 & k-5 \end{pmatrix}$

$$\text{Matrix of cofactors} = \begin{pmatrix} -2 & 3 & -1 \\ 12 & -h-14 & 10-h \\ -2 & 7-h & h-5 \end{pmatrix}$$

$$\text{Matrix of cofactors}^T = \begin{pmatrix} -2 & 12 & -2 \\ 3 & -h-14 & 7-h \\ -1 & 10-h & h-5 \end{pmatrix}$$

$$M^{-1} = \frac{1}{8-2h} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -h-14 & 7-h \\ -1 & 10-h & h-5 \end{pmatrix}$$

b) $$M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

$h=2$: $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -16 & 5 \\ -1 & 8 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 12p-6 \\ 13-16p \\ 8p-7 \end{pmatrix}$$

asks for coordinates.

$$\text{so } (x, y, z) = \left(\frac{6p-3}{2}, \frac{13-16p}{4}, \frac{8p-7}{4} \right)$$

ci) A $4x + 5y + 7z = 1$
B $x + y + z = q$
C $2x + y - z = 2$

We want to eliminate same variable twice.
i.e. B + C to eliminate z
and A + 7C to eliminate z

Note I have tried to use B only once to only get one equation in terms of q.

B + C : $3x + 2y = q + 2$

7C: $14x + 7y - 7z = 14$

A + 7C $18x + 12y = 15 \Rightarrow 3x + 2y = \frac{5}{2}$

So $q + 2 = \frac{5}{2} \Rightarrow q = \frac{1}{2}$

cii) All we need to do is find two points on our line and then the rest follows.

We know $3x + 2y = 5/2$ so we pick two pairs of x & y s.t. $3x + 2y = 5/2$

$(0, 5/4)$ and $(5/6, 0)$ are the easiest.

We sub back into any of A, B or C:

Subbing $(0, 5/4)$ into C gives $5/4 - z = 2$
$\Rightarrow z = -3/4$

Subbing $(5/6, 0)$ into C gives $5/3 - z = 2$
$z = -1/3$

So $\underbrace{(0, 5/4, -3/4)}_{A}$ & $\underbrace{(5/6, 0, -1/3)}_{B}$ are on our line.

$$\vec{AB} = \begin{pmatrix} 5/6 \\ -5/4 \\ 5/12 \end{pmatrix} \quad \text{so } \underline{r} = \begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix} + \lambda \begin{pmatrix} 5/6 \\ -5/4 \\ 5/12 \end{pmatrix}$$

$\times \frac{12}{5}$

which I can rescale to $\begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

remember that I can't rescale this, as it's a fixed point, whereas the direction vector I can because it's a direction.