London Academy of Maths

Series - Standard Formulae

Worked Solutions

*

1 Find $\sum_{r=1}^{n}(2r-1)^2$, expressing your answer in a fully factorised form. **[6]**

$$\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}(4r^2-4r+1)$$

$$= 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + \sum_{r=1}^{n}1$$

$$= 4\left(\frac{1}{6}n(n+1)(2n+1)\right) - 4\left(\frac{1}{2}n(n+1)\right) + n$$

$$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n$$

$$= \frac{1}{3}n\left(2(n+1)(2n+1) - 6(n+1) + 3\right)$$

$$= \frac{1}{3}n\left(4n^2+6n+2-6n-6+3\right)$$

$$= \frac{1}{3}n\left(4n^2-1\right)$$

$$= \boxed{\frac{1}{3}n(2n-1)(2n+1)}$$

*

2 Find $\sum_{r=1}^{n}(3r^2 - 3r + 2)$, expressing your answer in a fully factorised form. **[7]**

$$\sum_{r=1}^{n}(3r^2-3r+2) = 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 2$$

$$= 3\left(\frac{1}{6}n(n+1)(2n+1)\right) - 3\left(\frac{1}{2}n(n+1)\right) + 2n$$

$$= \frac{1}{2}n(n+1)(2n+1) - \frac{3}{2}n(n+1) + 2n$$

$$= \frac{1}{2}n\left((n+1)(2n+1) - 3(n+1) + 4\right)$$

$$= \frac{1}{2}n\left(2n^2+3n+1-3n-3+4\right)$$

$$= \frac{1}{2}n\left(2n^2+2\right)$$

$$= \frac{1}{2}n\left(2(n^2+1)\right)$$

$$= \boxed{n(n^2+1)}$$

*

3 Find $\sum_{r=1}^{n}(4r^3 - 3r^2 + r)$, giving your answer in a fully factorised form. **[6]**

$$\sum_{r=1}^{n}(4r^3-3r^2+r) = 4\sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$= 4\left(\frac{1}{4}n^2(n+1)^2\right) - 3\left(\frac{1}{6}n(n+1)(2n+1)\right) + \frac{1}{2}n(n+1)$$

$$= n^2(n+1)^2 - \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{2}n(n+1)\left(2n(n+1) - (2n+1) + 1\right)$$

$$= \frac{1}{2}n(n+1)\left(2n^2+2n-2n-1+1\right)$$

$$= \frac{1}{2}n(n+1)(2n^2)$$

$$= n(n+1)n^2$$

$$= \boxed{n^3(n+1)}$$

4 Find $\sum_{r=1}^{n}(3r+1)(r-1)$, giving your answer in a fully factorised form. **[5]**

$$\sum_{r=1}^{n}(3r+1)(r-1) = \sum_{r=1}^{n}(3r^2-2r-1)$$

$$= 3\sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= 3\left(\frac{1}{6}n(n+1)(2n+1)\right) - 2\left(\frac{1}{2}n(n+1)\right) - n$$

$$= \frac{1}{2}n(n+1)(2n+1) - n(n+1) - n$$

$$= \frac{1}{2}n\left((n+1)(2n+1) - 2(n+1) - 2\right)$$

$$= \frac{1}{2}n\left(2n^2+3n+1-2n-2-2\right)$$

$$= \frac{1}{2}n\left(2n^2+n-3\right)$$

$$= \boxed{\frac{1}{2}n\left(2n+3\right)\left(n-1\right)}$$

*

5 Find $\sum_{r=1}^{n}(r^2-r-8)$, giving your answer in a fully factorised form. **[5]**

$$\sum_{r=1}^{n}(r^2-r-8) = \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 8$$

$$= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - 8n$$

$$= \frac{1}{6}n\left((n+1)(2n+1) - 3(n+1) - 48\right)$$

$$= \frac{1}{6}n\left(2n^2+3n+1-3n-3-48\right)$$

$$= \frac{1}{6}n\left(2n^2-50\right)$$

$$= \frac{1}{6}n\left(2(n^2-25)\right)$$

$$= \boxed{\frac{1}{3}n(n-5)(n+5)}$$

6 Find $\sum_{r=1}^{n}(r+1)(r-1)$, expressing your answer in a fully factorised form. [6]

$$\sum_{r=1}^{n}(r+1)(r-1) = \sum_{r=1}^{n}(r^2-1)$$

$$= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1$$

$$= \frac{1}{6}n(n+1)(2n+1) - n$$

$$= \frac{1}{6}n\left((n+1)(2n+1) - 6\right)$$

$$= \frac{1}{6}n\left(2n^2+3n+1-6\right)$$

$$= \frac{1}{6}n\left(2n^2+3n-5\right)$$

$$= \boxed{\frac{1}{6}n(2n+5)(n-1)}$$

7 Use standard series formulae to find $\sum_{r=1}^{n} r(r^2+1)$, factorising your answer as far as possible. [6]

$$\sum_{r=1}^{n} r(r^2+1) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{2}n(n+1)$$

$$= \frac{1}{4}n(n+1)\left(n(n+1)+2\right)$$

$$= \boxed{\frac{1}{4}n(n+1)(n^2+n+2)}$$

8 Using the standard formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that $\sum_{r=1}^{n} [(r+1)(r-2)] = \frac{1}{3}n(n^2-7)$. [6]

$$\sum_{r=1}^{n} [(r+1)(r-2)] = \sum_{r=1}^{n} (r^2 - r - 2)$$

$$= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 2$$

$$= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - 2n$$

$$= \frac{1}{6}n\left((n+1)(2n+1) - 3(n+1) - 12\right)$$

$$= \frac{1}{6}n\left(2n^2 + 3n + 1 - 3n - 3 - 12\right)$$

$$= \frac{1}{6}n\left(2n^2 - 14\right)$$

$$= \frac{1}{6}n(2)(n^2-7)$$

$$= \boxed{\frac{1}{3}n(n^2-7)}$$

9 Use standard series formulae to show that $\sum_{r=1}^{n}(r+2)(r-3) = \frac{1}{3}n(n^2-19)$. **[6]**

$$\sum_{r=1}^{n}(r+2)(r-3) = \sum_{r=1}^{n}(r^2-r-6)$$

$$= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 6$$

$$= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - 6n$$

$$= \frac{1}{6}n\left((n+1)(2n+1) - 3(n+1) - 36\right)$$

$$= \frac{1}{6}n\left(2n^2+3n+1-3n-3-36\right)$$

$$= \frac{1}{6}n\left(2n^2-38\right)$$

$$= \frac{1}{6}n(2)\left(n^2-19\right)$$

$$= \boxed{\frac{1}{3}n\left(n^2-19\right)}$$

10 Using the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ show that

$$\sum_{r=1}^{n} r(r^2 - 3) = \tfrac{1}{4}n(n+1)(n+3)(n-2). \qquad [6]$$

$$\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} (r^3 - 3r)$$

$$= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 - 3\left(\frac{1}{2}n(n+1)\right)$$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{3}{2}n(n+1)$$

$$= \frac{1}{4}n(n+1)\left(n(n+1) - 6\right)$$

$$= \frac{1}{4}n(n+1)(n^2+n-6)$$

$$= \boxed{\frac{1}{4}n(n+1)(n+3)(n-2)}$$

11 **(i)** Use standard series formulae to show that

$$\sum_{r=1}^{n}[r(r-1)-1] = \frac{1}{3}n(n+2)(n-2).$$ **[5]**

i) $\sum_{r=1}^{n}(r(r-1)-1) = \sum_{r=1}^{n}(r^2-r-1)$

$= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$

$= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - n$

$= \frac{1}{6}n\left((n+1)(2n+1) - 3(n+1) - 6\right)$

$= \frac{1}{6}n\left(2n^2+3n+1-3n-3-6\right)$

$= \frac{1}{6}n(2n^2-8)$

$= \frac{1}{6}n(2)(n^2-4)$

$= \frac{1}{3}n(n^2-4)$

$= \boxed{\frac{1}{3}n(n+2)(n-2)}$

12 Use standard series formulae to find $\sum_{r=1}^{n} r(r-2)$, factorising your answer as far as possible. **[5]**

$$\sum_{r=1}^{n} r(r-2) = \sum_{r=1}^{n} (r^2 - 2r)$$

$$= \sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r$$

$$= \frac{1}{6}n(n+1)(2n+1) - 2\left(\frac{1}{2}n(n+1)\right)$$

$$= \frac{1}{6}n(n+1)(2n+1) - n(n+1)$$

$$= \frac{1}{6}n(n+1)(2n+1-6)$$

$$= \frac{1}{6}n(n+1)(2n-5)$$

*

13 Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that $\sum_{r=1}^{10} r(3r-2) = 1045$. **[3]**

$$\sum_{r=1}^{10} r(3r-2) = \sum_{r=1}^{10} (3r^2 - 2r)$$

$$= 3\sum_{r=1}^{10} r^2 \quad -2\sum_{r=1}^{10} r$$

$$= 3\left(\frac{1}{6}(10)(10+1)(2(10)+1)\right) - 2\left(\frac{1}{2}(10)(10+1)\right)$$

$$= 3\left(\frac{1}{6}\right)(10)(11)(21) \quad - 2\left(\frac{1}{2}\right)(10)(11)$$

$$= 1155 \quad - 110$$

$$= 1045$$

*

14 In this question you must show detailed reasoning.

Determine the value of $\sum_{r=1}^{50} r^2(16-r)$. **[3]**

$$\sum_{r=1}^{50} r^2(16-r) = \sum_{r=1}^{50}(16r^2 - r^3)$$

$$= 16\sum_{r=1}^{50} r^2 - \sum_{r=1}^{50} r^3$$

$$= 16\left(\frac{1}{6}(50)(50+1)(2(50)+1)\right) - \frac{1}{4}(50)^2(50+1)^2$$

$$= 16\left(\frac{1}{6}\right)(50)(51)(101) - \frac{1}{4}(50)^2(51)^2$$

$$= 686800 - 1625625$$

$$= \boxed{-938825}$$

*

15 In this question you must show detailed reasoning.

Determine the value of $\sum_{r=1}^{100}(2r+3)^2$. **[3]**

$$\sum_{r=1}^{100}(2r+3)^2 = \sum_{r=1}^{100}(4r^2+12r+9)$$

$$= 4\sum_{r=1}^{100} r^2 + 12\sum_{r=1}^{100} r + \sum_{r=1}^{100} 9$$

$$= 4\left(\tfrac{1}{6}(100)(100+1)(2(100)+1)\right) + 12\left(\tfrac{1}{2}(100)(100+1)\right) + 9\times 100$$

$$= 4\left(\tfrac{1}{6}\right)(100)(101)(201) + 12\left(\tfrac{1}{2}\right)(100)(101) + 900$$

$$= 1353400 + 60600 + 900$$

$$= \boxed{1{,}414{,}900}$$

16 Find $\sum_{r=1}^{n}(r+1)(r+5)$. Give your answer in a fully factorised form. **[4]**

$$\sum_{r=1}^{n}(r+1)(r+5) = \sum_{r=1}^{n}(r^2+6r+5)$$

$$= \sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r + \sum_{r=1}^{n} 5$$

$$= \frac{1}{6}n(n+1)(2n+1) + 6\left(\frac{1}{2}n(n+1)\right) + 5n$$

$$= \frac{1}{6}n(n+1)(2n+1) + 3n(n+1) + 5n$$

$$= \frac{1}{6}n\left((n+1)(2n+1) + 18(n+1) + 30\right)$$

$$= \frac{1}{6}n\left(2n^2+3n+1+18n+18+30\right)$$

$$= \frac{1}{6}n\left(2n^2+21n+49\right)$$

$$= \boxed{\frac{1}{6}n(2n+7)(n+7)}$$

17 Using standard summation formulae, find $\sum_{r=1}^{n}(r^2 - 3r)$, giving your answer in fully factorised form. **[3]**

$$\sum_{r=1}^{n}(r^2-3r) = \sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r$$

$$= \frac{1}{6}n(n+1)(2n+1) - 3\left(\frac{1}{2}n(n+1)\right)$$

$$= \frac{1}{6}n(n+1)(2n+1) - \frac{3}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1)\left(2n+1-9\right)$$

$$= \frac{1}{6}n(n+1)(2n-8)$$

$$= \frac{1}{6}n(n+1)\,2(n-4)$$

$$= \boxed{\frac{1}{3}n(n+1)(n-4)}$$

18 Find $\sum_{r=1}^{n}(2r^2-1)$, expressing your answer in fully factorised form. **[4]**

$$\sum_{r=1}^{n}(2r^2-1) = 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1$$

$$= 2\left(\frac{1}{6}n(n+1)(2n+1)\right) - n$$

$$= \frac{1}{3}n(n+1)(2n+1) - n$$

$$= \frac{1}{3}n\left((n+1)(2n+1) - 3\right)$$

$$= \frac{1}{3}n\left(2n^2+3n-2\right)$$

$$= \boxed{\frac{1}{3}n(2n-1)(n+2)}$$

*

19 Find $\sum_{r=1}^{n}(4r^3 + 6r^2 + 2r)$, expressing your answer in a fully factorised form. **[6]**

$$\sum_{r=1}^{n}(4r^3+6r^2+2r) = 4\sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$$

$$= 4\left(\frac{1}{4}n^2(n+1)^2\right) + 6\left(\frac{1}{6}n(n+1)(2n+1)\right) + 2\left(\frac{1}{2}n(n+1)\right)$$

$$= n^2(n+1)^2 + n(n+1)(2n+1) + n(n+1)$$

$$= n(n+1)\left(n(n+1) + 2n+1 + 1\right)$$

$$= n(n+1)(n^2+3n+2)$$

$$= n(n+1)(n+1)(n+2)$$

$$= \boxed{n(n+1)^2(n+2)}$$

20 Find $\sum_{r=1}^{n}(r-1)(r+1)$, giving your answer in a fully factorised form. **[6]**

$$\sum_{r=1}^{n}(r-1)(r+1) = \sum_{r=1}^{n}(r^2 - 1)$$

$$= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1$$

$$= \frac{1}{6}n(n+1)(2n+1) - n$$

$$= \frac{1}{6}n\left((n+1)(2n+1) - 6\right)$$

$$= \frac{1}{6}n\left(2n^2 + 3n - 5\right)$$

$$= \boxed{\frac{1}{6}n(n-1)(2n+5)}$$

21 Find $\sum_{r=1}^{n}(3r^2-5)$, expressing your answer in a fully factorised form. **[4]**

$$\sum_{r=1}^{n}(3r^2-5) = 3\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 5$$

$$= 3\left(\frac{1}{6}n(n+1)(2n+1)\right) - 5n$$

$$= \frac{1}{2}n(n+1)(2n+1) - 5n$$

$$= \frac{1}{2}n\left((n+1)(2n+1) - 10\right)$$

$$= \frac{1}{2}n\left(2n^2+3n-9\right)$$

$$= \boxed{\frac{1}{2}n(2n-3)(n+3)}$$

**

22 **(a)** Using standard summation formulae, write down an expression in terms of n for $\sum_{r=1}^{2n} r^3$. **[1]**

(b) Hence show that $\sum_{r=n+1}^{2n} r^3 = \frac{1}{4}n^2(an+b)(cn+d)$, where a, b, c and d are integers to be determined. **[5]**

a) $$\sum_{r=1}^{2n} r^3 = \frac{1}{4}(2n)^2(2n+1)^2 = \frac{1}{4}(4n^2)(2n+1)^2 = n^2(2n+1)^2$$

b) $$\sum_{r=n+1}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n} r^3$$

$$= n^2(2n+1)^2 - \frac{1}{4}n^2(n+1)^2$$

$$= \frac{1}{4}n^2\left(4(2n+1)^2 - (n+1)^2\right)$$

$$= \frac{1}{4}n^2\left(4(4n^2+4n+1) - (n^2+2n+1)\right)$$

$$= \frac{1}{4}n^2(16n^2+16n+4-n^2-2n-1)$$

$$= \frac{1}{4}n^2(15n^2+14n+3)$$

$$= \boxed{\frac{1}{4}n^2(3n+1)(5n+3)}$$

 **

23 **(i)** Show that, when $n = 5$, $\sum_{r=n+1}^{2n} r^2 = 330$. **[1]**

(ii) Find, in terms of n, a fully factorised expression for $\sum_{r=n+1}^{2n} r^2$. **[4]**

i) when $n=5$, $\sum_{r=n+1}^{2n} r^2 = \sum_{r=6}^{10} r^2$

$$= \sum_{r=1}^{10} r^2 - \sum_{r=1}^{5} r^2$$

$$= \frac{1}{6}10(10+1)(2(10)+1) - \frac{1}{6}5(5+1)(2(5)+1)$$

$$= 385 - 55$$

$$= \boxed{330}$$

ii) $\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$

$$= \frac{1}{6}(2n)(2n+1)(2(2n)+1) - \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{1}{3}n(2n+1)(4n+1) - \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{1}{6}n(2n+1)\left(2(4n+1) - (n+1)\right)$$

$$= \frac{1}{6}n(2n+1)(8n+2-n-1)$$

$$= \boxed{\frac{1}{6}n(2n+1)(7n+1)}$$

 **

24 Find $\sum_{r=1}^{n} r(r+1)(r-2)$, expressing your answer in a fully factorised form. **[6]**

$$\sum_{r=1}^{n} r(r+1)(r-2) = \sum_{r=1}^{n} (r^3 - r^2 - 2r)$$

$$= \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) - 2\left(\frac{1}{2}n(n+1)\right)$$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) - n(n+1)$$

$$= \frac{1}{12}n(n+1)\left(3n(n+1) - 2(2n+1) - 12\right)$$

$$= \frac{1}{12}n(n+1)(3n^2 - n - 14)$$

$$= \boxed{\frac{1}{12}n(n+1)(3n-7)(n+2)}$$

**

25 Find $\sum_{r=1}^{n} r(r^2 - 3)$, expressing your answer in a fully factorised form. **[6]**

$$\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 - 3\left(\frac{1}{2}n(n+1)\right)$$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{3}{2}n(n+1)$$

$$= \frac{1}{4}n(n+1)\left(n(n+1) - 6\right)$$

$$= \frac{1}{4}n(n+1)(n^2+n-6)$$

$$= \boxed{\frac{1}{4}n(n+1)(n+3)(n-2)}$$

**

26 Find $\sum_{r=1}^{n} r^2(r-1)$, expressing your answer in a fully factorised form. [6]

$$\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n}(r^3 - r^2)$$

$$= \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{1}{12}n(n+1)\left(3n(n+1) - 2(2n+1)\right)$$

$$= \frac{1}{12}n(n+1)(3n^2 - n - 2)$$

$$= \boxed{\frac{1}{12}n(n+1)(3n+2)(n-1)}$$

**

27 Find $\sum_{r=1}^{2n}(3r^2 - \frac{1}{2})$, expressing your answer in a fully factorised form. **[6]**

$$\sum_{r=1}^{2n}\left(3r^2 - \frac{1}{2}\right) = 3\sum_{r=1}^{2n} r^2 - \sum_{r=1}^{2n}\frac{1}{2}$$

$$= 3\left(\frac{1}{6}(2n)(2n+1)\cdot(2(2n)+1)\right) - \frac{1}{2}(2n)$$

$$= n(2n+1)(4n+1) - n$$

$$= n\left((2n+1)(4n+1) - 1\right)$$

$$= n(8n^2 + 6n + 1 - 1)$$

$$= n(8n^2 + 6n)$$

$$= \boxed{2n^2(4n+3)}$$

 **

28 **(i)** Find $\sum_{r=1}^{n} r(r^2+r-7)$, giving your answer in a fully factorised form. **[5]**

i) $\sum_{r=1}^{n} r(r^2+r-7) = \sum_{r=1}^{n} (r^3+r^2-7r)$

$= \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2 - 7\sum_{r=1}^{n} r$

$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1) - 7\left(\frac{1}{2}n(n+1)\right)$

$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1) - \frac{7}{2}n(n+1)$

$= \frac{1}{12}n(n+1)\left(3n(n+1) + 2(2n+1) - 42\right)$

$= \frac{1}{12}n(n+1)\left(3n^2+7n-40\right)$

$= \boxed{\frac{1}{12}n(n+1)(3n-8)(n+5)}$

**

29 **(i)** Show that $\sum_{r=n}^{2n} r^3 = \frac{3}{4}n^2(n+1)(5n+1)$. **[4]**

(ii) Hence find $\sum_{r=n}^{2n} r(r^2-2)$, giving your answer in a fully factorised form. **[5]**

i) $\sum_{r=n}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n-1} r^3$

$= \frac{1}{4}(2n)^2(2n+1)^2 - \frac{1}{4}(n-1)^2(n-1+1)^2$

$= \frac{1}{4}(4n^2)(2n+1)^2 - \frac{1}{4}(n-1)^2 n^2$

$= n^2(2n+1)^2 - \frac{1}{4}n^2(n-1)^2$

$= \frac{1}{4}n^2\left(4(2n+1)^2 - (n-1)^2\right)$

$= \frac{1}{4}n^2\left(4(4n^2+4n+1) - (n^2-2n+1)\right)$

$= \frac{1}{4}n^2(16n^2+16n+4-n^2+2n-1)$

$= \frac{1}{4}n^2(15n^2+18n+3)$

$= \frac{1}{4}n^2(3(5n^2+6n+1))$

$= \frac{3}{4}n^2(5n^2+6n+1)$

$= \boxed{\frac{3}{4}n^2(5n+1)(n+1)}$

**

ii) $$\sum_{r=n}^{2n} r(r^2-2) = \sum_{r=n}^{2n} r^3 - 2\sum_{r=n}^{2n} r$$

$$= \frac{3}{4}n^2(5n+1)(n+1) - 2\left(\sum_{r=1}^{2n} r - \sum_{r=1}^{n-1} r\right)$$

$$= \frac{3}{4}n^2(5n+1)(n+1) - 2\left(\tfrac{1}{2}(2n)(2n+1) - \tfrac{1}{2}(n-1)(n-1+1)\right)$$

$$= \frac{3}{4}n^2(5n+1)(n+1) - 2\left(n(2n+1) - \tfrac{1}{2}n(n-1)\right)$$

$$= \frac{3}{4}n^2(5n+1)(n+1) - 2\left(\tfrac{1}{2}n\right)\left(2(2n+1) - (n-1)\right)$$

$$= \frac{3}{4}n^2(5n+1)(n+1) - n\left(4n+2-n+1\right)$$

$$= \frac{3}{4}n^2(5n+1)(n+1) - n(3n+3)$$

$$= \frac{3}{4}n^2(5n+1)(n+1) - 3n(n+1)$$

$$= \frac{3}{4}n(n+1)\left(n(5n+1) - 4\right)$$

$$= \frac{3}{4}n(n+1)(5n^2+n-4)$$

$$= \frac{3}{4}n(n+1)(5n-4)(n+1)$$

$$= \boxed{\frac{3}{4}n(n+1)^2(5n-4)}$$

As you can see, where possible first priority is always to factorise in these types of questions!

 **

30 Use standard series formulae to show that $\sum_{r=1}^{n} r^2(3-4r) = \frac{1}{2}n(n+1)(1-2n^2)$. **[5]**

$$\sum_{r=1}^{n} r^2(3-4r) = \sum_{r=1}^{n}(3r^2 - 4r^3)$$

$$= 3\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r^3$$

$$= 3\left(\frac{1}{6}n(n+1)(2n+1)\right) - 4\left(\frac{1}{4}n^2(n+1)^2\right)$$

$$= \frac{1}{2}n(n+1)(2n+1) - n^2(n+1)^2$$

$$= \frac{1}{2}n(n+1)\left(2n+1 - 2n(n+1)\right)$$

$$= \frac{1}{2}n(n+1)(2n+1-2n^2-2n)$$

$$= \boxed{\frac{1}{2}n(n+1)(1-2n^2)}$$

 **

31 **(i)** Using the standard summation formulae, find an expression for $\sum_{r=1}^{n}(1-2r)^2$ in terms of n. Give your answer in a fully factorised form. **[6]**

(ii) Hence evaluate $\sum_{r=25}^{75}(1-2r)^2$. **[2]**

i) $$\sum_{r=1}^{n}(1-2r)^2 = \sum_{r=1}^{n}(1-4r+4r^2)$$

$$= \sum_{r=1}^{n}1 - 4\sum_{r=1}^{n}r + 4\sum_{r=1}^{n}r^2$$

$$= n - 4\left(\frac{1}{2}n(n+1)\right) + 4\left(\frac{1}{6}n(n+1)(2n+1)\right)$$

$$= n - 2n(n+1) + \frac{2}{3}n(n+1)(2n+1)$$

$$= \frac{1}{3}n\left(3 - 6(n+1) + 2(n+1)(2n+1)\right)$$

$$= \frac{1}{3}n\left(3 - 6n - 6 + 4n^2 + 6n + 2\right)$$

$$= \frac{1}{3}n\left(4n^2 - 1\right)$$

$$= \boxed{\frac{1}{3}n(2n-1)(2n+1)}$$

ii) $$\sum_{r=25}^{75}(1-2r)^2 = \sum_{r=1}^{75}(1-2r)^2 - \sum_{r=1}^{24}(1-2r)^2$$

$$= \frac{1}{3}(75)(149)(151) - \frac{1}{3}(24)(47)(49)$$

$$= \boxed{544051}$$

**

32 **(i)** Show that $\sum_{r=1}^{n}(2r-1) = n^2$. **[3]**

(ii) Show that $\dfrac{\sum_{r=1}^{n}(2r-1)}{\sum_{r=n+1}^{2n}(2r-1)} = k$, where k is a constant to be determined. **[4]**

i) $\sum_{r=1}^{n}(2r-1) = 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$

$= 2\left(\frac{1}{2}n(n+1)\right) - n$

$= n(n+1) - n$

$= n(n+1-1)$

$= n(n) = n^2$

◻

ii) $\dfrac{\sum_{r=1}^{n}(2r-1)}{\sum_{r=n+1}^{2n}(2r-1)} = \dfrac{n^2}{\sum_{r=1}^{2n}(2r-1) - \sum_{r=1}^{n}(2r-1)}$

$= \dfrac{n^2}{(2n)^2 - n^2}$

$= \dfrac{n^2}{3n^2} = \dfrac{1}{3}$

So $\boxed{k = \frac{1}{3}}$

 **

33 **(a)** Using standard summation formulae, show that $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$. **[4]**

(b) Use induction to prove the result in part **(a)**. **[6]**

a) $\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} (r^2+2r)$

$= \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) + 2\left(\frac{1}{2}n(n+1)\right)$

$= \frac{1}{6}n(n+1)(2n+1) + n(n+1)$

$= \frac{1}{6}n(n+1)\left(2n+1+6\right)$

$= \boxed{\frac{1}{6}n(n+1)(2n+7)}$

b) Base case:

let $n=1$, $LHS = \sum_{r=1}^{1} r(r+2) = 1(1+2) = 3$

$RHS = \frac{1}{6}(1)(1+1)(2(1)+7) = \frac{1}{6}(1)(2)(9)$

$= \frac{18}{6} = 3$

LHS = RHS

So the result is true for $n=1$.

Inductive Hypothesis

Assume result is true for $n=k$.

i.e. $\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$

P.T.O

Inductive Step

$$\sum_{r=1}^{k+1} r(r+2) = \sum_{r=1}^{k} r(r+2) + (k+1)(k+3)$$

by splitting up the sum of the first k terms and the $(k+1)^{th}$ term.

$$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$

$$= \frac{1}{6}(k+1)\left(k(2k+7) + 6(k+3)\right)$$

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6k + 18)$$

$$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

$$= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+7)$$

so the result is true for $n=1$,
if it's true for $n=k$ it's also true for $n=k+1$
so by induction it must be true for all positive integers n.

 ***

34 **(i)** Use standard series to show that

$$\sum_{r=1}^{n} r^2(2r-p) = \frac{1}{6}n(n+1)(3n^2+(3-2p)n-p),$$

where p is a constant. **[4]**

(ii) Given that the coefficients of n^3 and n^4 in the expression for $\sum_{r=1}^{n} r^2(2r-p)$ are equal, find the value of p. **[2]**

i) $\sum_{r=1}^{n} r^2(2r-p) = \sum_{r=1}^{n}(2r^3 - pr^2)$

$= 2\sum_{r=1}^{n} r^3 - p\sum_{r=1}^{n} r^2$

$= 2\left(\frac{1}{4}n^2(n+1)^2\right) - p\left(\frac{1}{6}n(n+1)(2n+1)\right)$

$= \frac{1}{2}n^2(n+1)^2 - \frac{p}{6}n(n+1)(2n+1)$

$= \frac{1}{6}n(n+1)\left(3n(n+1) - p(2n+1)\right)$

$= \frac{1}{6}n(n+1)\left(3n^2+3n-2pn-p\right)$

$= \frac{1}{6}n(n+1)\left(3n^2+(3-2p)n-p\right)$

ii) $\frac{1}{6}n(n+1)\left(3n^2+(3-2p)n-p\right) = \frac{1}{6}(n^2+n)\left(3n^2+(3-2p)n-p\right)$

$= \frac{1}{6}\left(3n^4+(3-2p)n^3+3n^3+\dots\right)$

Coefficient of $n^4 = \frac{1}{6}\times 3$

Coefficient of $n^3 = \frac{1}{6}\left((3-2p)+3\right)$

So $3 = 3-2p+3 \Rightarrow 2p = 3 \Rightarrow \boxed{p = \frac{3}{2}}$

***

35 Given that $\sum_{r=1}^{n}(ar^3 + br) \equiv n(n-1)(n+1)(n+2)$, find the values of the constants a and b. **[6]**

$$\sum_{r=1}^{n}(ar^3+br) = a\sum_{r=1}^{n} r^3 + b\sum_{r=1}^{n} r$$

$$= a\,\frac{1}{4}n^2(n+1)^2 + b\,\frac{1}{2}n(n+1)$$

$$= n(n+1)\left(\frac{1}{4}an(n+1) + \frac{1}{2}b\right)$$

$$= n(n+1)\left(\frac{1}{4}an^2 + \frac{1}{4}an + \frac{1}{2}b\right)$$

Notice how I don't factor out $\frac{1}{4}$ since on the rhs there's no constant.

by comparing $n(n+1)\left(\frac{1}{4}an^2 + \frac{1}{4}an + \frac{1}{2}b\right)$ with $n(n-1)(n+1)(n+2)$

we see that $\frac{1}{4}an^2 + \frac{1}{4}an + \frac{1}{2}b \equiv (n-1)(n+2)$

$$\equiv n^2+n-2$$

compare coefficients:

$[n^2]$: $\frac{1}{4}a = 1 \Rightarrow \boxed{a = 4}$

$[n]$: $\frac{1}{2}b = -2 \Rightarrow \boxed{b = -4}$

Note, plenty of ways to do this question.

***

36 **(i)** Using the standard formulae for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$, prove that

$$\sum_{r=1}^{n} r^2(r+1) = \tfrac{1}{12}n(n+1)(n+2)(3n+1).$$ [5]

(ii) Prove the same result by mathematical induction. [8]

i) $\sum_{r=1}^{n} r^2(r+1) = \sum_{r=1}^{n}(r^3+r^2)$

$= \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2$

$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)$

$= \frac{1}{12}n(n+1)\Big(3n(n+1) + 2(2n+1)\Big)$

$= \frac{1}{12}n(n+1)(3n^2+7n+2)$

$= \frac{1}{12}n(n+1)(3n+1)(n+2)$

ii) Base case:

when $n=1$, LHS $= \sum_{r=1}^{n} r^2(r+1) = \sum_{r=1}^{1} r^2(r+1) = 1^2(1+1) = 1\times 2 = 2$

RHS $= \frac{1}{12}(1)(1+1)(3(1)+1)(1+2) = \frac{1}{12}(1)(2)(4)(3) = \frac{24}{12} = 2$

LHS = RHS, so the statement is true for $n=1$.

Inductive Hypothesis

assume the result is true for $n=k$.

i.e. $\sum_{r=1}^{k} r^2(r+1) = \frac{1}{12}k(k+1)(k+2)(3k+1)$

***

Inductive step

When $n = k+1$,

Since it's the sum of the first k terms, plus the $(k+1)^{th}$ term

$$\sum_{r=1}^{k+1} r^2(r+1) = \sum_{r=1}^{k} r^2(r+1) + (k+1)^2(k+1+1)$$

$$= \frac{1}{12}k(k+1)(k+2)(3k+1) + (k+1)^2(k+2)$$ by Inductive Hypothesis

$$= \frac{1}{12}(k+1)(k+2)\Big(k(3k+1) + 12(k+1)\Big)$$

$$= \frac{1}{12}(k+1)(k+2)\left(3k^2+13k+12\right)$$

$$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+4)$$

a lot of students forget this last step.

$$= \frac{1}{12}(k+1)\left((k+1)+1\right)\left((k+1)+2\right)\left(3(k+1)+1\right)$$

Therefore, since the result is true for $n=1$, and if it's true for $n=k$, we've proved it's true for $n=k+1$.

So, by induction the result is true for all positive integers.

37 In this question you must show detailed reasoning.

Determine the smallest value of n for which $\dfrac{1^2+2^2+\ldots+n^2}{1+2+\ldots+n} > 341$. **[4]**

$$\frac{1^2+2^2+\ldots+n^2}{1+2+\ldots+n} = \frac{\sum_{r=1}^{n} r^2}{\sum_{r=1}^{n} r}$$

$$= \frac{\frac{1}{6}\cancel{n}\cancel{(n+1)}(2n+1)}{\frac{1}{2}\cancel{n}\cancel{(n+1)}}$$

$$= \frac{\frac{1}{6}(2n+1)}{\frac{1}{2}}$$

note I can only do this since $n \neq 0$ or $n \neq -1$ as it must be a positive integer. Otherwise I would be losing solutions.

$$= \frac{1}{3}(2n+1) > 341$$

$$2n+1 > 1023$$

$$2n > 1022$$

$$n > 511$$

$$\boxed{n = 512}$$

****

38 You are given that $\sum_{r=1}^{n}(ar+b) = n^2$ for all n, where a and b are constants.

By finding $\sum_{r=1}^{n}(ar+b)$ in terms of a, b and n, determine the values of a and b. **[6]**

$$\sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n} r + \sum_{r=1}^{n} b$$

$$= a\left(\tfrac{1}{2}n(n+1)\right) + bn$$

$$= \tfrac{1}{2}an^2 + \left(\tfrac{1}{2}a+b\right)n$$

Rare case of where I'll expand

so, $\tfrac{1}{2}an^2 + \left(\tfrac{1}{2}a+b\right)n = n^2$

by comparing coefficients,

$[n^2]$: $\tfrac{1}{2}a = 1 \Rightarrow \boxed{a=2}$

$[n]$: $\tfrac{1}{2}a + b = 0 \Rightarrow 1+b=0 \Rightarrow \boxed{b=-1}$

****

39 Find an expression for $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \ldots + n(n+1)^2$ in terms of n. Give your answer in fully factorised form. **[3]**

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \ldots + n(n+1)^2$$

$$= \sum_{r=1}^{n} r(r+1)^2$$

$$= \sum_{r=1}^{n} r(r^2+2r+1)$$

$$= \sum_{r=1}^{n} (r^3+2r^2+r)$$

$$= \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 + 2\left(\frac{1}{6}n(n+1)(2n+1)\right) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{12}n(n+1)\left(3n(n+1) + 4(2n+1) + 6\right)$$

$$= \frac{1}{12}n(n+1)\left(3n^2+3n+8n+4+6\right)$$

$$= \frac{1}{12}n(n+1)\left(3n^2+11n+10\right)$$

$$= \boxed{\frac{1}{12}n(n+1)(3n+5)(n+2)}$$

****

40 Using standard summation of series formulae, determine the sum of the first n terms of the series

$$(1 \times 2 \times 4) + (2 \times 3 \times 5) + (3 \times 4 \times 6) + \ldots,$$

where n is a positive integer. Give your answer in fully factorised form. **[6]**

$$(1 \times 2 \times 4) + (2 \times 3 \times 5) + (3 \times 4 \times 6) + \ldots$$

$$= \sum_{r=1}^{n} r(r+1)(r+3)$$

$$= \sum_{r=1}^{n} (r^3 + 4r^2 + 3r)$$

$$= \sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 + 4\left(\frac{1}{6}n(n+1)(2n+1)\right) + 3\left(\frac{1}{2}n(n+1)\right)$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{2}{3}n(n+1)(2n+1) + \frac{3}{2}n(n+1)$$

$$= \frac{1}{12}n(n+1)\left(3n(n+1) + 8(2n+1) + 18\right)$$

$$= \frac{1}{12}n(n+1)\left(3n^2 + 19n + 26\right)$$

$$= \boxed{\frac{1}{12}n(n+1)(3n+13)(n+2)}$$

 ****

41 In this question you must show detailed reasoning.

The series S is defined as being the sum of the squares of all positive **odd** integers from 1^2 to 779^2.

Determine the value of S. **[5]**

$2r-1 = 779 \quad \Rightarrow r = 390$

$$S = \sum_{r=1}^{390} (2r-1)^2$$

$$= \sum_{r=1}^{390} (4r^2 - 4r + 1)$$

$$= 4\sum_{r=1}^{390} r^2 - 4\sum_{r=1}^{390} r + \sum_{r=1}^{390} 1$$

$$= 4\left(\frac{1}{6}(390)(390+1)(2(390)+1)\right) - 4\left(\frac{1}{2}(390)(390+1)\right) + 390$$

$$= 4\left(\frac{1}{6}\right)(390)(391)(781) - 4\left(\frac{1}{2}\right)(390)(391) + 390$$

$$= 79396460 - 304980 + 390$$

$$= \boxed{79{,}091{,}870}$$