London Academy of Maths

# Vectors

Question Paper

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**1** The plane $x+2y+cz=4$ is perpendicular to the plane $2x-cy+6z=9$, where $c$ is a constant. Find the value of $c$. **[3]**

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**2** Find the cartesian equation of the plane which contains the three points $(1, 0, -1)$, $(2, 2, 1)$ and $(1, 1, 2)$.

**[5]**

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**3** **(a)** Find $\mathbf{M}^{-1}$, where $\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ -2 & 1 & 2 \end{pmatrix}$. **[1]**

**(b)** Hence find, in terms of the constant $k$, the point of intersection of the planes

$$x + 2y + 3z = 19,$$
$$-x + y + 2z = 4,$$
$$-2x + y + 2z = k.$$ **[3]**

**(c)** **In this question you must show detailed reasoning.**

Find the acute angle between the planes $x + 2y + 3z = 19$ and $-x + y + 2z = 4$. **[4]**

**4** **In this question you must show detailed reasoning.**

Find the angle between the vector $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and the plane $-x+3y+2z=8$. **[5]**

**5** **(a)** Show that the vector $\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ is parallel to the plane $2x+y-3z=10$. **[3]**

**(b)** Determine the acute angle between the planes $2x+y-3z=10$ and $x-y-3z=3$. **[4]**

**6** Three planes have the following equations.

$$2x - 3y + z = -3,$$
$$x - 4y + 2z = 1,$$
$$-3x - 2y + 3z = 14.$$

**(a)** **(i)** Write the system of equations in matrix form. **[1]**

**(ii)** Hence find the point of intersection of the planes. **[2]**

**(b)** **In this question you must show detailed reasoning.**

Find the acute angle between the planes $2x - 3y + z = -3$ and $x - 4y + 2z = 1$. **[4]**

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**7** The equations of two non-intersecting lines, $l_1$ and $l_2$, are

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

Find the shortest distance between lines $l_1$ and $l_2$. **[5]**

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**8** The lines $l_1$ and $l_2$ have equations $\dfrac{x-3}{1}=\dfrac{y-5}{2}=\dfrac{z+2}{-3}$ and $\dfrac{x-4}{2}=\dfrac{y+2}{-1}=\dfrac{z-7}{4}$.

**(i)** Find the shortest distance between $l_1$ and $l_2$. **[5]**

**(ii)** Find a cartesian equation of the plane which contains $l_1$ and is parallel to $l_2$. **[2]**

**9** The equation of a plane $\Pi$ is $x - 2y - z = 30$.

**(i)** Find the acute angle between the line $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}$ and $\Pi$. **[4]**

**(ii)** Determine the geometrical relationship between the line $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ and $\Pi$. **[4]**

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**10** **(a)** A plane $\Pi$ has the equation $\mathbf{r}.\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = 15$. $C$ is the point $(4, -5, 1)$.

Find the shortest distance between $\Pi$ and $C$. **[3]**

**(b)** Lines $l_1$ and $l_2$ have the following equations.

$$l_1: \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$$

$$l_2: \mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Find, in exact form, the distance between $l_1$ and $l_2$. **[5]**

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**11** The equation of a plane is $4x+2y+z=7$.
The point $A$ has coordinates (9, 6, 1) and the point $B$ is the reflection of $A$ in the plane.

Find the coordinates of the point $B$. **[6]**

**12** **(a)** Show that the three planes with equations

$$x + \lambda y + 3z = -12$$
$$2x + y + 5z = -11$$
$$x - 2y + 2z = -9$$

where $\lambda$ is a constant, meet at a unique point except for one value of $\lambda$ which is to be determined. **[3]**

**(b)** In the case $\lambda = -2$, use matrices to find the point of intersection P of the planes, showing your method clearly. **[3]**

The line $l$ has equation $\dfrac{x-1}{2} = \dfrac{y-1}{-1} = \dfrac{z+2}{-2}$.

**(c)** Find a vector equation of $l$. **[2]**

**(d)** Find the shortest distance between the point P and $l$. **[4]**

**(e)** **(i)** Show that $l$ is parallel to the plane $x - 2y + 2z = -9$. **[3]**

**(ii)** Find the distance between $l$ and the plane $x - 2y + 2z = -9$. **[2]**

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**13** The equation of the plane $\Pi$ is $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$.

**(a)** Find the acute angle between $\Pi$ and the plane with equation $\mathbf{r}.\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = 4$. **[4]**

The point $A$ has coordinates $(9, -7, 20)$.

The point $F$ is the point of intersection between $\Pi$ and the perpendicular from $A$ to $\Pi$.

**(b)** Determine the coordinates of $F$. **[4]**

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**14** The plane P has normal vector $2\mathbf{i} + a\mathbf{j} - \mathbf{k}$, where $a$ is a positive constant, and the point $(3, -1, 1)$ lies in P. The plane $x - z = 3$ makes an angle of $45°$ with P.

Find the cartesian equation of P. **[7]**

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**15** A vector **v** has magnitude 1 unit. The angle between **v** and the positive $z$-axis is 60°, and **v** is parallel to the plane $x-2y=0$.

Given that $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, where $a$, $b$ and $c$ are all positive, find **v**. **[7]**

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**16** The plane Π has equation $3x - 5y + z = 9$.

**(i)** Show that Π contains

- the point $(4,1,2)$

and

- the vector $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. **[4]**

**(ii)** Determine the equation of a plane which is perpendicular to Π and which passes through $(4,1,2)$. **[3]**

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**17** The equations of two intersecting lines $l_1$ and $l_2$ are

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \qquad l_2 : \mathbf{r} = \begin{pmatrix} 7 \\ 9 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

where $a$ is a constant.

The equation of the plane $\Pi$ is

$$\mathbf{r}.\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = -14.$$

$l_1$ and $\Pi$ intersect at $Q$.

$l_2$ and $\Pi$ intersect at $R$.

**(a)** Verify that the coordinates of $R$ are (13, 3, –14). **[2]**

**(b)** Determine the exact value of the length of $QR$. **[7]**

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**18** The line $l_1$ has equation $\mathbf{r}=\begin{pmatrix}1\\-3\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\-2\end{pmatrix}$.

The plane $\Pi$ has equation $\mathbf{r}.\begin{pmatrix}2\\-5\\-3\end{pmatrix}=4$.

**(a)** Find the position vector of the point of intersection of $l_1$ and $\Pi$. **[3]**

**(b)** Find the acute angle between $l_1$ and $\Pi$. **[3]**

$A$ is the point on $l_1$ where $\lambda = 1$.

$l_2$ is the line with the following properties.

- $l_2$ passes through $A$
- $l_2$ is perpendicular to $l_1$
- $l_2$ is parallel to $\Pi$

**(c)** Find, in vector form, the equation of $l_2$. **[3]**

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**19 In this question you must show detailed reasoning.**

Find a vector **v** which has the following properties.

- It is a unit vector.
- It is parallel to the plane $2x+2y+z=10$.
- It makes an angle of 45° with the normal to the plane $x+z=5$. **[8]**

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**20** The point P (4, 1, 0) is equidistant from the plane $2x + y + 2z = 0$ and the line $\dfrac{x-3}{2} = \dfrac{y-1}{b} = \dfrac{z+5}{3}$, where $b > 0$.

Determine the value of $b$. **[10]**

**21** The plane Π has equation $2x - y + 2z = 4$. The point P has coordinates (8, 4, 5).

**(a)** Calculate the shortest distance from P to Π. **[2]**

The line L has equation $\frac{x-2}{3} = \frac{y}{2} = \frac{z+3}{4}$.

**(b)** Verify that P lies on L. **[2]**

**(c)** Find the coordinates of the point of intersection of L and Π. **[3]**

**(d)** Determine the acute angle between L and Π. **[4]**

**(e)** Use the results of parts **(b)**, **(c)** and **(d)** to verify your answer to part **(a)**. **[3]**

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**22** The points $P$, $Q$ and $R$ have coordinates (0, 2, 3), (2, 0, 1) and (1, 3, 0) respectively.

The acute angle between the line segments $PQ$ and $PR$ is $\theta$.

**(a)** Show that $\sin\theta = \frac{2}{11}\sqrt{22}$. **[3]**

The triangle $PQR$ lies in the plane $\Pi$.

**(b)** Determine an equation for $\Pi$, giving your answer in the form $ax + by + cz = d$, where $a$, $b$, $c$ and $d$ are integers. **[3]**

The point $S$ has coordinates $(5, 3, -1)$.

**(c)** By finding the shortest distance between $S$ and the plane $\Pi$, show that the volume of the tetrahedron $PQRS$ is $\frac{14}{3}$.

[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height] **[4]**

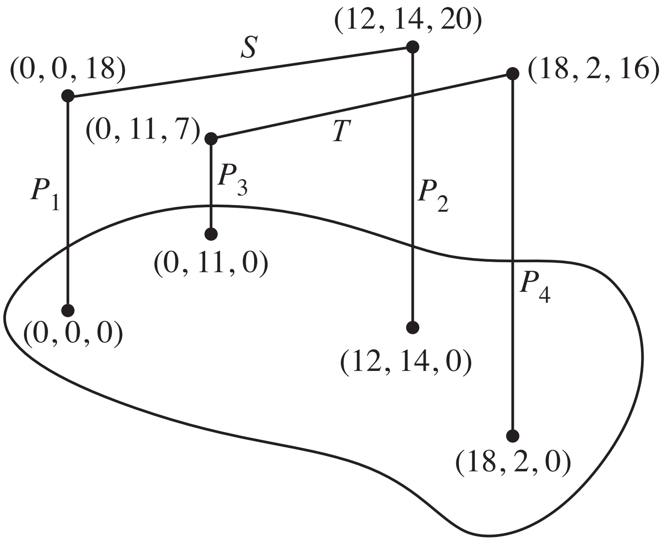
The tetrahedron $PQRS$ is transformed to the tetrahedron $P'Q'R'S'$ by a rotation about the $y$-axis.

The $x$-coordinate of $S'$ is $2\sqrt{2}$.

**(d)** By using the matrix for a rotation by angle $\theta$ about the $y$-axis, as given in the Formulae Booklet, determine in exact form the possible coordinates of $R'$. **[5]**

**23** A 3-D coordinate system, whose units are metres, is set up to model a construction site. The construction site contains four vertical poles $P_1$, $P_2$, $P_3$ and $P_4$. The floor of the construction site is modelled as lying in the *x*-*y* plane and the poles are modelled as vertical line segments. One end of each pole lies on the floor of the construction site, and the other end of each pole is modelled by the points (0, 0, 18), (12, 14, 20), (0, 11, 7) and (18, 2, 16) respectively.

A wire, *S*, runs from the top of $P_1$ to the top of $P_2$. A second wire, *T*, runs from the top of $P_3$ to the top of $P_4$. The wires are modelled by straight lines segments. The layout of the construction site is illustrated on the diagram below which is **not** drawn to scale.




A vector equation of the line **segment** that represents the wire *S* is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 18 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 7 \\ 1 \end{pmatrix}, 0 \leqslant \lambda \leqslant 2.$$

**(a)** Find, in the same form, a vector equation of the line **segment** that represents the wire *T*. The components of the direction vector should be integers whose only positive common factor is 1. **[2]**

For the construction site to be considered safe, it must pass two tests.

Test 1: The wires *S* and *T* need to be at least 5 metres apart at all positions on *S* and *T*.

**(b)** By using an appropriate formula, determine whether the construction site passes Test 1. **[2]**

A security camera is placed at a point *Q* on wire *S*.

Test 2: To ensure sufficient visibility of the construction site, the distance between the security camera and the top of $P_3$ must be at least 19 m.

**(c)** Determine whether it is possible to find point *Q* on *S* such that the construction site passes Test 2. **[3]**

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**24** The coordinates of the points $A$ and $B$ are $(3, -2, -1)$ and $(13, 10, 9)$ respectively.

- The plane $\Pi_A$ contains $A$ and the plane $\Pi_B$ contains $B$.
- The planes $\Pi_A$ and $\Pi_B$ are parallel.
- The $x$ and $y$ components of any normal to plane $\Pi_A$ are equal.
- The shortest distance between $\Pi_A$ and $\Pi_B$ is 2.

There are **two** possible solution planes for $\Pi_A$ which satisfy the above conditions.

Determine the acute angle between these two possible solution planes. **[8]**