Vectors

Worked Solutions

1 The plane $x+2y+cz=4$ is perpendicular to the plane $2x-cy+6z=9$, where c is a constant. Find the value of c. **[3]**

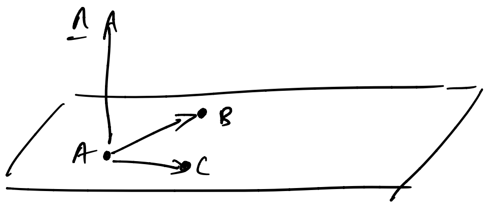
if the two planes are perpendicular, their normals must also be perpendicular.

$$\begin{pmatrix}1\\2\\c\end{pmatrix}\cdot\begin{pmatrix}2\\-c\\6\end{pmatrix}=0 \quad \Rightarrow \quad 2-2c+6c=0$$

$$c=-\frac{1}{2}$$

2 Find the cartesian equation of the plane which contains the three points $(1, 0, -1)$, $(2, 2, 1)$ and $(1, 1, 2)$.

[5]



$A(1, 0, -1)$

$B(2, 2, 1)$

$C(1, 1, 2)$

Idea: Find $\vec{AB}$ and $\vec{AC}$ (assuming they aren't parallel). These vectors both lie in our plane, so if we cross them we can find a vector perpendicular to both. This will be our normal.

$$\vec{AB} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad \vec{AC} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ -1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

So, $4x - 3y + z = d$

sub either A, B, C to find d.

$4(1) - 3(0) - 1 = d \implies d = 3$

so $\boxed{4x - 3y + z = 3}$ is our equation

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3 **(a)** Find $\mathbf{M}^{-1}$, where $\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ -2 & 1 & 2 \end{pmatrix}$. **[1]**

(b) Hence find, in terms of the constant k, the point of intersection of the planes

$$x + 2y + 3z = 19,$$
$$-x + y + 2z = 4,$$
$$-2x + y + 2z = k.$$ **[3]**

(c) **In this question you must show detailed reasoning.**

Find the acute angle between the planes $x + 2y + 3z = 19$ and $-x + y + 2z = 4$. **[4]**

a) $M^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -8 & 5 \\ -1 & 5 & -3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 19 \\ 4 \\ k \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -8 & 5 \\ -1 & 5 & -3 \end{pmatrix} \begin{pmatrix} 19 \\ 4 \\ k \end{pmatrix}$

$= \begin{pmatrix} 4 - k \\ 6 + 5k \\ 1 - 3k \end{pmatrix}$

So $(4 - k, 6 + 5k, 1 - 3k)$ is the point of intersection.

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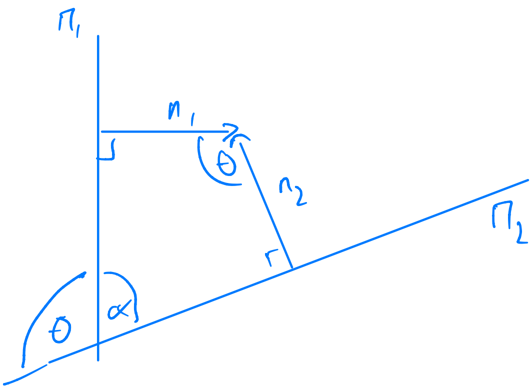
c) Theory:



By the dot product, $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

If θ is obtuse, then I do $180 - \theta$ to find α, as this will be acute.

If θ is acute, then I'm done.

These are the normals.

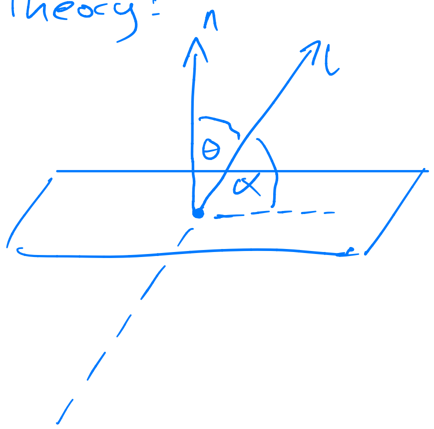
$$\cos\theta = \frac{\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\2\end{pmatrix}}{\sqrt{1^2+2^2+3^2}\sqrt{1^2+1^2+2^2}} = \frac{7}{\sqrt{14}\sqrt{6}}$$

$$\theta = \boxed{40.2^\circ}$$

4 In this question you must show detailed reasoning.

Find the angle between the vector $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and the plane $-x+3y+2z=8$. **[5]**

Theory:



We want α. Using the dot product,

$\cos\theta = \frac{\underline{n}\cdot\underline{b}}{|\underline{n}||\underline{b}|}$ where $\underline{b}$ is the direction vector of l.

① If θ is acute, then $\alpha = 90-\theta$ and we have found the angle.

② If θ is obtuse, then we need to do $180-\theta$ to find the acute angle between the normal and the line and then 90 - the result to find α.

$$\cos\theta = \frac{\begin{pmatrix}3\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}}{\sqrt{3^2+2^2+1^2}\sqrt{1^2+3^2+2^2}} = \frac{5}{\sqrt{14}\sqrt{14}} = \frac{5}{14}$$

$\theta = 69.1^\circ$

$\alpha = 90-\theta = \boxed{20.9^\circ}$

5 **(a)** Show that the vector $\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ is parallel to the plane $2x+y-3z=10$. **[3]**

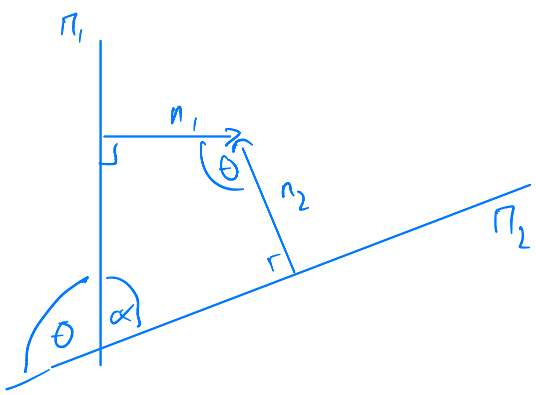
(b) Determine the acute angle between the planes $2x+y-3z=10$ and $x-y-3z=3$. **[4]**

a) $\begin{pmatrix}1\\4\\2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-3\end{pmatrix} = 2+4-6=0$

So $\begin{pmatrix}1\\4\\2\end{pmatrix}$ is perpendicular to the normal vector of the plane

$\Rightarrow \begin{pmatrix}1\\4\\2\end{pmatrix}$ is parallel to the plane.

b) Theory:



By the dot product, $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

If θ is obtuse, then I do $180-\theta$ to find α, as this will be acute.

If θ is acute, then I'm done.

$$\cos\theta = \frac{\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-3\end{pmatrix}}{\sqrt{2^2+1^2+3^2}\sqrt{1^2+1^2+3^2}} = \frac{10}{\sqrt{14}\sqrt{11}}$$

$\theta = \boxed{36.3^\circ}$

6 Three planes have the following equations.

$$2x - 3y + z = -3,$$
$$x - 4y + 2z = 1,$$
$$-3x - 2y + 3z = 14.$$

(a) (i) Write the system of equations in matrix form. **[1]**

(ii) Hence find the point of intersection of the planes. **[2]**

(b) In this question you must show detailed reasoning.

Find the acute angle between the planes $2x - 3y + z = -3$ and $x - 4y + 2z = 1$. **[4]**

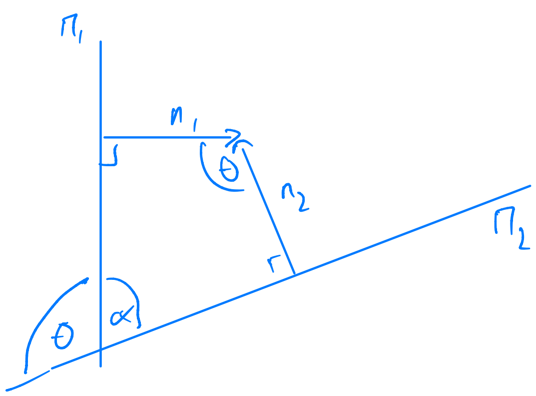
a) i) $\begin{pmatrix} 2 & -3 & 1 \\ 1 & -4 & 2 \\ -3 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 14 \end{pmatrix}$

ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -4 & 2 \\ -3 & -2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} -3 \\ 1 \\ 14 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$

So $\boxed{(-1, 2, 5)}$

b)

Theory:



By the dot product, $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

If θ is obtuse, then I do $180 - \theta$ to find α, as this will be acute.

If θ is acute, then I'm done.

P.T.O

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$$\cos\theta = \frac{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}}{\sqrt{2^2+3^2+1^2}\sqrt{1^2+4^2+2^2}} = \frac{16}{\sqrt{14}\,\sqrt{21}}$$

$\boxed{\theta = 21.1^\circ}$ to 3sf

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7 The equations of two non-intersecting lines, l_1 and l_2, are

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

Find the shortest distance between lines l_1 and l_2. **[5]**

If they are skew, we can just use the formula booklet.

First check parallel:

$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ so l_1 and l_2 are not parallel

We are told they are also non-intersecting, so they are therefore skew.

The distance between skew lines is $D = \dfrac{|(\mathbf{b}-\mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$, where **a** and **b** are position vectors of points on each line and **n** is a mutual perpendicular to both lines

$$\underline{n} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 1 & -1 & 4 \end{vmatrix} = 2\underline{i} - 10\underline{j} - 3\underline{k}$$

$$D = \frac{\left| \left(\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ -10 \\ -3 \end{pmatrix} \right|}{\sqrt{2^2 + 10^2 + 3^2}}$$

$$= \frac{\left| \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -10 \\ -3 \end{pmatrix} \right|}{\sqrt{113}} = \frac{2+6}{\sqrt{113}} = \boxed{\frac{8}{\sqrt{113}}}$$

8 The lines l_1 and l_2 have equations $\frac{x-3}{1}=\frac{y-5}{2}=\frac{z+2}{-3}$ and $\frac{x-4}{2}=\frac{y+2}{-1}=\frac{z-7}{4}$.

(i) Find the shortest distance between l_1 and l_2. **[5]**

(ii) Find a cartesian equation of the plane which contains l_1 and is parallel to l_2. **[2]**

i) $l_1: \underline{r}=\begin{pmatrix}3\\5\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-3\end{pmatrix}$ $\quad l_2: \underline{r}=\begin{pmatrix}4\\-2\\7\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\4\end{pmatrix}$

If the lines are skew, we can just use the formula booklet.

First check parallel:

not parallel since $\begin{pmatrix}1\\2\\-3\end{pmatrix}\neq\begin{pmatrix}2\\-1\\4\end{pmatrix}$

not intersecting otherwise shortest distance would be 0.

The distance between skew lines is $D=\frac{|(\mathbf{b}-\mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$, where **a** and **b** are position vectors of points on each line and **n** is a mutual perpendicular to both lines

$$\underline{n}=\begin{pmatrix}1\\2\\-3\end{pmatrix}\times\begin{pmatrix}2\\-1\\4\end{pmatrix}=\begin{vmatrix}i & j & k\\1 & 2 & -3\\2 & -1 & 4\end{vmatrix}=\begin{pmatrix}5\\-10\\-5\end{pmatrix}$$

can scale normal down

$$\text{so} \quad D=\frac{\left|\left(\begin{pmatrix}3\\5\\-2\end{pmatrix}-\begin{pmatrix}4\\-2\\7\end{pmatrix}\right)\cdot\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right|}{\sqrt{1^2+2^2+1^2}}$$

$$=\frac{\left|\begin{pmatrix}-1\\7\\-9\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right|}{\sqrt{6}}=\frac{|-6|}{\sqrt{6}}=\boxed{\sqrt{6}}$$

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b) Vector equation:

$$\underline{r} = \underbrace{\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}}_{\text{"contains } l_1\text{"}} + \mu \underbrace{\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}}_{\text{"parallel to } l_2\text{"}}$$

To get the cartesian equation we need to cross the direction vectors to get the normal.

From part (a), $n = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

So $x - 2y - z = d$

we know $(3, 5, -2)$ lies on the plane, so

$$3 - 2(5) - (-2) = d \implies d = -5$$

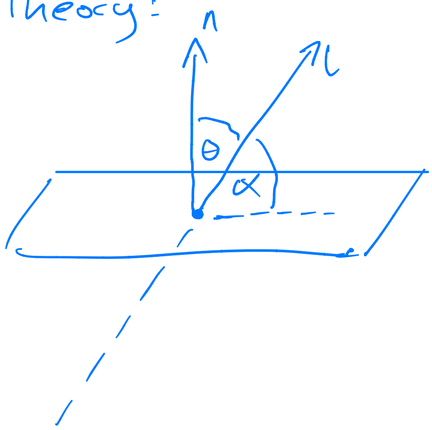
So $\boxed{x - 2y - z = -5}$

9 The equation of a plane Π is $x - 2y - z = 30$.

(i) Find the acute angle between the line $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}$ and Π. **[4]**

(ii) Determine the geometrical relationship between the line $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ and Π. **[4]**

i) Theory:



We want α. Using the dot product,

$\cos\theta = \dfrac{\underline{n} \cdot \underline{b}}{|\underline{n}|\,|\underline{b}|}$ where $\underline{b}$ is the direction vector of l.

① If θ is acute, then $\alpha = 90 - \theta$ and we have found the angle.

② If θ is obtuse, then we need to do $180 - \theta$ to find the acute angle between the normal and the line and then 90 - the result to find α.

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}}{\sqrt{1^2+2^2+1^2}\,\sqrt{5^2+3^2+2^2}} = \frac{-13}{\sqrt{6}\,\sqrt{38}}$$

$\theta = 149.4^\circ$

$180 - \theta = 30.6^\circ$

$\alpha = 90 - 30.6^\circ = 59.4^\circ$

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ii) First check if the plane and line are parallel.

$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 3 + 2 - 5 = 0$$

so the normal of the plane is perpendicular to the line

$\Longrightarrow$ the plane itself is parallel to the line.

Now, check if the line actually lies in the plane as well.

Sub in any point on the line (e.g $(1, 4, 2)$) into the equation of the plane!

$$1 - 2(4) - 2 = -9 \neq 30$$

So the line is parallel to the plane, but doesn't lie in the plane.

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10 **(a)** A plane Π has the equation $\mathbf{r}.\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = 15$. C is the point $(4, -5, 1)$.

Find the shortest distance between Π and C. **[3]**

(b) Lines l_1 and l_2 have the following equations.

$$l_1 : \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Find, in exact form, the distance between l_1 and l_2. **[5]**

a)

The distance between a point and a plane is $D = \dfrac{|\mathbf{b}.\mathbf{n} - p|}{|\mathbf{n}|}$, where $\mathbf{b}$ is the position vector of the point and the equation of the plane is given by $\mathbf{r}.\mathbf{n} = p$

$$D = \frac{\left|\begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} - 15\right|}{\sqrt{3^2 + 6^2 + 2^2}}$$

$$= \frac{|12 - 30 - 2 - 15|}{\sqrt{49}} = \frac{35}{7} = \boxed{5}$$

b) P.T.O

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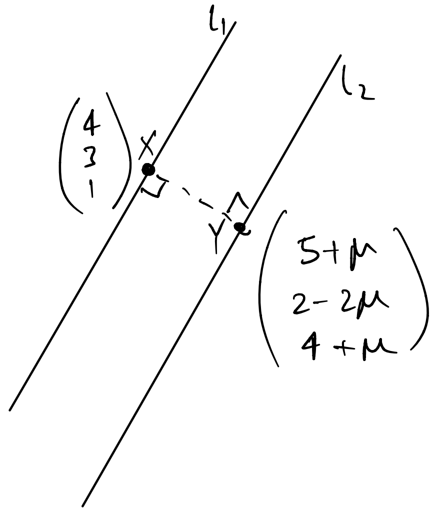
b) If they are skew, we can just use the formula booklet.

First check parallel:

$$\begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

So l_1 and l_2 are parallel and therefore not skew.



Notice that any point on l_1 which is joined by a vector perpendicular to l_1, to l_2 will be the shortest distance. So I can pick any point on l_1. Easiest is $(4,3,1)$

Now we need the general position vector of l_2 which is $\begin{pmatrix} 5+\mu \\ 2-2\mu \\ 4+\mu \end{pmatrix}$

Now, $\overrightarrow{XY} / \overrightarrow{YX}$ is perpendicular to l_1 / l_2 so I can dot $\overrightarrow{XY}$ or $\overrightarrow{YX}$ with the direction vector of l_1 or l_2 and set it equal to 0.

$$\overrightarrow{XY} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1+\mu \\ -1-2\mu \\ 3+\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1+\mu+2+4\mu+3+\mu = 0$$

$$\Rightarrow 6\mu + 6 = 0$$

$$\Rightarrow \mu = -1$$

$$\therefore \overrightarrow{XY} = \begin{pmatrix} 1-1 \\ -1+2 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\therefore |\overrightarrow{XY}| = \sqrt{0^2+1^2+2^2} = \boxed{\sqrt{5}}$$

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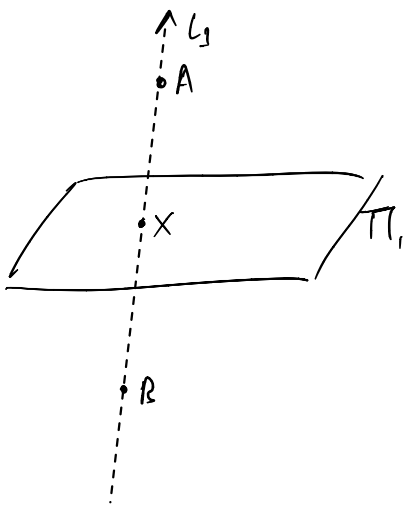
11 The equation of a plane is $4x+2y+z=7$.
The point A has coordinates (9, 6, 1) and the point B is the reflection of A in the plane.

Find the coordinates of the point B. **[6]**



let Π_1 be the plane: $4x+2y+z=7$

and l_1 the line that goes through A.

B is a reflection of A so l_1 must be perpendicular to the plane.

Therefore the vector equation of l_1 is

$$\underline{r} = \begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

by definition, the normal vector is just $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ where $n_1 x + n_2 y + n_3 z = d$

To find B, first I'll find X.

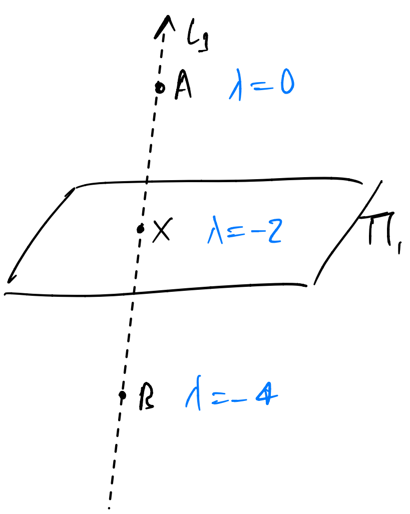
X is the intersection of the line and plane so I sub the general position vector of the line which is $\begin{pmatrix} 9+4\lambda \\ 6+2\lambda \\ 1+\lambda \end{pmatrix}$

into the plane: $4(9+4\lambda) + 2(6+2\lambda) + 1+\lambda = 7$

$$21\lambda + 49 = 7$$

$$\Rightarrow 21\lambda = -42$$

$$\lambda = -2$$



$\lambda = 0$ at A because $\begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix}$

which is the position vector of A. $\lambda = -2$ at X as we found above. Therefore $\lambda = -4$ at B since A and B must be equidistant from X. P.T.O

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Sub $\lambda = -4$ in to find B:

$$\vec{OB} = \begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix} - 4 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ -3 \end{pmatrix}$$

$$\therefore \boxed{B = (-7, -2, -3)}$$

Have to write as coordinates!

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12 (a) Show that the three planes with equations

$$x + \lambda y + 3z = -12$$
$$2x + y + 5z = -11$$
$$x - 2y + 2z = -9$$

where λ is a constant, meet at a unique point except for one value of λ which is to be determined. **[3]**

(b) In the case $\lambda = -2$, use matrices to find the point of intersection P of the planes, showing your method clearly. **[3]**

The line l has equation $\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z+2}{-2}$.

(c) Find a vector equation of l. **[2]**

(d) Find the shortest distance between the point P and l. **[4]**

(e) (i) Show that l is parallel to the plane $x - 2y + 2z = -9$. **[3]**

(ii) Find the distance between l and the plane $x - 2y + 2z = -9$. **[2]**

a) see consistency of solutions if you're not sure about this

$\det M \neq 0 \iff$ planes intersect at a unique point

$$M = \begin{pmatrix} 1 & \lambda & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix}$$

$$\det M = 1(2+10) - \lambda(4-5) + 3(-4-1)$$
$$= 12 + \lambda - 15 = \lambda - 3$$

if $\lambda = 3$, then $\det M = 0$ and the planes <u>don't</u> intersect at a single point.

So unique point provided that $\lambda \neq 3$.

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b) $$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

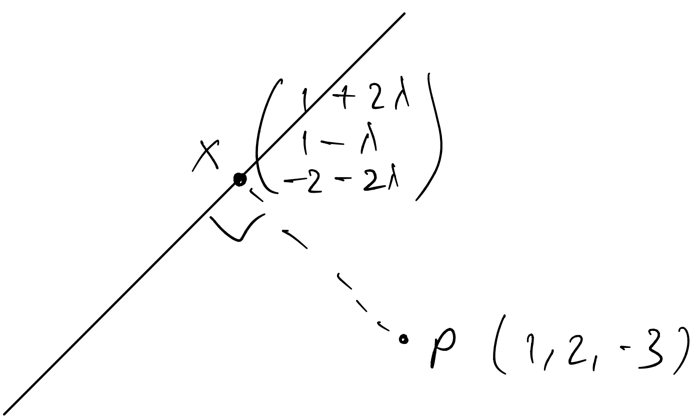
Point of intersection : $P \boxed{(1, 2, -3)}$

c) $$\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z+2}{-2}$$

$x = 1 + 2\lambda$, $y = 1 - \lambda$, $z = -2 - 2\lambda$

$$\underline{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

d)



$$\overrightarrow{PX} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$$

Since $\overrightarrow{PX}$ is perpendicular to the line so we can dot it with the direction vector of the line. P.T.O

$$\vec{PX} = \begin{pmatrix} 2\lambda \\ -1-\lambda \\ 1-2\lambda \end{pmatrix}$$

so, $\begin{pmatrix} 2\lambda \\ -1-\lambda \\ 1-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$

$$4\lambda + 1 + \lambda - 2 + 4\lambda = 0$$

$$9\lambda = 1 \qquad \Rightarrow \lambda = 1/9$$

sub $\lambda = 1/9$ into $\vec{PX}$:

$\vec{PX} = \begin{pmatrix} 2/9 \\ -10/9 \\ 7/9 \end{pmatrix}$ $\qquad |\vec{PX}| = \sqrt{\left(\frac{2}{9}\right)^2 + \left(\frac{-10}{9}\right)^2 + \left(\frac{7}{9}\right)^2}$

$$= \boxed{\frac{\sqrt{17}}{3}}$$

e) i) $x - 2y + 2z = -9$

if the line is parallel to the plane, then the line must be perpendicular to the normal.

$$\underbrace{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}}_{\text{normal}} \cdot \underbrace{\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}}_{\text{direction vector of line}} = 2 + 2 - 4 = 0$$

$\Rightarrow$ line l is parallel to plane

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e ii)

The distance between a point and a plane is $D = \frac{|\mathbf{b}.\mathbf{n} - p|}{|\mathbf{n}|}$, where $\mathbf{b}$ is the position vector of the point and the equation of the plane is given by $\mathbf{r}.\mathbf{n} = p$

We can just take any point on the line, since the line and plane are parallel.

$$D = \frac{\left| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - -9 \right|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{4}{\sqrt{9}} = \boxed{\frac{4}{3}}$$

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13 The equation of the plane Π is $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$.

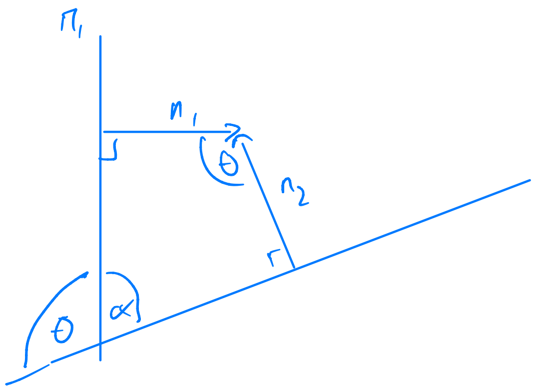
(a) Find the acute angle between Π and the plane with equation $\mathbf{r} . \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = 4$. **[4]**

The point A has coordinates $(9, -7, 20)$.

The point F is the point of intersection between Π and the perpendicular from A to Π.

(b) Determine the coordinates of F. **[4]**

a) Theory:



By the dot product, $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

If θ is obtuse, then I do $180-\theta$ to find α, as this will be acute.

If θ is acute, then I'm done.

To find the normal of Π, I cross the two direction vectors:

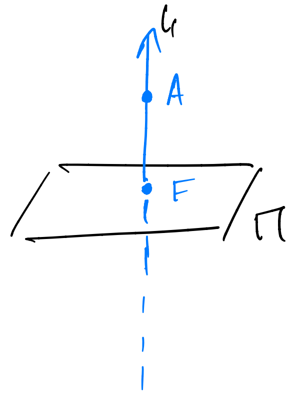
$$n = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 4 & 4 & 3 \\ -2 & 3 & 1 \end{vmatrix} = \begin{pmatrix} -5 \\ -10 \\ 20 \end{pmatrix}$$

by rescaling

$$\text{so} \quad \cos\theta = \frac{\begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}}{\sqrt{1^2+2^2+4^2}\sqrt{2^2+3^2}} = \frac{10}{\sqrt{21}\sqrt{13}}$$

$$\text{so} \quad \theta = \boxed{52.8^\circ} \text{ to 3sf.}$$

b)



let l_1 be the line that goes through A
and is perpendicular to Π.

$$l_1: \ \underline{r} = \begin{pmatrix} 9 \\ -7 \\ 20 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$$

Since the direction vector of l_1 is parallel to the normal vector of the plane.

To find F, I sub $\begin{pmatrix} 9 - \lambda \\ -7 - 2\lambda \\ 20 + 4\lambda \end{pmatrix}$ into Π:

But first I need the cartesian equation of Π:

$$-x - 2y + 4z = d$$

sub $(-1, 2, 1)$ in to find d (since this point lies on the plane)

$$1 - 4 + 4 = d \implies d = 1$$

$$-x - 2y + 4z = 1$$

$$-(9 - \lambda) - 2(-7 - 2\lambda) + 4(20 + 4\lambda) = 1$$

$$21\lambda + 85 = 1$$

$$\lambda = -4$$

sub $\lambda = -4$ into l_1 to find F:

$$\boxed{F = (13, 1, 4)}$$

***

14 The plane P has normal vector $2\mathbf{i} + a\mathbf{j} - \mathbf{k}$, where a is a positive constant, and the point $(3, -1, 1)$ lies in P. The plane $x - z = 3$ makes an angle of 45° with P.

Find the cartesian equation of P. **[7]**

(1) The plane P has normal vector $2\mathbf{i} + a\mathbf{j} - \mathbf{k}$,

$\Rightarrow$ Plane P has cartesian equation: $2x + ay - z = d$

(2) "and the point $(3,-1,1)$ lies in P"

$\Rightarrow$ we can sub $(3,-1,1)$ into the equation:

$$2(3) + a(-1) - 1 = d$$
$$5 - a = d$$

(3) The plane $x - z = 3$ makes an angle of 45° with P

$$\Rightarrow \quad \frac{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\a\\-1\end{pmatrix}}{\sqrt{1^2+1^2}\ \sqrt{2^2+a^2+1^2}} = \cos(45)$$

$$\frac{3}{\sqrt{2}\ \sqrt{5+a^2}} = \frac{\sqrt{2}}{2}$$

$$3 = \sqrt{5+a^2}$$

$$9 = 5 + a^2$$

$$a^2 = 4$$

$a = \pm 2$ $\quad$ a is a positive constant $\Rightarrow$ $a = 2$.

So by (2), $\quad 5 - 2 = d \Rightarrow d = 3$

and by (1), $\quad \boxed{2x + 2y - z = 3}$

 ***

15 A vector **v** has magnitude 1 unit. The angle between **v** and the positive z-axis is 60°, and **v** is parallel to the plane $x-2y=0$.

Given that $\mathbf{v} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, where a, b and c are all positive, find **v**. **[7]**

① vector v has magnitude 1

$\Rightarrow \quad a^2+b^2+c^2=1$

② The angle between **v** and the positive z-axis is 60°

$$\cos(60) = \frac{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{a^2+b^2+c^2} \times 1}$$

$$\frac{1}{2} = \frac{c}{1 \times 1} \qquad \text{since } a^2+b^2+c^2=1$$

$\Rightarrow c = \frac{1}{2}$

③ "v is parallel to the plane $x-2y=0$"

$\Rightarrow$ v is perpendicular to $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ (since it's the normal)

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 0 \quad \Rightarrow \quad a-2b=0$$

$a = 2b$

sub $a=2b$, $c=\frac{1}{2}$ into $a^2+b^2+c^2=1$

$$4b^2+b^2+\frac{1}{4}=1$$

$$5b^2 = \frac{3}{4}$$

$$b^2 = \frac{3}{20} \qquad b = \sqrt{\frac{3}{20}} = \frac{\sqrt{15}}{10} \quad \text{since } b>0.$$

$$a = 2b = 2\frac{\sqrt{15}}{10} = \frac{\sqrt{15}}{5} \quad \text{or } \left(\sqrt{\frac{3}{5}}\right)$$

***

so $\boxed{\underline{v} = \frac{\sqrt{15}}{5}\underline{i} + \frac{\sqrt{15}}{10}\underline{j} + \frac{1}{2}\underline{k}}$

***

16 The plane Π has equation $3x - 5y + z = 9$.

(i) Show that Π contains

- the point $(4,1,2)$

and

- the vector $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. **[4]**

(ii) Determine the equation of a plane which is perpendicular to Π and which passes through $(4,1,2)$. **[3]**

i) sub $(4,1,2)$ into plane:

$LHS = 3(4) - 5(1) + 2 = 12 - 5 + 2 = 9 = RHS$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} = 3 - 5 + 2 = 0$$

So $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is perpendicular to the normal, so it must lie in the plane.

ii) We need to find the normal of this new plane. But if $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ lies in the old plane it must be perpendicular to the new plane.

So $x + y + 2z = d$

Sub $(4,1,2)$ in:

$4 + 1 + 4 = d \qquad d = 9$

So $\boxed{x + y + 2z = 9}$

****

17 The equations of two intersecting lines l_1 and l_2 are

$$l_1\colon \mathbf{r} = \begin{pmatrix}1\\0\\a\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-3\end{pmatrix} \qquad l_2\colon \mathbf{r} = \begin{pmatrix}7\\9\\-2\end{pmatrix} + \mu\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

where a is a constant.

The equation of the plane Π is

$$\mathbf{r}.\begin{pmatrix}1\\5\\3\end{pmatrix} = -14.$$

l_1 and Π intersect at Q.

l_2 and Π intersect at R.

(a) Verify that the coordinates of R are (13, 3, –14). **[2]**

(b) Determine the exact value of the length of QR. **[7]**

a) Sub $\begin{pmatrix}7-\mu\\9+\mu\\-2+2\mu\end{pmatrix}$ into Π :

$$\begin{pmatrix}7-\mu\\9+\mu\\-2+2\mu\end{pmatrix}\cdot\begin{pmatrix}1\\5\\3\end{pmatrix} = -14$$

$$\Rightarrow \quad 7-\mu+45+5\mu-6+6\mu = -14$$

$$10\mu + 46 = -14$$

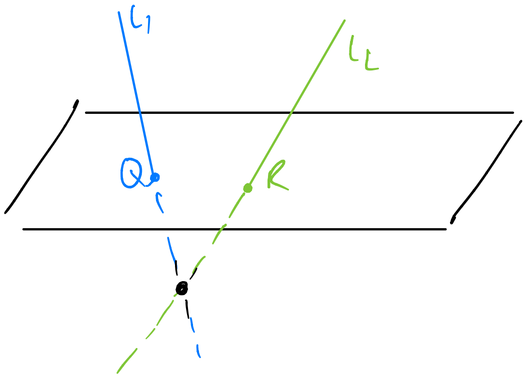
$$\mu = -6$$

Sub $\mu = -6$ into l_2 :

$R = (13, 3, -14)$

P.T.O

b) ****



We need to find point Q, by "subbing" in l_1 into Π and finding the value of λ. But, if I do this I'll get an equation with 2 unknowns. So first I should find a.

We know l_1 & l_2 intersect so:

$$\begin{pmatrix} 1+2\lambda \\ \lambda \\ a-3\lambda \end{pmatrix} = \begin{pmatrix} 7-\mu \\ 9+\mu \\ -2+2\mu \end{pmatrix} \Rightarrow \begin{array}{ll} 1+2\lambda = 7-\mu & (A) \\ \lambda = 9+\mu & (B) \end{array}$$

$(A)+(B)$: $\quad 1+3\lambda = 16$

$\Rightarrow \lambda = 5$

Sub $\lambda = 5$ into (B):

$\mu = -4$

Sub $\lambda = 5, \mu = -4$ into (C):

$a - 3(5) = -2 + 2(-4)$

$a = 5$

Now, I can "sub" l_1 into Π:

$$\begin{pmatrix} 1+2\lambda \\ \lambda \\ 5-3\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = -14$$

$1 + 2\lambda + 5\lambda + 15 - 9\lambda = -14$

$-2\lambda = -30$

$\lambda = 15$

****

Sub $\lambda = 15$ into l_1 to find Q:

$$Q = (31, 15, -40)$$

$$|QR| = \sqrt{(31-13)^2 + (15-3)^2 + (-40--14)^2}$$

$$= \boxed{2\sqrt{286}} \quad = \left(\sqrt{1144}\right)$$

 ****

18 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.

The plane Π has equation $\mathbf{r}.\begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} = 4$.

(a) Find the position vector of the point of intersection of l_1 and Π. **[3]**

(b) Find the acute angle between l_1 and Π. **[3]**

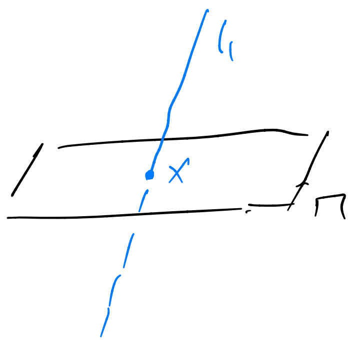
A is the point on l_1 where $\lambda = 1$.

l_2 is the line with the following properties.

- l_2 passes through A
- l_2 is perpendicular to l_1
- l_2 is parallel to Π

(c) Find, in vector form, the equation of l_2. **[3]**

a)



To find X, "sub" l_1 into Π:

$$\begin{pmatrix} 1+3\lambda \\ -3+2\lambda \\ 3-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} = 4$$

$$2+6\lambda+15-10\lambda-9+6\lambda = 4$$

$$2\lambda + 8 = 4$$

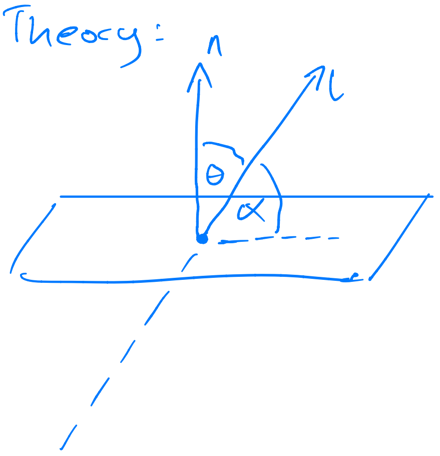
$$\lambda = -2$$

sub $\lambda = -2$ into l_1:

$$X = \begin{pmatrix} -5 \\ -7 \\ 7 \end{pmatrix}$$

****

b) Theory:



We want α. Using the dot product,

$$\cos\theta = \frac{\underline{n}\cdot\underline{b}}{|\underline{n}||\underline{b}|}$$ where $\underline{b}$ is the direction vector of l.

① If θ is acute, then $\alpha = 90 - \theta$ and we have found the angle.

② If θ is obtuse, then we need to do $180-\theta$ to find the acute angle between the normal and the line and then 90 - the result to find α.

$$\cos\theta = \frac{\begin{pmatrix}2\\-5\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-2\end{pmatrix}}{\sqrt{2^2+5^2+3^2}\sqrt{3^2+2^2+2^2}} = \frac{2}{\sqrt{38}\sqrt{17}}$$

$\theta = 85.49°$ to 2dp.

so $\alpha = 90 - \theta = \boxed{4.51°}$ to 2dp

c) $\lambda = 1$: $r = \begin{pmatrix}4\\-1\\1\end{pmatrix}$ so $A = (4,-1,1)$

① l_2 is perpendicular to l_1

② l_2 is parallel to $\Pi \Rightarrow l_2$ is perpendicular to $\underline{n}$

so ① and ② $\Rightarrow$ l_2 is perpendicular to l_1 and $\underline{n}$.

To find a vector perpendicular to l_1 and $\underline{n}$, we can cross the direction vector of l_1 and n:

****

$$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 3 & 2 & -2 \\ 2 & -5 & -3 \end{vmatrix} = \begin{pmatrix} -16 \\ 5 \\ -19 \end{pmatrix}$$

So l_2 is $\underline{r} = \boxed{\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -16 \\ 5 \\ -19 \end{pmatrix}}$

or any multiple of this

****

19 In this question you must show detailed reasoning.

Find a vector **v** which has the following properties.

- It is a unit vector.
- It is parallel to the plane $2x+2y+z=10$.
- It makes an angle of 45° with the normal to the plane $x+z=5$. **[8]**

let $\underline{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

(1) It is a unit vector.

$\Rightarrow \quad |\underline{v}| = 1$

$\Rightarrow \quad a^2+b^2+c^2 = 1$ (1) (since we can square both sides to get rid of $\sqrt{\ }$)

(2) It is parallel to the plane $2x+2y+z=10$.

$\Rightarrow$ v is perpendicular to the normal of the plane

$\Rightarrow \quad v \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 0$

$\Rightarrow \quad 2a+2b+c=0$ (2)

(3) It makes an angle of 45° with the normal to the plane $x+z=5$.

$$\Rightarrow \quad \cos(45) = \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}}{\sqrt{2}\ \sqrt{a^2+b^2+c^2}}$$

$$\frac{\sqrt{2}}{2} = \frac{a+c}{\sqrt{2} \times 1}$$

$a+c = 1$ (3)

****

$$a^2+b^2+c^2=1 \quad (1)$$

$$2a+2b+c=0 \quad (2)$$

$$a+c=1 \quad (3)$$

$(2)-(3)$: $\quad a+2b=-1$

$$\Rightarrow b=\frac{-1-a}{2}$$

sub $b=\frac{-1-a}{2}$ and $c=1-a$ into (1):

$$a^2+\left(\frac{-1-a}{2}\right)^2+(1-a)^2=1$$

$$a^2+\frac{a^2+2a+1}{4}+a^2-2a+\cancel{1}=\cancel{1} \quad \Big\} \times 4$$

$$4a^2+a^2+2a+1+4a^2-8a=0$$

$$9a^2-6a+1=0$$

$$(3a-1)^2=0 \quad \Rightarrow a=1/3$$

so $b=\frac{-1-1/3}{2}=\frac{-2}{3}$ and $c=1-\frac{1}{3}=\frac{2}{3}$

so $\underline{v}=\begin{pmatrix}1/3\\-2/3\\2/3\end{pmatrix}$

20 The point P (4, 1, 0) is equidistant from the plane $2x+y+2z=0$ and the line $\frac{x-3}{2}=\frac{y-1}{b}=\frac{z+5}{3}$, where $b>0$.

Determine the value of b. **[10]**

Distance from point to plane:

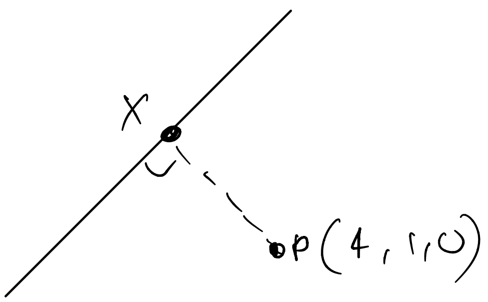
The distance between a point and a plane is $D=\frac{|\mathbf{b}.\mathbf{n}-p|}{|\mathbf{n}|}$, where **b** is the position vector of the point and the equation of the plane is given by $\mathbf{r}.\mathbf{n}=p$

$$D=\frac{\left|\begin{pmatrix}4\\1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix}-0\right|}{\sqrt{2^2+1^2+2^2}}=\frac{9}{\sqrt{9}}=\sqrt{9}$$

Distance from point to line:

first get in vector form:

$$r=\begin{pmatrix}3\\1\\-5\end{pmatrix}+\lambda\begin{pmatrix}2\\b\\3\end{pmatrix}$$



X has position vector $\begin{pmatrix}3+2\lambda\\1+b\lambda\\-5+3\lambda\end{pmatrix}$

$$\vec{PX}\cdot\begin{pmatrix}2\\b\\3\end{pmatrix}=0$$

$$\begin{pmatrix}2\lambda-1\\b\lambda\\-5+3\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\b\\3\end{pmatrix}=0$$

$$4\lambda-2+b^2\lambda-15+9\lambda=0$$

$$\Rightarrow \lambda(13+b^2)=17$$

$$\lambda=\frac{17}{13+b^2}$$

****

Now, we want $|\overrightarrow{PX}| = \sqrt{9}$

before we sub $\lambda = \frac{17}{13+b^2}$ let's find $|\overrightarrow{PX}|$ in terms of λ

(as this should be tidier to work with)

$$|\overrightarrow{PX}| = \sqrt{(2\lambda-1)^2+(b\lambda)^2+(-5+3\lambda)^2}$$

$$= \sqrt{4\lambda^2-4\lambda+1+b^2\lambda^2+25-30\lambda+9\lambda^2}$$

$$= \sqrt{(13+b^2)\lambda^2-34\lambda+26}$$

Sub in $\lambda = \frac{17}{13+b^2}$ and set $|\overrightarrow{PX}| = \sqrt{9}$

$$9 = (13+b^2)\left(\frac{17}{13+b^2}\right)^2 - 34\left(\frac{17}{13+b^2}\right) + 26$$

$$-17 = \frac{1}{13+b^2}\left(17^2 - 34\times 17\right)$$

$$-17 = \frac{-289}{13+b^2}$$

$13+b^2 = 17$ $\Rightarrow b^2 = 4$

$\Rightarrow \boxed{b = 2}$ since $b > 0$

 ****

21 The plane Π has equation $2x-y+2z=4$. The point P has coordinates (8, 4, 5).

(a) Calculate the shortest distance from P to Π. **[2]**

The line L has equation $\frac{x-2}{3}=\frac{y}{2}=\frac{z+3}{4}$.

(b) Verify that P lies on L. **[2]**

(c) Find the coordinates of the point of intersection of L and Π. **[3]**

(d) Determine the acute angle between L and Π. **[4]**

(e) Use the results of parts **(b)**, **(c)** and **(d)** to verify your answer to part **(a)**. **[3]**

a) The distance between a point and a plane is $D=\frac{|\mathbf{b}.\mathbf{n}-p|}{|\mathbf{n}|}$, where $\mathbf{b}$ is the position vector of the point and the equation of the plane is given by $\mathbf{r}.\mathbf{n}=p$

$$D=\frac{\left|\begin{pmatrix}8\\4\\5\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\2\end{pmatrix}-4\right|}{\sqrt{2^2+1^2+2^2}}=\frac{|16-4+10-4|}{3}=\frac{18}{3}=\boxed{6}$$

b) $\frac{8-2}{3}=2 \qquad \frac{4}{2}=2 \qquad \frac{5+3}{4}=2$

so $P(8,4,5)$ lies on the line

c) $\left.\begin{array}{l}x=2+3\lambda\\ y=2\lambda\\ z=-3+4\lambda\end{array}\right\}$ sub into Π

$$2(2+3\lambda)-2\lambda+2(-3+4\lambda)=4$$

$$12\lambda-2=4 \quad \Rightarrow \lambda=1/2$$

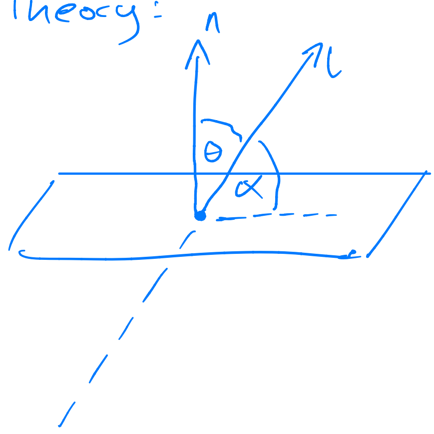
sub $\lambda=1/2$ back into line :

****

Point of intersection is $\boxed{\left(\frac{7}{2}, 1, -1\right)}$

d)

Theory:



We want α. Using the dot product,

$$\cos\theta = \frac{\underline{n} \cdot \underline{b}}{|\underline{n}|\,|\underline{b}|}$$

where $\underline{b}$ is the direction vector of l.

① If θ is acute, then $\alpha = 90 - \theta$ and we have found the angle.

② If θ is obtuse, then we need to do $180 - \theta$ to find the acute angle between the normal and the line and then $90 -$ the result to find α.

$$\cos\theta = \frac{\begin{pmatrix}3\\2\\4\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\2\end{pmatrix}}{\sqrt{3^2+2^2+4^2}\,\sqrt{2^2+1^2+2^2}}$$

direction vector of line

normal vector of plane

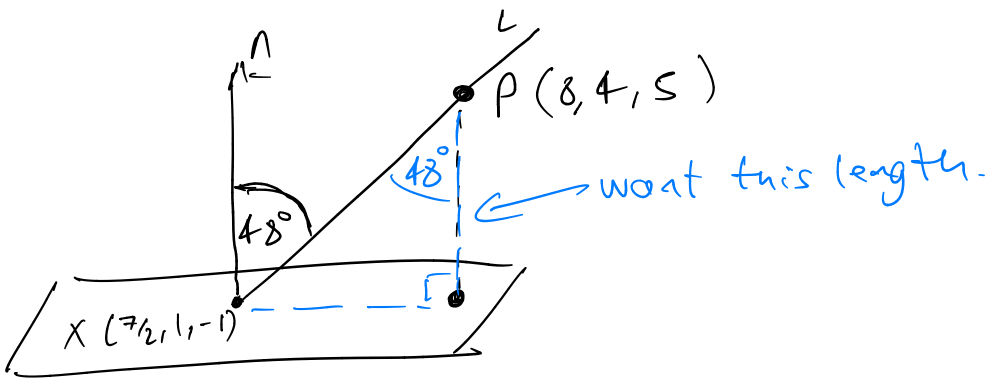
$$= \frac{12}{\sqrt{29} \times 3}$$

So $\theta = \cos^{-1}\left(\frac{12}{3\sqrt{29}}\right) = 42.0^\circ$

$$\alpha = 90 - \theta = 90 - 42 = \boxed{48.0^\circ}$$

****

e)



let X be $(7/2, 1, -1)$

If I find $|\vec{PX}|$ then I can use basic trig (SOHCAHTOA)

$$|\vec{PX}| = \sqrt{(8-7/2)^2 + (4-1)^2 + (5--1)^2} = \frac{3\sqrt{29}}{2}$$

let Y be the length I want to find.

$$\cos(48) = \frac{Y}{|\vec{PX}|}$$

but I know $\cos(48) = \frac{12}{3\sqrt{29}}$ from (d)

$$\text{so} \quad Y = \frac{12}{3\sqrt{29}} \times |\vec{PX}| = \frac{12}{\cancel{3\sqrt{29}}} \times \frac{\cancel{3\sqrt{29}}}{2} = \frac{12}{2} = 6$$

$\square$

22 The points P, Q and R have coordinates $(0, 2, 3)$, $(2, 0, 1)$ and $(1, 3, 0)$ respectively.

The acute angle between the line segments PQ and PR is θ.

(a) Show that $\sin\theta = \frac{2}{11}\sqrt{22}$. **[3]**

The triangle PQR lies in the plane Π.

(b) Determine an equation for Π, giving your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. **[3]**

The point S has coordinates $(5, 3, -1)$.

(c) By finding the shortest distance between S and the plane Π, show that the volume of the tetrahedron $PQRS$ is $\frac{14}{3}$.

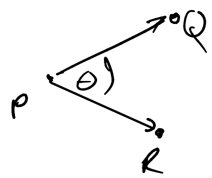
[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height] **[4]**

The tetrahedron $PQRS$ is transformed to the tetrahedron $P'Q'R'S'$ by a rotation about the y-axis.

The x-coordinate of S' is $2\sqrt{2}$.

(d) By using the matrix for a rotation by angle θ about the y-axis, as given in the Formulae Booklet, determine in exact form the possible coordinates of R'. **[5]**

a)



Method 1: (Cross/Vector Product)

$|a \times b| = |a||b| \sin\theta$ Cross/Vector product formula.

$|\vec{PQ} \times \vec{PR}| = |\vec{PQ}||\vec{PR}| \sin\theta$

$$\left|\begin{pmatrix}2\\-2\\-2\end{pmatrix} \times \begin{pmatrix}1\\1\\-3\end{pmatrix}\right| = \sqrt{2^2+2^2+2^2}\sqrt{1^2+1^2+3^2}\sin\theta$$

$$\left\|\begin{vmatrix} i & j & k \\ 2 & -2 & -2 \\ 1 & 1 & -3 \end{vmatrix}\right\| = \sqrt{12}\sqrt{11}\sin\theta$$

These mean magnitude

These mean determinant

$$|8i + 4j + 4k| = \sqrt{12}\sqrt{11}\sin\theta$$

$$\sqrt{8^2+4^2+4^2} = \sqrt{12}\sqrt{11}\sin\theta$$

*****

$$\sqrt{96} = \sqrt{12}\sqrt{11}\sin\theta$$

$$\sin\theta = \frac{\sqrt{96}}{\sqrt{12}\sqrt{11}} = \frac{\cancel{\sqrt{12}}\sqrt{8}}{\cancel{\sqrt{12}}\sqrt{11}} = \frac{2\sqrt{2}}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}}$$

$$= \frac{2\sqrt{22}}{11} = \frac{2}{11}\sqrt{22}$$

Method 2 : Dot/Scal Product

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\Rightarrow \cos\theta = \frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}||\vec{PR}|}$$

$$= \frac{\begin{pmatrix}2\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}}{\sqrt{2^2+2^2+2^2}\sqrt{1^2+1^2+3^2}}$$

$$\cos\theta = \frac{6}{\sqrt{12}\sqrt{11}}$$

From Year 1 single maths, $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$= \pm\sqrt{1 - \frac{36}{12\times 11}} = \pm\sqrt{1-\frac{3}{11}} = \pm\sqrt{\frac{8}{11}}$$

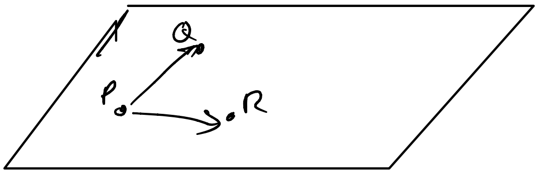
$$= \pm\frac{2}{11}\sqrt{22}$$

Now, $\sin\theta > 0$ if θ is acute (by inspection of its graph)

So $\sin\theta = \frac{2}{11}\sqrt{22}$

*****

b)



we need the normal vector to the plane. But this is just the cross/vector product of $\vec{PQ}$ and $\vec{PR}$ (or any two non parallel vectors in the plane)

$$\vec{PQ} \times \vec{PR} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$$ by (a) method 1.

So $8x + 4y + 4z = d$

P, Q and R all lie on the plane so must satisfy this equation.
So we need to sub any of them in to find d.

$8(0) + 4(2) + 4(3) = d$

$d = 20$

So $\boxed{8x + 4y + 4z = 20}$

$\Rightarrow \boxed{2x + y + z = 5}$

c) Shortest distance between plane and S: use formula booklet

The distance between a point and a plane is $D = \dfrac{|\mathbf{b}.\mathbf{n} - p|}{|\mathbf{n}|}$, where $\mathbf{b}$ is the position vector of the point and the equation of the plane is given by $\mathbf{r}.\mathbf{n} = p$

$$D = \frac{\left| \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - 5 \right|}{\sqrt{2^2 + 1^2 + 1^2}} = \frac{7}{\sqrt{6}}$$

*****

$$\text{Volume} = \frac{1}{3} \times \text{area of } PQR \times \frac{7}{\sqrt{6}}$$

$$= \frac{1}{3} \times \frac{1}{2} |\vec{PQ}||\vec{PR}| \sin\theta \ \frac{7}{\sqrt{6}}$$

$$= \frac{1}{3} \times \frac{1}{2} \times \underbrace{\sqrt{12}\sqrt{11} \times \frac{2}{11}\sqrt{22}}_{\text{from (a)}} \times \frac{7}{\sqrt{6}}$$

$$= \boxed{\frac{14}{3}}$$

d) This is a application of matrices question.

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \quad 5\cos\theta - \sin\theta = 2\sqrt{2}$$

The most mathematically sound way to solve this type of equation is to use the R-addition formulae. However we need the exact coordinates which makes this challenging.

Instead we move $\sin\theta$ to RHS and square both sides.

Note: This can be problematic because squaring can introduce extra solutions. e.g.

$$x = 5$$
$$x^2 = 25$$
$$x = \pm 5$$

So we'll need to check if the solutions are valid at the end.

$$5\cos\theta = 2\sqrt{2} + \sin\theta$$

$$25\cos^2\theta = \sin^2\theta + 4\sqrt{2}\sin\theta + 8$$

$$25(1 - \sin^2\theta) = \sin^2\theta + 4\sqrt{2}\sin\theta + 8$$

*****

$$26\sin^2\theta + 4\sqrt{2}\sin\theta - 17 = 0$$

$$\sin\theta = \frac{\sqrt{2}}{2} \quad \text{or} \quad -\frac{17\sqrt{2}}{26}$$

Now, $5\cos\theta = 2\sqrt{2} + \sin\theta$

so $\cos\theta = \dfrac{2\sqrt{2} + \frac{\sqrt{2}}{2}}{5} = \dfrac{\sqrt{2}}{2}$

or $\cos\theta = \dfrac{2\sqrt{2} - \frac{17\sqrt{2}}{26}}{5} = \dfrac{7\sqrt{2}}{26}$

Subbing these pairs in gives valid solutions.

$$OR' = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ 3 \\ -\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 3 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} \quad \text{if } \sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$$

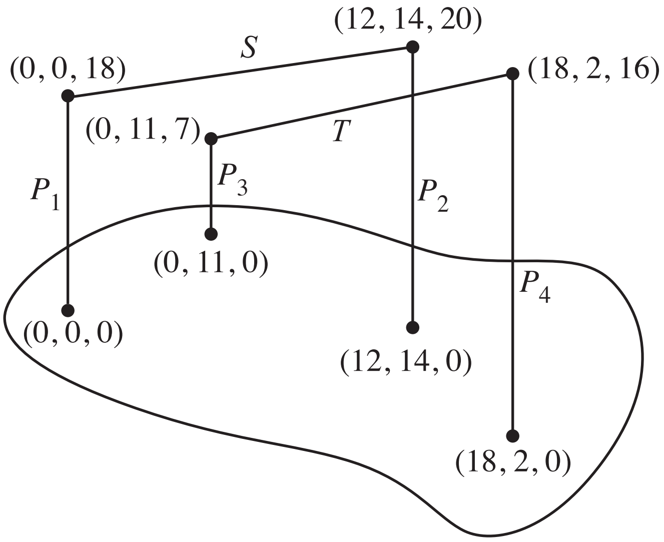
or $\begin{pmatrix} \frac{7\sqrt{2}}{26} \\ 3 \\ \frac{17\sqrt{2}}{26} \end{pmatrix}$ if $\cos\theta = \dfrac{7\sqrt{2}}{26}$ and $\sin\theta = -\dfrac{17\sqrt{2}}{26}$

so $\boxed{R' = \left(\frac{\sqrt{2}}{2}, 3, -\frac{\sqrt{2}}{2}\right) \text{ or } \left(\frac{7\sqrt{2}}{26}, 3, \frac{17\sqrt{2}}{26}\right)}$

 *****

23 A 3-D coordinate system, whose units are metres, is set up to model a construction site. The construction site contains four vertical poles P_1, P_2, P_3 and P_4. The floor of the construction site is modelled as lying in the *x*-*y* plane and the poles are modelled as vertical line segments. One end of each pole lies on the floor of the construction site, and the other end of each pole is modelled by the points (0, 0, 18), (12, 14, 20), (0, 11, 7) and (18, 2, 16) respectively.

A wire, *S*, runs from the top of P_1 to the top of P_2. A second wire, *T*, runs from the top of P_3 to the top of P_4. The wires are modelled by straight lines segments. The layout of the construction site is illustrated on the diagram below which is **not** drawn to scale.



A vector equation of the line **segment** that represents the wire *S* is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 18 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 7 \\ 1 \end{pmatrix}, 0 \leqslant \lambda \leqslant 2.$$

(a) Find, in the same form, a vector equation of the line **segment** that represents the wire *T*. The components of the direction vector should be integers whose only positive common factor is 1. **[2]**

For the construction site to be considered safe, it must pass two tests.

Test 1: The wires *S* and *T* need to be at least 5 metres apart at all positions on *S* and *T*.

(b) By using an appropriate formula, determine whether the construction site passes Test 1. **[2]**

A security camera is placed at a point *Q* on wire *S*.

Test 2: To ensure sufficient visibility of the construction site, the distance between the security camera and the top of P_3 must be at least 19 m.

(c) Determine whether it is possible to find point *Q* on *S* such that the construction site passes Test 2. **[3]**

a) the direction vector is the vector from top of P_3 to top of P_4 *****

$$\begin{pmatrix} 18 \\ 2 \\ 16 \end{pmatrix} - \begin{pmatrix} 0 \\ 11 \\ 7 \end{pmatrix} = \begin{pmatrix} 18 \\ -9 \\ 9 \end{pmatrix}$$

We can scale down the direction vector always.

So
$$\boxed{\underline{r} = \begin{pmatrix} 0 \\ 11 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad 0 \le \lambda \le 9}$$

b) Lines are skew (they don't intersect by info given and aren't parallel since $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ is not a multiple of $\begin{pmatrix} 6 \\ 7 \\ 1 \end{pmatrix}$)

So we can use shortest distance formula:

The distance between skew lines is $D = \dfrac{|(\mathbf{b} - \mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$, where **a** and **b** are position vectors of points on each line and **n** is a mutual perpendicular to both lines

$$n = \begin{pmatrix} 6 \\ 7 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 6 & 7 & 1 \\ 2 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 8 \\ -4 \\ -20 \end{pmatrix}$$

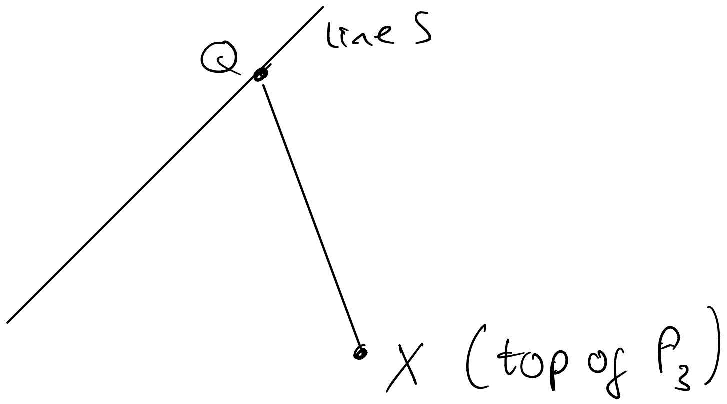
$$D = \frac{\left| \left(\begin{pmatrix} 0 \\ 0 \\ 18 \end{pmatrix} - \begin{pmatrix} 0 \\ 11 \\ 7 \end{pmatrix} \right) \cdot \begin{pmatrix} 8 \\ -4 \\ -20 \end{pmatrix} \right|}{\sqrt{8^2 + 4^2 + 20^2}} = \frac{\left| \begin{pmatrix} 0 \\ -11 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -4 \\ -20 \end{pmatrix} \right|}{4\sqrt{30}} = \frac{176}{4\sqrt{30}}$$

$$= 8.03 > 5$$

So Yes, the site passes the test.

*****

c)



$(0, 11, 7)$

$\vec{OQ} = \begin{pmatrix} 6\lambda \\ 7\lambda \\ 18+\lambda \end{pmatrix}$ since Q lies on S.

$\vec{XQ} = \begin{pmatrix} 6\lambda \\ 7\lambda-11 \\ \lambda+11 \end{pmatrix}$

$$|\vec{XQ}| = \sqrt{(6\lambda)^2 + (7\lambda-11)^2 + (\lambda+11)^2}$$

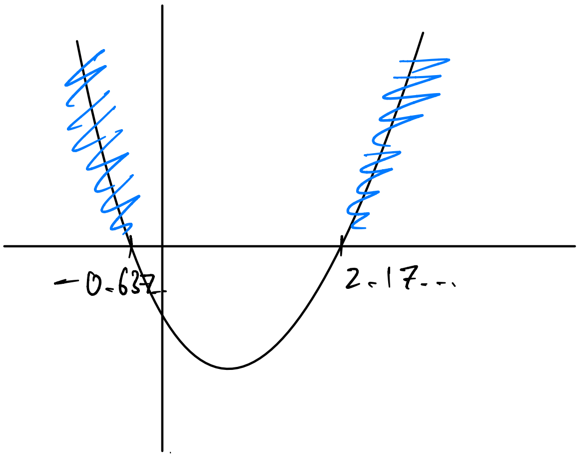
$$= \sqrt{36\lambda^2 + 49\lambda^2 - 154\lambda + 121 + \lambda^2 + 22\lambda + 121}$$

$$= \sqrt{86\lambda^2 - 132\lambda + 242}$$

we want $|\vec{XQ}| \geq 19$

i.e. $86\lambda^2 - 132\lambda + 242 \geq 361$ ✓ 19^2

$$86\lambda^2 - 132\lambda - 119 \geq 0$$



$\Rightarrow \lambda \leq -0.637\ldots$ or $\lambda \geq 2.17\ldots$

but we know $0 \leq \lambda \leq 2$

so no, it's not possible

******

24 The coordinates of the points A and B are $(3, -2, -1)$ and $(13, 10, 9)$ respectively.

- The plane Π_A contains A and the plane Π_B contains B.
- The planes Π_A and Π_B are parallel.
- The x and y components of any normal to plane Π_A are equal.
- The shortest distance between Π_A and Π_B is 2.

There are **two** possible solution planes for Π_A which satisfy the above conditions.

Determine the acute angle between these two possible solution planes. **[8]**

Let Π_A: $a_1x + b_1y + c_1z = d_1$

Let Π_B: $a_2x + b_2y + c_2z = d_2$

The plane Π_A contains A and the plane Π_B contains B.

$\Rightarrow$ $(3,-2,-1)$ satisfies $a_1x + b_1y + c_1z = d_1$ and same for B

so $3a_1 - 2b_1 - c_1 = d_1$

and $13a_2 + 10b_2 + 9c_2 = d_2$

The planes Π_A and Π_B are parallel.

$\Rightarrow$ they have the same normal vectors

i.e. $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$

so we have $3a_1 - 2b_1 - c_1 = d_1$

& $13a_1 + 10b_1 + 9c_1 = d_2$

The x and y components of any normal to plane Π_A are equal.

$\Rightarrow a_1 = b_1$ (and $a_2 = b_2$)

so $3a_1 - 2a_1 - c_1 = d_1 \Rightarrow a_1 - c_1 = d_1$

and $13a_1 + 10a_1 + 9c_1 = d_2 \Rightarrow 23a_1 + 9c_1 = d_2$

The shortest distance between Π_A and Π_B is 2.

to find shortest distance between two parallel planes just take any point on Π_B and use the formula.

$$D = \frac{\left| \begin{pmatrix} 13 \\ 10 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_1 \\ c_1 \end{pmatrix} - d_1 \right|}{\sqrt{a_1^2 + a_1^2 + c_1^2}} = \frac{|23a_1 + 9c_1 - d_1|}{\sqrt{2a_1^2 + c_1^2}} = 2$$

but $d_1 = a_1 - c_1$ from earlier, so

$$\frac{|23a_1 + 9c_1 - a_1 + c_1|}{\sqrt{2a_1^2 + c_1^2}} = 2$$

$$|22a_1 + 10c_1| = 2\sqrt{2a_1^2 + c_1^2}$$

$$|11a_1 + 5c_1| = \sqrt{2a_1^2 + c_1^2}$$

$$(11a_1 + 5c_1)^2 = 2a_1^2 + c_1^2$$

$$119a_1^2 + 110a_1c_1 + 24c_1^2 = 0$$

$$(17a_1 + 6c_1)(7a_1 + 4c_1) = 0$$

so $c_1 = \frac{-17}{6}a_1$ or $c_1 = \frac{-7}{4}a_1$

so we have Π_A: $a_1x + a_1y - \frac{17}{6}a_1 z = d_1$

and Π_B: $a_1x + a_1y - \frac{7}{4}a_1 z = d_2$

$$\cos\theta = \frac{\begin{pmatrix} a_1 \\ a_1 \\ -\frac{17}{6}a_1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_1 \\ -\frac{7}{4}a_1 \end{pmatrix}}{\sqrt{a_1^2 + a_1^2 + \left(\frac{17}{6}a_1\right)^2}\sqrt{a_1^2 + a_1^2 + \left(\frac{7}{4}a_1\right)^2}}$$

$$= \frac{\left(1 + 1 + \frac{17}{6} \times \frac{7}{4}\right) \cancel{a_1^2}}{\cancel{a_1^2}\sqrt{1+1+\left(\frac{17}{6}\right)^2}\sqrt{1+1+\left(\frac{7}{4}\right)^2}} = \frac{\frac{167}{24}}{\frac{57}{8}}$$

$$= \frac{167}{171}$$

$$\boxed{\theta = 12.42^\circ}$$