London Academy of Maths

Complex Trigonometry: De Moivre's Theorem and Identities

Question Paper

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1 In this question you must show detailed reasoning.

(a) Use de Moivre's theorem to determine constants A, B and C such that
$\sin^4\theta \equiv A\cos 4\theta + B\cos 2\theta + C$. **[5]**

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2 By expanding $\left(z^2+\frac{1}{z^2}\right)^3$, where $z=\mathrm{e}^{\mathrm{i}\theta}$, show that $4\cos^3 2\theta = \cos 6\theta + 3\cos 2\theta$. **[5]**

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3 **(i)** Express $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ and show that

$$\sin^4\theta \equiv \tfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$ **[4]**

(ii) Hence find the exact value of $\int_0^{\frac{1}{6}\pi} \sin^4\theta \, d\theta$. **[4]**

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4 Use de Moivre's theorem to find the constants A, B and C in the identity $\sin^5\theta \equiv A\sin\theta + B\sin 3\theta + C\sin 5\theta$. **[4]**

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5 Show that $\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta$, where a, b and c are constants to be determined. **[7]**

6 **(i)** By expressing $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\cos^5\theta \equiv \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta).$$ [5]

(ii) Hence solve the equation $\cos 5\theta + 5\cos 3\theta + 9\cos\theta = 0$ for $0 \leqslant \theta \leqslant \pi$. [4]

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7 In this question you must show detailed reasoning.

(a) By writing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ show that

$\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$. **[5]**

(b) Hence show that $\sin\frac{1}{8}\pi = \frac{1}{2}\sqrt[6]{20 - 14\sqrt{2}}$. **[3]**

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8 **(i)** By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^5\theta \equiv \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$ **[4]**

(ii) Hence solve the equation

$$\sin 5\theta + 4\sin\theta = 5\sin 3\theta$$

for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. **[4]**

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9 **(i)** Starting with the result

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

show that

(A) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ **[2]**

(B) $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$. **[2]**

(ii) Using the result in part **(i)** (A), obtain the values of the constants a, b, c and d in the identity

$$\cos 6\theta \equiv a\cos^6\theta + b\cos^4\theta + c\cos^2\theta + d\,.$$ **[6]**

(iii) Using the result in part **(i)** (B), obtain the values of the constants P, Q, R and S in the identity

$$\cos^6\theta \equiv P\cos 6\theta + Q\cos 4\theta + R\cos 2\theta + S\,.$$ **[5]**

(iv) Show that $\cos\frac{\pi}{12} = \left(\frac{26+15\sqrt{3}}{64}\right)^{\frac{1}{6}}$. **[3]**

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10 **(a)** Given that $z = \cos\theta + \mathrm{i}\sin\theta$, express $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$ in simplified trigonometric form. **[2]**

(b) By considering $\left(z+\dfrac{1}{z}\right)^3\left(z-\dfrac{1}{z}\right)^3$, find constants A and B such that

$\sin^3\theta\cos^3\theta = A\sin 6\theta + B\sin 2\theta$. **[6]**

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11 **(i)** **(a)** Verify, without using a calculator, that $\theta = \frac{1}{8}\pi$ is a solution of the equation $\sin 6\theta = \sin 2\theta$. [1]

(b) By sketching the graphs of $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, or otherwise, find the other solution of the equation $\sin 6\theta = \sin 2\theta$ in the interval $0 < \theta < \frac{1}{2}\pi$. [2]

(ii) Use de Moivre's theorem to prove that

$$\sin 6\theta \equiv \sin 2\theta(16\cos^4\theta - 16\cos^2\theta + 3).$$ [5]

(iii) Hence show that one of the solutions obtained in part **(i)** satisfies $\cos^2\theta = \frac{1}{4}(2 - \sqrt{2})$, and justify which solution it is. [3]

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12 **(i)** By expressing $\sin\theta$ in terms of $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$, show that

$$\sin^6\theta \equiv -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10).$$ **[5]**

(ii) Replace θ by $\left(\frac{1}{2}\pi - \theta\right)$ in the identity in part **(i)** to obtain a similar identity for $\cos^6\theta$. **[3]**

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \left(\sin^6\theta - \cos^6\theta\right) \mathrm{d}\theta$. **[4]**

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13 **(i)** By expressing $\cos\theta$ in terms of $e^{i\theta}$ show that

$$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$ **[4]**

(ii) Hence solve, for $0 \leqslant \theta \leqslant \pi$,

$$\cos 6\theta + 6\cos 4\theta + 2\cos 2\theta = 3.$$ **[5]**

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14 **(i)** Use de Moivre's theorem to express $\cos 4\theta$ as a polynomial in $\cos \theta$. **[4]**

(ii) Hence prove that $\cos 4\theta \cos 2\theta \equiv 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1$. **[1]**

(iii) Use part **(ii)** to show that the only roots of the equation $\cos 4\theta \cos 2\theta = 1$ are $\theta = n\pi$, where n is an integer. **[3]**

(iv) Show that $\cos 4\theta \cos 2\theta = -1$ only when $\cos \theta = 0$. **[3]**

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15 In this question you must show detailed reasoning.

(a) Show that $\mathrm{Re}\left(\mathrm{e}^{4\mathrm{i}\theta}(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta})^4\right) = a\cos 4\theta \cos^4\theta$, where a is an integer to be determined. **[3]**

(b) Hence show that $\cos\frac{1}{12}\pi = \frac{1}{2}\sqrt[4]{b+c\sqrt{3}}$, where b and c are integers to be determined. **[6]**

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16 **(i)** Use de Moivre's theorem to prove that

$$\tan 5\theta \equiv \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$ **[4]**

(ii) Solve the equation $\tan 5\theta = 1$, for $0 \leqslant \theta < \pi$. **[3]**

(iii) Show that the roots of the equation

$$t^4 - 4t^3 - 14t^2 - 4t + 1 = 0$$

may be expressed in the form $\tan\alpha$, stating the exact values of α, where $0 \leqslant \alpha < \pi$. **[5]**

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17 **(i)** Use de Moivre's theorem to show that $\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **[5]**

(ii) Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos\alpha$ where $0 \leqslant \alpha \leqslant \pi$. **[4]**

(iii) Hence find the exact value of $\cos\frac{1}{10}\pi$. **[3]**

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18 **(i)** By expressing $\sin\theta$ and $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, prove that

$$\sin^3\theta\cos^2\theta \equiv -\tfrac{1}{16}(\sin 5\theta - \sin 3\theta - 2\sin\theta).$$ **[6]**

(ii) Hence show that all the roots of the equation

$$\sin 5\theta = \sin 3\theta + 2\sin\theta$$

are of the form $\theta = \dfrac{n\pi}{k}$, where n is any integer and k is to be determined. **[3]**

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19 **(i)** Use de Moivre's theorem to show that $\tan 4\theta \equiv \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. **[4]**

(ii) Hence find the exact roots of $t^4 + 4\sqrt{3}t^3 - 6t^2 - 4\sqrt{3}t + 1 = 0$. **[5]**

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20 **(i)** Use de Moivre's theorem to show that

$$\sin 6\theta \equiv \cos\theta\left(6\sin\theta - 32\sin^3\theta + 32\sin^5\theta\right).$$ **[5]**

(ii) Hence show that, for $\sin 2\theta \neq 0$,

$$-1 \leqslant \frac{\sin 6\theta}{\sin 2\theta} < 3.$$ **[7]**

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21 **(i)** Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. **[3]**

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1).$$ **[5]**

(iii) Hence find the exact value of

$$\cos\left(\tfrac{1}{12}\pi\right)\cos\left(\tfrac{5}{12}\pi\right)\cos\left(\tfrac{7}{12}\pi\right)\cos\left(\tfrac{11}{12}\pi\right),$$

justifying your answer. **[5]**

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22 **(i)** Use de Moivre's theorem to find an expression for $\cot 7\theta$ in terms of $\cot\theta$ and hence find the exact roots of the equation $u^6 - 21u^4 + 35u^2 - 7 = 0$. **[7]**

(ii) State the exact roots of the equation $v^3 - 21v^2 + 35v - 7 = 0$, justifying your answer. Hence find the exact value of

$$\frac{\cot^2\left(\frac{1}{14}\pi\right)\cot^2\left(\frac{3}{14}\pi\right)+\cot^2\left(\frac{3}{14}\pi\right)\cot^2\left(\frac{5}{14}\pi\right)+\cot^2\left(\frac{5}{14}\pi\right)\cot^2\left(\frac{1}{14}\pi\right)}{\cot\left(\frac{1}{14}\pi\right)\cot\left(\frac{3}{14}\pi\right)\cot\left(\frac{5}{14}\pi\right)}.$$ **[4]**