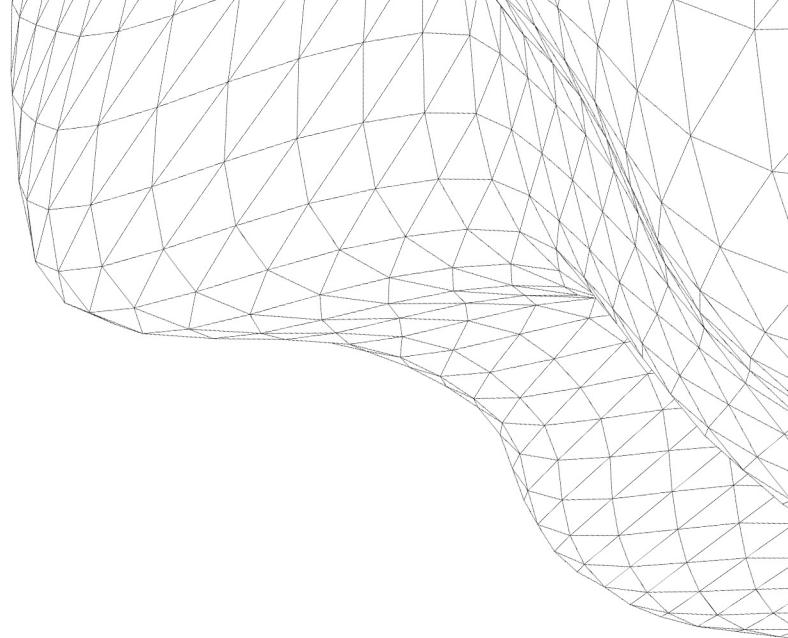
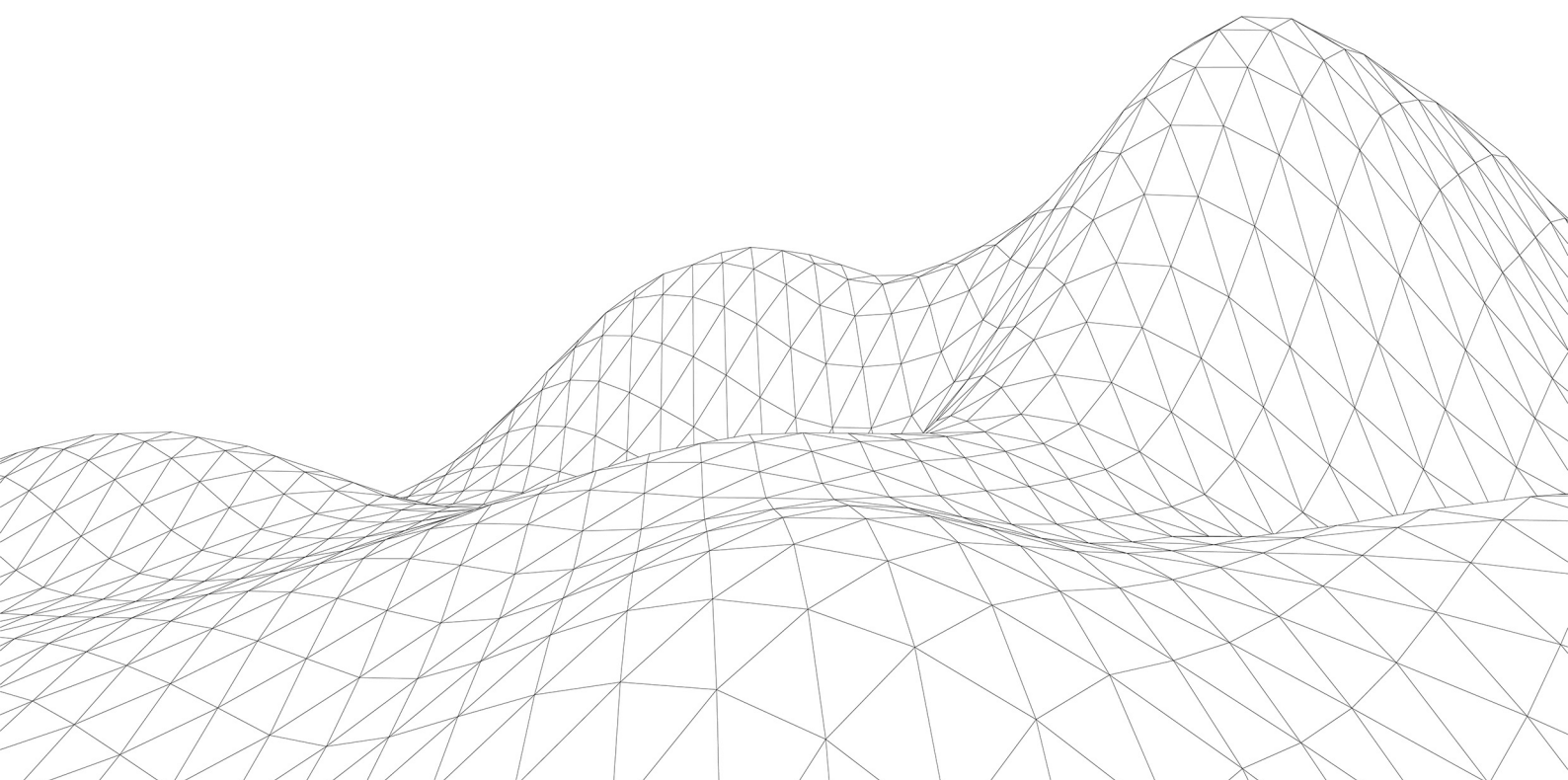




Complex Number Basics

Worked Solutions



1 Show that $\frac{5}{2-4i} = \frac{1}{2} + i$.

[2]

$$\frac{5}{2-4i} = \frac{5(2+4i)}{(2-4i)(2+4i)}$$

$$= \frac{10 + 20i}{4 + 16}$$

$$= \frac{10}{20} + \frac{20}{20}i$$

$$= \frac{1}{2} + i$$

2 The complex number $x + iy$ is denoted by z . Express $3zz^* - |z|^2$ in terms of x and y .

[3]

Recall $|z|^2 = z \cdot z^*$ and $|z|^2 = x^2 + y^2$

$$\begin{aligned}\Rightarrow 3zz^* - |z|^2 &= 3|z|^2 - |z|^2 \\ &= 2|z|^2 \\ &= 2x^2 + 2y^2\end{aligned}$$

- 3 The complex numbers z and w are given by $z = 5 - 2i$ and $w = 3 + 7i$. Giving your answers in the form $x + iy$ and showing clearly how you obtain them, find

(i) $4z - 3w$, [2]

(ii) z^*w . [2]

$$\begin{aligned} \text{i)} \quad 4z - 3w &= 4(5 - 2i) - 3(3 + 7i) \\ &= 20 - 8i - 9 - 21i \\ &= \boxed{11 - 29i} \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad z^*w &= (5 + 2i)(3 + 7i) \\ &= 15 - 14 + 35i + 6i \\ &= \boxed{1 + 41i} \end{aligned}$$

- 4 The complex numbers a and b are given by $a = 7 + 6i$ and $b = 1 - 3i$. Showing clearly how you obtain your answers, find

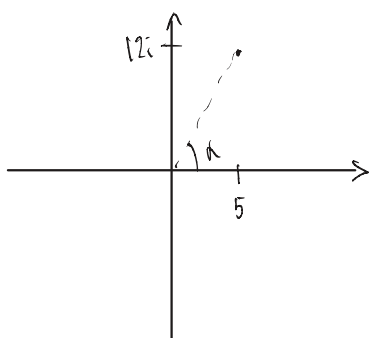
(i) $|a - 2b|$ and $\arg(a - 2b)$, [4]

(ii) $\frac{b}{a}$, giving your answer in the form $x + iy$. [3]

$$i) \quad a - 2b = 7 + 6i - 2(1 - 3i) = 5 + 12i$$

$$|a - 2b| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$

$$|a - 2b| = 13$$



$$\alpha = \arctan\left(\frac{12}{5}\right)$$

$$= 1.176$$

$$ii) \quad \frac{b}{a} = \frac{1 - 3i}{7 + 6i} = \frac{(1 - 3i)(7 - 6i)}{(7 + 6i)(7 - 6i)}$$

$$= \frac{7 - 18 - 21i - 6i}{49 + 36}$$

$$= \frac{-11 - 27i}{85}$$

$$= -\frac{11}{85} - \frac{27}{85}i$$

- 5 The complex numbers z and w are given by $z = 6 - i$ and $w = 5 + 4i$. Giving your answers in the form $x + iy$ and showing clearly how you obtain them, find

(i) $z + 3w$, [2]

(ii) $\frac{z}{w}$. [3]

$$\begin{aligned} \text{i)} \quad 6 - i + 3(5 + 4i) &= 6 - i + 15 + 12i \\ &= 21 + 11i \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \frac{6 - i}{5 + 4i} &= \frac{(6 - i)(5 - 4i)}{(5 + 4i)(5 - 4i)} \\ &= \frac{30 - 5i - 24i - 4}{25 + 16} \\ &= \frac{26 - 29i}{41} \\ &= \frac{26}{41} - \frac{29}{41}i \end{aligned}$$

- 6 Express $\frac{2+3i}{5-i}$ in the form $x+iy$, showing clearly how you obtain your answer.

[4]

$$\begin{aligned}\frac{2+3i}{5-i} &= \frac{(2+3i)(5+i)}{(5-i)(5+i)} \\ &= \frac{10 - 3 + 2i + 15i}{26} \\ &= \frac{7 + 17i}{26} \\ &= \boxed{\frac{7}{26} + \frac{17}{26}i}\end{aligned}$$

7 The complex number z_1 is $1+i$ and the complex number z_2 has modulus 4 and argument $\frac{\pi}{3}$.

(i) Express z_2 in the form $a+bi$, giving a and b in exact form.

[2]

(ii) Express $\frac{z_2}{z_1}$ in the form $c+di$, giving c and d in exact form.

[2]

$$\begin{aligned} \text{i)} \quad z_2 &= 4 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right) \\ &= 4 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) \\ &= \boxed{2 + 2\sqrt{3}i} \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \frac{z_2}{z_1} &= \frac{2 + 2\sqrt{3}i}{1+i} \\ &= \frac{(2 + 2\sqrt{3}i)(1-i)}{(1+i)(1-i)} \\ &= \frac{2 + 2\sqrt{3}i - 2i + 2\sqrt{3}}{2} \\ &= \boxed{1 + \sqrt{3} + (\sqrt{3} - 1)i} \end{aligned}$$

8 Find real numbers a and b such that $(a-3i)(5-i) = b-17i$.

[5]

$$(a-3i)(5-i) = b-17i$$

$$5a - 15i - ai - 3 = b - 17i$$

$$5a - b - 3 + (2-a)i = 0$$

$$\Rightarrow a = 2 \quad \text{and} \quad 5 \times 2 - b - 3 = 0$$
$$b = 10 - 3 = 7.$$

$$a = 2, \quad b = 7$$

9 In this question you must show detailed reasoning.

Given that $z_1 = 3 + 2i$ and $z_2 = -1 - i$, find the following, giving each in the form $a + bi$.

(i) $z_1^* z_2$ [2]

(ii) $\frac{z_1 + 2z_2}{z_2}$ [2]

$$\begin{aligned} \text{i)} \quad (3 - 2i)(-1 - i) &= -(3 - 2i + 3i + 2) \\ &= \boxed{-5 - i} \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \frac{(3 + 2i) + 2(-1 - i)}{(-1 - i)} &= \frac{\cancel{3} + \cancel{2i} - \cancel{2} - \cancel{2i}}{(-1 - i)} \frac{(-1 + i)}{(-1 + i)} \\ &= \frac{-1 + i}{1 + 1} \\ &= \boxed{-\frac{1}{2} + \frac{1}{2}i} \end{aligned}$$

- 10 The complex numbers z and w are given by $z = 4 + 3i$ and $w = 6 - i$. Giving your answers in the form $x + iy$ and showing clearly how you obtain them, find

(i) $3z - 4w$, [2]

(ii) $\frac{z^*}{w}$. [4]

$$\begin{aligned} \text{i)} \quad 3z - 4w &= 3(4 + 3i) - 4(6 - i) \\ &= 12 + 9i - 24 + 4i \\ &= -12 + 13i \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \frac{z^*}{w} &= \frac{4 - 3i}{6 - i} \\ &= \frac{(4 - 3i)(6 + i)}{(6 - i)(6 + i)} \\ &= \frac{24 + 3 + 4i - 18i}{37} \\ &= \frac{27}{37} - \frac{14}{37}i \end{aligned}$$

- 11 The complex number w has modulus 6 and argument $\frac{2\pi}{3}$. Find $\frac{\sqrt{3}+2i}{w}$, giving your answer in the form $x+iy$, where x and y are exact real numbers. [5]

$$\begin{aligned}w &= 6 \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right) \\&= 6 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\&= -3 + 3\sqrt{3}i\end{aligned}$$

$$\begin{aligned}\frac{\sqrt{3}+2i}{w} &= \frac{\sqrt{3}+2i}{-3+3\sqrt{3}i} \\&= \frac{(\sqrt{3}+2i)(-3-3\sqrt{3}i)}{(-3+3\sqrt{3}i)(-3-3\sqrt{3}i)} \\&= \frac{-3\sqrt{3}-6i-9i+6\sqrt{3}}{36} \\&= \frac{3\sqrt{3}}{36} - \frac{15i}{36} \\&= \boxed{\frac{\sqrt{3}}{12} - \frac{5}{12}i}\end{aligned}$$

12 The complex number $2 - i$ is denoted by z .

(i) Find $|z|$ and $\arg z$.

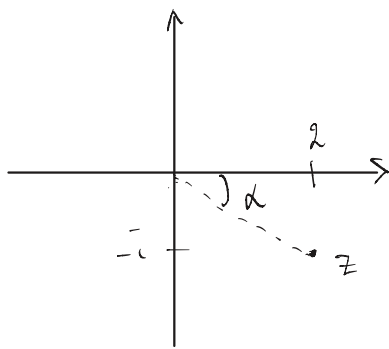
[2]

(ii) Given that $az + bz^* = 4 - 8i$, find the values of the real constants a and b .

[5]

$$i) \quad |z| = \sqrt{2^2 + (-1)^2} = \sqrt{4 + 1} = \sqrt{5}$$

$$|z| = \sqrt{5}$$



$$\alpha = \arctan\left(\frac{-1}{2}\right) = -0.464$$

$$\arg(z) = -\alpha = -0.464$$

$$\arg(z) = -0.464$$

$$ii) \quad a(2 - i) + b(2 + i) = 4 - 8i$$

$$2a - ai + 2b + bi = 4 - 8i$$

Equating real and imaginary parts we get

$$\textcircled{1} \quad 2a + 2b = 4$$

$$\textcircled{2} \quad -a + b = -8$$

$$\frac{1}{2} \times \textcircled{1} \quad \begin{cases} a + b = 2 \\ -a + b = -8 \end{cases}$$

$$\frac{1}{2} \times \textcircled{1} + \textcircled{2} \quad \begin{cases} 2b = -6 \\ a = 2 - b \end{cases}$$

$$\begin{aligned} b &= -3 \\ a &= 5 \end{aligned}$$

- 13 The complex number z satisfies the equation $z + 2iz^* = 12 + 9i$. Find z , giving your answer in the form $x + iy$. [5]

$$\text{Let } z = a + bi, \text{ then}$$

$$a + bi + 2i(a - bi) = 12 + 9i$$

$$a + 2b + bi + 2ai = 12 + 9i$$

$$a + 2b + (2a + b)i = 12 + 9i$$

$$\begin{cases} \textcircled{1} & 2a + b = 9 \\ \textcircled{2} & a + 2b = 12 \end{cases}$$

$$\begin{cases} 2 \times \textcircled{1} & 4a + 2b = 18 \\ \textcircled{2} & a + 2b = 12 \end{cases}$$

$$\begin{cases} 2 \times \textcircled{1} - \textcircled{2} & 3a = 6 \\ \textcircled{2} & 2b = 12 - a \end{cases}$$

$$\begin{cases} a = 2 \\ b = 5 \end{cases}$$

$$z = 2 + 5i$$

- 14** The complex number $a + 5i$, where a is positive, is denoted by z . Given that $|z| = 13$, find the value of a and hence find $\arg z$. **[4]**

$$|z| = \sqrt{a^2 + 25} = 13$$

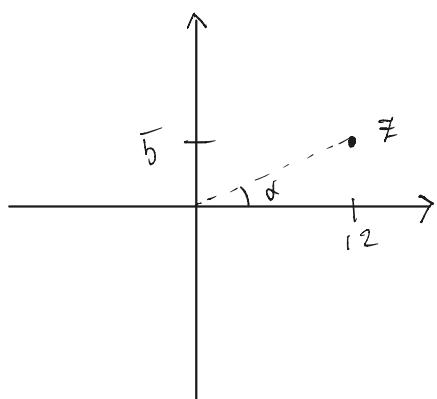
$$\Rightarrow a^2 + 25 = 169$$

$$a^2 = 144$$

$$a = \begin{cases} 12 \\ -12 \end{cases}$$

a is positive, so $a = 12$.

$$\alpha = \arctan\left(\frac{5}{12}\right) = 0.395$$



- 15 The complex number z has modulus $2\sqrt{3}$ and argument $-\frac{1}{3}\pi$. Giving your answers in the form $x+iy$, where x and y are exact real numbers, and showing clearly how you obtain them, find

(i) z , [2]

(ii) $\frac{1}{(z^* - 5i)^2}$. [5]

$$i) \quad z = 2\sqrt{3} \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right)$$

$$= 2\sqrt{3} \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right)$$

$$= \sqrt{3} - 3i$$

$$ii) \quad \frac{1}{(z^* - 5i)^2} = \frac{1}{(\sqrt{3} + 3i - 5i)^2}$$

$$= \frac{1}{(\sqrt{3} - 2i)^2}$$

$$= \frac{1}{3 - 4\sqrt{3}i - 4}$$

$$= \frac{1}{-1 - 4\sqrt{3}i}$$

$$= - \frac{1 - 4\sqrt{3}i}{(1 + 4\sqrt{3}i)(1 - 4\sqrt{3}i)}$$

$$= - \frac{1 - 4\sqrt{3}i}{49}$$

$$= -\frac{1}{49} + \frac{4\sqrt{3}}{49}i$$

16 The complex number $a + ib$ is denoted by z and the complex number $c + id$ is denoted by w .

It is given that $z^2 = z * w$.

(i) Show that $2ab = ad - bc$.

[4]

(ii) Given that the real part of $w = 0$, find the values of b in terms of a .

[6]

$$i) \quad (a + bi)^2 = (a - bi)(c + id)$$

$$a^2 - b^2 + 2abi = ac + bd - cb i + adi$$

Equating imaginary parts we get

$$2ab = ad - bc$$

$$ii) \quad \text{Re}(w) = 0, \text{ i.e. } c = 0$$

Real ① $\left\{ \begin{array}{l} a^2 - b^2 = bd \\ 2ab = ad \end{array} \right.$

Imaginary ②

Dividing $\frac{①}{②}$

$$\Rightarrow \frac{a^2 - b^2}{2ab} = \frac{b}{a}$$

$$\Rightarrow a^3 - b^2 a = 2ab^2$$

$$3ab^2 = a^3$$

$$b^2 = \frac{a^2}{3}$$

$$\Rightarrow b = \pm \frac{a}{\sqrt{3}}$$

17 In this question you must show detailed reasoning.

The complex numbers z_1 and z_2 are given by $z_1 = 3 - 7i$ and $z_2 = 2 + 4i$.

(a) Express each of the following as exact numbers in the form $a + bi$.

(i) $3z_1 + 4z_2$ [1]

(ii) $z_1 z_2$ [2]

(iii) $\frac{z_1}{z_2}$ [2]

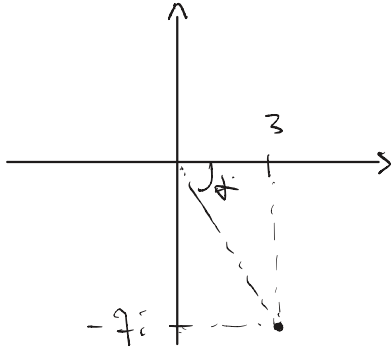
(b) Write z_1 in modulus-argument form giving the modulus in exact form and the argument correct to 3 significant figures. [3]

a) i) $3(3 - 7i) + 4(2 + 4i) = 9 - 21i + 8 + 16i$
 $= 17 - 5i$

ii) $(3 - 7i)(2 + 4i) = 6 - 14i + 12i + 28$
 $= 34 - 2i$

iii) $\frac{3 - 7i}{2 + 4i} = \frac{(3 - 7i)(2 - 4i)}{(2 + 4i)(2 - 4i)}$
 $= \frac{6 - 28 - 14i - 12i}{4 + 16}$
 $= -\frac{22}{20} - \frac{26}{20}i$
 $= -\frac{11}{10} - \frac{13}{10}i$

$$b) \quad |z_1| = \sqrt{3^2 + (-7)^2} = \sqrt{9 + 49} \\ = \sqrt{58}$$



$$\alpha = \arctan\left(\frac{|-7|}{3}\right) = 1.17$$

$$\arg(z_1) = -1.17.$$

$$z_1 = \sqrt{58} \left(\cos(-1.17) + i \sin(-1.17) \right)$$

18 You are given that $z = 3 - 4i$.

(a) Find

- $|z|$,
- $\arg(z)$,
- z^* .

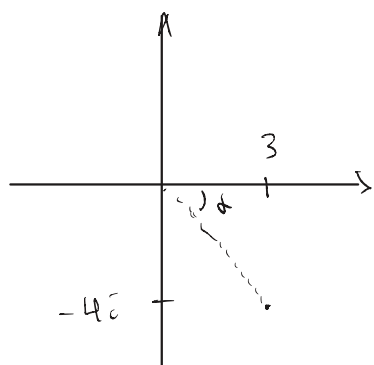
[3]

On an Argand diagram the complex number w is represented by the point A and w^* is represented by the point B .

(b) Describe the geometrical relationship between the points A and B .

[2]

$$a) \quad |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$



$$\alpha = \arctan\left(\frac{|-4|}{3}\right) = 0.927$$

$$\arg(z) = -0.927$$

$$z^* = 3 + 4i$$

$$|z| = 5$$

$$\arg(z) = -0.927$$

$$z^* = 3 + 4i$$

b) Point B is a reflection of point A in the real axis. Indeed, if $A = (x, y)$, then

$$B = (x, -y).$$

19 In this question you must show detailed reasoning.

In this question the principal argument of a complex number lies in the interval $[0, 2\pi)$.

Complex numbers z_1 and z_2 are defined by $z_1 = 3 + 4i$ and $z_2 = -5 + 12i$.

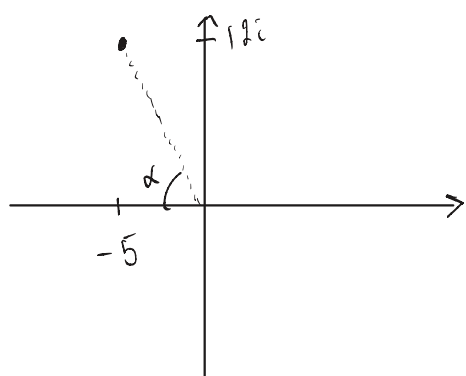
(a) Determine $z_1 z_2$, giving your answer in the form $a + bi$. [2]

(b) Express z_2 in modulus-argument form. [3]

(c) Verify, by direct calculation, that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. [3]

$$\begin{aligned} a) \quad z_1 z_2 &= (3 + 4i)(-5 + 12i) \\ &= -15 - 20i + 36i - 48 \\ &= -63 + 16i \end{aligned}$$

$$\begin{aligned} b) \quad |z_2| &= \sqrt{5^2 + 12^2} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

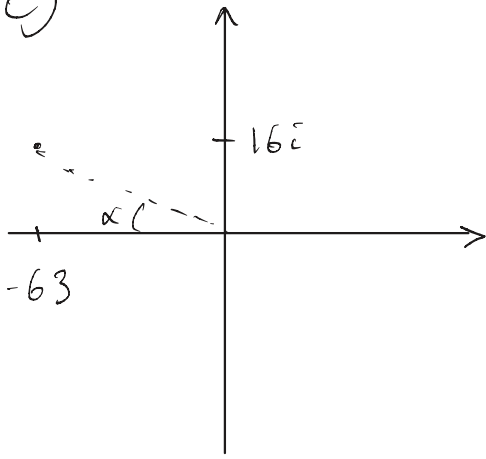


$$\alpha = \arctan\left(\frac{12}{-5}\right) = 1.176\dots$$

$$\arg(z_2) = \pi - \alpha = 1.97$$

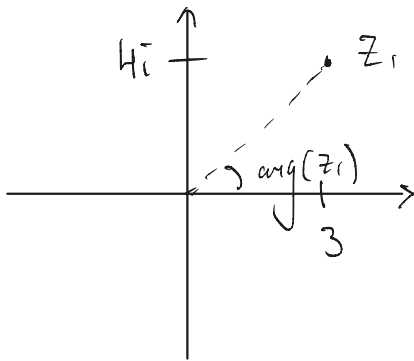
$$z_2 = 13 (\cos(1.97) + i \sin(1.97))$$

e)



$$\alpha = \arctan\left(\frac{16}{63}\right) = 0.2487\dots$$

$$\arg(z_1 z_2) = \pi - \alpha = 2.8929\dots$$



$$\arg(z_1) = \arctan\left(\frac{4}{3}\right) = 0.9273$$

$$\arg(z_1) + \arg(z_2) = 0.9273 + 1.9656.$$

$$= 2.8929\dots$$

$$= \arg(z_1 z_2).$$

20 In this question you must show detailed reasoning.

(a) Express $\frac{8+i}{2-i}$ in the form $a+bi$ where a and b are real. [2]

(b) Solve the equation $4x^2 - 8x + 5 = 0$. Give your answer(s) in the form $c+di$ where c and d are real. [2]

$$\begin{aligned}
 a) \quad \frac{8+i}{2-i} &= \frac{(8+i)(2+i)}{(2-i)(2+i)} \\
 &= \frac{16-1+8i+2i}{4+1} \\
 &= \frac{15+10i}{5} \\
 &= \boxed{3+2i}
 \end{aligned}$$

$$b) \quad 4x^2 - 8x + 5 = 0$$

$$\begin{aligned}
 x &= \frac{8 \pm \sqrt{8^2 - 4 \times 4 \times 5}}{2 \times 4} \\
 &= \frac{8 \pm \sqrt{-16}}{2 \times 4} \\
 &= \frac{8 \pm 4i}{8} \\
 &= \boxed{1 \pm \frac{1}{2}i}
 \end{aligned}$$

21 In this question you must show detailed reasoning.

(a) Solve the equation $2z^2 - 10z + 25 = 0$ giving your answers in the form $a + bi$. [2]

(b) Solve the equation $3\omega - 2 = i(5 + 2\omega)$ giving your answer in the form $a + bi$. [4]

a) $2z^2 - 10z + 25$

$$z = \frac{10 \pm \sqrt{10^2 - 4 \times 2 \times 25}}{2 \times 2}$$

$$= \frac{10 \pm 10i}{4}$$

$$= \boxed{\frac{5}{2} \pm \frac{5}{2}i}$$

b) $3\omega - 2 = i(5 + 2\omega)$

$$3\omega - 2 = 5i + 2i\omega$$

$$(3 - 2i)\omega = 2 + 5i$$

$$\omega = \frac{2 + 5i}{3 - 2i}$$

$$= \left(\frac{2 + 5i}{3 - 2i} \right) \left(\frac{3 + 2i}{3 + 2i} \right)$$

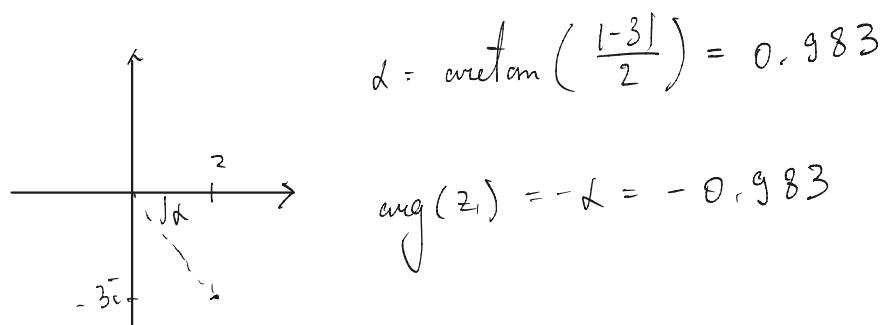
$$= \frac{6 - 10 + 15i + 4i}{9 + 4} = \boxed{-\frac{4}{13} + \frac{19}{13}i}$$

22 In this question you must show detailed reasoning.

The complex numbers z_1 and z_2 are given by $z_1 = 2 - 3i$ and $z_2 = a + 4i$ where a is a real number.

- (i) Express z_1 in modulus-argument form, giving the modulus in exact form and the argument correct to 3 significant figures. [3]
- (ii) Find $z_1 z_2$ in terms of a , writing your answer in the form $c + id$. [2]
- (iii) The real and imaginary parts of a complex number on an Argand diagram are x and y respectively. Given that the point representing $z_1 z_2$ lies on the line $y = x$, find the value of a . [2]
- (iv) Given instead that $z_1 z_2 = (z_1 z_2)^*$ find the value of a . [2]

$$i) \quad |z_1| = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$$



$$z_1 = \sqrt{13} \left(\cos(-0.983) + i \sin(-0.983) \right)$$

$$ii) \quad z_1 z_2 = (2 - 3i)(a + 4i) = 2a + 12 + 8i - 3ai$$

$$= 2a + 12 + (8 - 3a)i$$

iii) For $x + iy$, the condition says $x = y$

$$2a + 12 = 8 - 3a$$

$$5a = -4$$

$$a = -\frac{4}{5}$$

iv) For any complex number w , the condition $w = w^*$ means that it is real: $a + bi = a - bi$
 $\Rightarrow b = -b \Rightarrow b = 0$.

$$8 - 3a = 0 \Rightarrow a = \frac{8}{3}, \text{ because}$$

$(z_1 z_2)^* = z_1^* z_2^*$ means z_1, z_2 is real, hence the imaginary part vanishes.

23 The complex number z satisfies the equation $5(z-i) = (-1+2i)z^*$.

Determine z , giving your answer in the form $a+bi$, where a and b are real.

[5]

Let $z = a+bi$, then

$$5(a+bi-i) = (-1+2i)(a-bi)$$

$$5a + (5b-5)i = -a + 2b + 2ai + bi$$

$$6a - 2b + (-2a + 4b - 5)i = 0$$

Separating real and imaginary parts we get

$$\begin{cases} 6a - 2b = 0 \\ 2a - 4b + 5 = 0 \end{cases} \Rightarrow \begin{cases} b = 3a \\ 2a - 12a + 5 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} b = 3a \\ a = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} b = -\frac{3}{2} \\ a = \frac{1}{2} \end{cases}$$

We get

$$z = \frac{1}{2} + \frac{3}{2}i$$

24 (a) The complex number $a + ib$ is denoted by z .

(i) Write down z^* . [1]

(ii) Find $\operatorname{Re}(iz)$. [2]

(b) The complex number w is given by $w = \frac{5+i\sqrt{3}}{2-i\sqrt{3}}$.

(i) In this question you must show detailed reasoning.

Express w in the form $x + iy$. [2]

(ii) Convert w to modulus-argument form. [2]

$$a) \quad i) \quad (a + ib)^* = a - ib$$

$$\begin{aligned} ii) \quad \operatorname{Re}(iz) &= \operatorname{Re}(i(a + ib)) \\ &= \operatorname{Re}(ia - b) \\ &= -b. \end{aligned}$$

$$\begin{aligned} b) \quad i) \quad \frac{5 + i\sqrt{3}}{2 - i\sqrt{3}} &= \frac{(5 + i\sqrt{3})(2 + i\sqrt{3})}{(2 - i\sqrt{3})(2 + i\sqrt{3})} \\ &= \frac{10 + i2\sqrt{3} + i5\sqrt{3} - 3}{7} \\ &= \frac{7 + i7\sqrt{3}}{7} \\ &= 1 + i\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{i)} \quad |w| &= \sqrt{1^2 + (\sqrt{3})^2} \\ &= \sqrt{4} \\ &= 2. \end{aligned}$$

$$\begin{aligned} \arg(w) &= \arctan\left(\frac{\sqrt{3}}{1}\right) \\ &= \frac{\pi}{3} \end{aligned}$$

$$w = 2 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right).$$

25 Two complex numbers are given by $u = -1 + i$ and $v = -2 - i$.

(a) (i) Find $u - v$ in the form $a + bi$, where a and b are real. [1]

(ii) In this question you must show detailed reasoning.

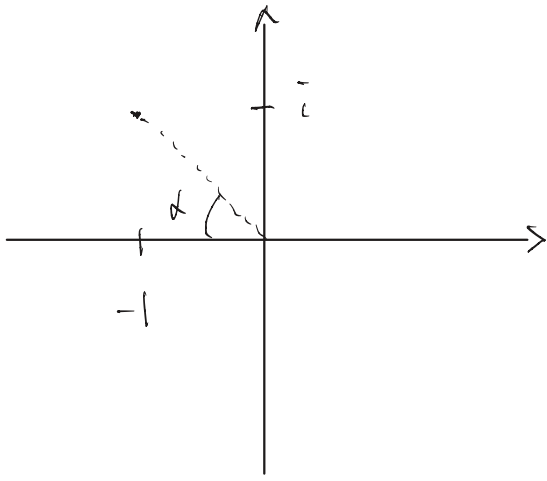
Find $\frac{u}{v}$ in the form $a + bi$, where a and b are real. [3]

(b) Express u in exact modulus-argument form. [3]

$$\begin{aligned} \text{a) } \quad i) \quad u - v &= -1 + i - (-2 - i) \\ &= -1 + i + 2 + i \\ &= 3 + 2i \end{aligned}$$

$$\begin{aligned} i) \quad \frac{u}{v} &= \frac{-1 + i}{-2 - i} \\ &= \frac{(-1 + i)(-2 + i)}{(-2 - i)(-2 + i)} \\ &= \frac{2 - 2i - i - 1}{4 + 1} \\ &= \frac{1 - 3i}{5} \\ &= \frac{1}{5} - i\frac{3}{5} \end{aligned}$$

$$\text{b) } |u| = \sqrt{1^2 + 1^2} = \sqrt{2}$$



$$\alpha = \arctan\left(\frac{1}{-1}\right) = \frac{\pi}{4}$$

$$\arg(z) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$z = \sqrt{2} \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right)$$

26 In this question you must show detailed reasoning.

The complex number $\sqrt{3} + i$ is denoted by z .

(a) By expanding $(\sqrt{3} + i)^5$, express z^5 in the form $a + bi$ where a and b are real and exact. [3]

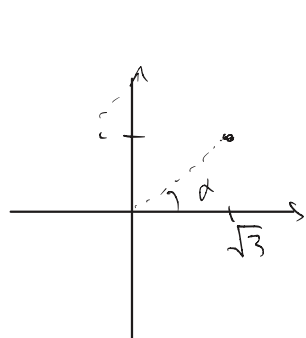
(b) (i) Express z in modulus-argument form. [3]

(ii) Hence find z^5 in modulus-argument form. [2]

(iii) Use this result to verify your answers to part (a). [2]

$$\begin{aligned} a) \quad (\sqrt{3} + i)^5 &= \sqrt{3}^5 + 5\sqrt{3}^4 \cdot i - 10\sqrt{3}^3 - 10\sqrt{3}^2 i + 5\sqrt{3} + i \\ &= 9\sqrt{3} + 45i - 30\sqrt{3} - 30i + 5\sqrt{3} + i \\ &= -16\sqrt{3} + 16i \end{aligned}$$

$$b) \quad i) \quad |z| = \sqrt{\sqrt{3}^2 + 1} = 2$$



$$\alpha = \arctan\left(\frac{i}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\Rightarrow z = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$ii) \quad |z^5| = |z|^5 = 32$$

$$\arg(z^5) = 5\arg(z) = \frac{5\pi}{6}$$

$$z = 32 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$$

$$(ii) \quad z = 32 \left(-\frac{\sqrt{3}}{2} + i \frac{1}{2} \right) = \boxed{-16\sqrt{3} + 16i}$$

27 The complex number z satisfies the equation $z^2 - 4iz^* + 11 = 0$.

Given that $\text{Re}(z) > 0$, find z in the form $a + bi$, where a and b are real numbers.

[4]

Let $z = a + bi$ and substitute

$$(a + bi)^2 - 4i(a - bi) + 11 = 0$$

$$a^2 + 2abi - b^2 - 4ai - 4b + 11 = 0$$

$$(a^2 - b^2 - 4b + 11) + (2ab - 4a)i = 0$$

We get a system of equations

$$\begin{cases} 2ab - 4a = 0 \\ a^2 - b^2 - 4b + 11 = 0 \end{cases} \Rightarrow \begin{cases} a(b - 2) = 0 \\ a^2 - b^2 - 4b + 11 = 0 \end{cases}$$

$\text{Re}(z) = a > 0$, forcing $b - 2 = 0$ i.e.

$$a(b - 2) = 0 \Rightarrow b = 2$$

$$\Rightarrow \begin{cases} b = 2 \\ a^2 - 2^2 - 4 \times 2 + 11 = 0 \end{cases} \Rightarrow \begin{cases} b = 2 \\ a^2 = 1 \end{cases} \Rightarrow \begin{cases} b = 2 \\ a = \pm 1 \end{cases}$$

$$a > 0 \Rightarrow$$

$$z = 1 + 2i$$

28 In this question you must show detailed reasoning.

The complex numbers z_1 and z_2 are given by $z_1 = -2 + 2i$ and $z_2 = 2\left(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi\right)$.

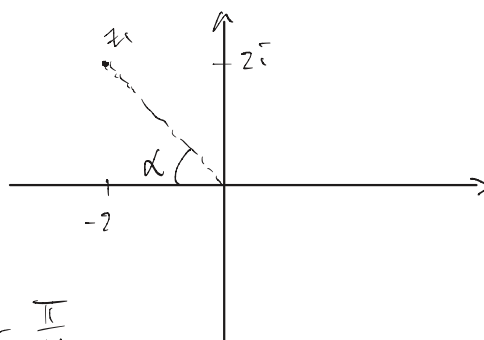
(a) Find the modulus and argument of z_1 . [2]

(b) Hence express $\frac{z_1}{z_2}$ in exact modulus-argument form. [4]

$$a) \quad |z_1| = \sqrt{(-2)^2 + (2)^2} = \boxed{\sqrt{8}}$$

$$z_1 = -2 + 2i$$

$$\alpha = \arctan\left(\frac{|2|}{|-2|}\right) = \frac{\pi}{4}$$



$$\Rightarrow \arg(z_1) = \pi - \alpha = \pi - \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$$

$$b) \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

$$= \frac{3\pi}{4} - \frac{\pi}{6}$$

$$= \frac{7\pi}{12}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{8}}{2} = \sqrt{2}$$

$$\boxed{\frac{z_1}{z_2} = \sqrt{2} \left(\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right) \right)}$$

29 In this question you must show detailed reasoning.

(a) Write the complex number $-24 + 7i$ in modulus-argument form.

[3]

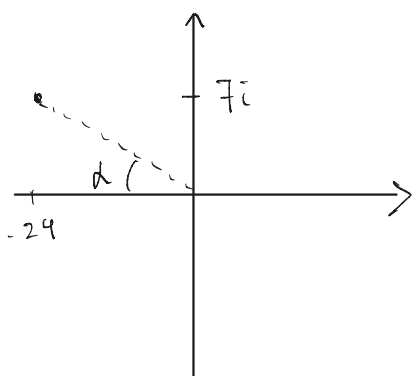
(b) Solve the simultaneous equations given below, giving your answers in cartesian form.

$$iz + 3w = -7i$$

$$-6z + 5iw = 3 + 13i$$

[4]

$$a) \quad |-24 + 7i| = \sqrt{(-24)^2 + 7^2} = \sqrt{625} = 25$$



$$\alpha = \arctan\left(\frac{|7|}{|-24|}\right) = 0.284$$

$$\arg(-24 + 7i) = 2.86$$

$$-24 + 7i = 25[\cos(2.86) + i\sin(2.86)]$$

$$b) \quad \begin{cases} \textcircled{1} \quad iz + 3w = -7i \\ \textcircled{2} \quad -6z + 5iw = 3 + 13i \end{cases}$$

$$\Rightarrow \begin{cases} -6z + 18iw = 42 \\ -6z + 5iw = 3 + 13i \end{cases}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow \begin{cases} 13iw = 39 - 13i \\ iz = -3w - 7i \end{cases} \Rightarrow \begin{cases} w = -1 - 3i \\ z = 2 - 3i \end{cases}$$

30 In this question you must show detailed reasoning.

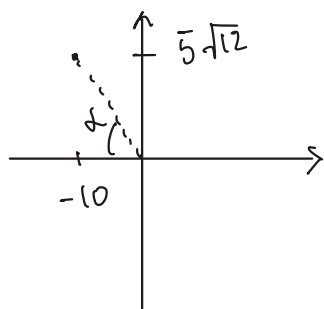
(a) Solve the equation $x^2 - 6x + 58 = 0$. Give your solutions in the form $a + bi$ where a and b are real numbers. [3]

(b) Determine, in exact form, $\arg(-10 + (5\sqrt{12})i)^5$. [3]

a) $x^2 - 6x + 58 = 0$, we use the quadratic formula

$$\begin{aligned} x_{1,2} &= \frac{6 \pm \sqrt{6^2 - 4 \times 58}}{2} \\ &= \frac{6 \pm \sqrt{-196}}{2} = \frac{6 \pm i14}{2} \\ &= \boxed{3 \pm i7} \end{aligned}$$

b) $\arg(z^5) = 5 \arg(z)$



$$\begin{aligned} \alpha &= \arctan\left(\frac{5\sqrt{12}}{10}\right) \\ &= \arctan(\sqrt{3}) = \frac{\pi}{3} \end{aligned}$$

Fun exercise:
Show by induction
that $\arg(z^n) = n \arg(z)$.

It is preferred
to give value θ
 $-\pi < \theta \leq \pi$

Thus $\arg(-10 + 5\sqrt{12}i) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Thus, $\arg(-10 + 5\sqrt{12}i)^5 = \frac{10\pi}{3} = \frac{4\pi}{3} = 2\pi - \frac{2\pi}{3}$

answer $\boxed{-\frac{2\pi}{3}}$

- 31 Use an algebraic method to find the square roots of $3 + (6\sqrt{2})i$. Give your answers in the form $x + iy$, where x and y are exact real numbers. [6]

$$(x + iy)^2 = 3 + 6\sqrt{2}i$$

$$x^2 + 2xyi - y^2 = 3 + 6\sqrt{2}i$$

Equating real and imaginary parts we have

Real ① $x^2 - y^2 = 3$

Imaginary ② $2xy = 6\sqrt{2}$

② $y = \frac{3\sqrt{2}}{x}$

① $x^2 - \frac{18}{x^2} = 3$

Subbing ② into ①

$x^2 \times$ ① $x^4 - 3x^2 - 18 = 0 \Rightarrow x^2 = \frac{3 \pm \sqrt{9 + 4 \times 18}}{2}$

$$= \frac{3 \pm 9}{2} = 6, -3$$

Since x is real x^2 must be positive, so $x^2 = 6$

$$x = \pm \sqrt{6}, y = \pm \sqrt{3}$$

$$z_1 = \sqrt{6} + \sqrt{3}i$$

$$z_2 = -\sqrt{6} - \sqrt{3}i$$

- 32 Use an algebraic method to find the square roots of the complex number $43 - (6\sqrt{10})i$. Give your answers in the form $x + iy$, where x and y are exact real numbers. [5]

$$(x + iy)^2 = 43 - (6\sqrt{10})i, \text{ where } x, y \text{ are real.}$$

$$x^2 - y^2 + 2xyi = 43 - 6\sqrt{10}i$$

Equating real and imaginary parts we get

$$\textcircled{1} \quad x^2 - y^2 = 43$$

$$\textcircled{2} \quad 2xy = -6\sqrt{10}$$

$$\textcircled{2} \quad y = -\frac{3\sqrt{10}}{x}$$

$$\textcircled{1} \quad x^2 - \frac{90}{x^2} = 43 \leftarrow \text{Subbing } \textcircled{2} \text{ into } \textcircled{1}$$

$$x^2 \times \textcircled{1}$$

$$x^4 - 43x^2 - 90 = 0 \Rightarrow x^2 = \frac{43 \pm \sqrt{43^2 + 4 \times 90}}{2}$$

Since x is real, x^2 must be positive real

$$x^2 = \frac{43 + 47}{2} = 45$$

$$x = \pm 3\sqrt{5}$$

$\Rightarrow$

$$y = \mp \sqrt{2}$$

The roots are

$$\begin{aligned} &3\sqrt{5} - \sqrt{2}i \\ &-3\sqrt{5} + \sqrt{2}i \end{aligned}$$

33 In this question you must show detailed reasoning.Solve the equation $4z^2 - 20z + 169 = 0$. Give your answers in modulus-argument form.**[5]**

$$4z^2 - 20z + 169 = 0$$

$$z = \frac{20 \pm \sqrt{20^2 - 16 \times 169}}{2 \times 4}$$

$$= \frac{20 \pm \sqrt{48^2 i}}{8}$$

$$= \frac{20 \pm 48i}{8}$$

$$= \frac{5}{2} \pm 6i$$

$$z_1 = \frac{5}{2} + 6i$$

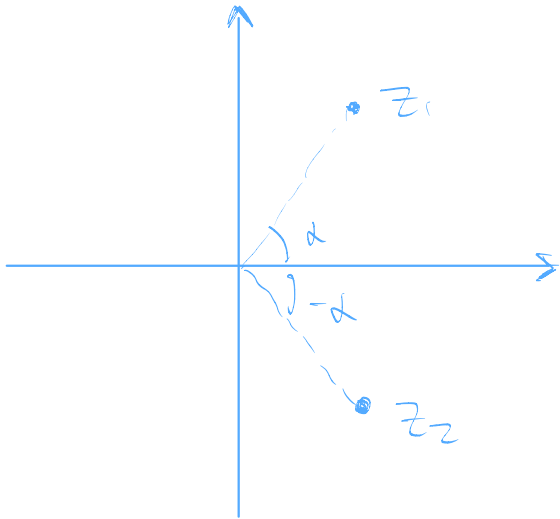
$$z_2 = \frac{5}{2} - 6i$$

$$|z_1| = \sqrt{\frac{25}{4} + 36}$$

$$= \sqrt{\frac{169}{4}}$$

$$= \frac{13}{2}$$

z_1 lies in I-quadrant, z_2 lies in IV-quadrant.



$$z_1 = \frac{5}{2} + 6i$$

$$\alpha = \arctan\left(\frac{6 \times 2}{5}\right)$$

$$= \arctan\left(\frac{12}{5}\right)$$

$$= 1.18$$

$$z_1 = \frac{13}{2} \left(\cos(1.18) + i \sin(1.18) \right)$$

$$z_2 = \frac{13}{2} \left(\cos(-1.18) + i \sin(-1.18) \right)$$

34 In this question you must show detailed reasoning.

(a) Use an algebraic method to find the square roots of $-16 + 30i$. [5]

(b) By finding the cube of one of your answers to part (a) determine a cube root of $\frac{-99 + 5i}{4}$.

Give your answer in the form $a + bi$. [2]

a) $(a + bi)^2 = -16 + 30i$, where a, b are real

$a^2 - b^2 + 2abi = -16 + 30i$, equating real and imaginary parts we get

Real ① $a^2 - b^2 = -16$
Imaginary ② $2ab = 30$
② $b = \frac{15}{a}$
① $a^2 - \frac{225}{a^2} = -16$ ← Subst ② into ①

$$a^4 + 16a^2 - 225 = 0 \Rightarrow a^2 = \frac{-16 \pm \sqrt{16^2 + 4 \times 225}}{2}$$

a is real $\Rightarrow a^2$ is positive real, so we pick positive solution

$$a^2 = \frac{-16 + 34}{2} = 9$$

$a = \pm 3$
 $b = \pm 5$, so the roots are

$$\begin{matrix} 3 + 5i \\ -3 - 5i \end{matrix}$$

$$\begin{aligned} \text{b) } (3+5i)^3 &= 27 + 3 \times 9 \times 5i - 3 \times 3 \times 25 - 125i \\ &= -198 - 10i \\ &= 2 \times (-99 - 5i) \end{aligned}$$

We use the fact that $(z^*)^3 = (z^3)^*$ to get

$$\Rightarrow (3 - 5i)^3 = 2 \times (-99 + 5i) \quad \downarrow \div 8$$
$$\div 8 \quad \downarrow \quad \frac{(3 - 5i)^3}{8} = \frac{-99 + 5i}{4}$$

Thus $\frac{3-5i}{2}$ is a cube root of $\frac{-99+5i}{4}$.

35 In this question you must show detailed reasoning.

Use an algebraic method to find the square roots of $-77 - 36i$.

[6]

$$(a + bi)^2 = -77 - 36i, \text{ where } a, b \text{ are real.}$$

$$a^2 - b^2 + 2abi = -77 - 36i, \text{ equating real and imaginary parts we get}$$

$$\begin{array}{l} \text{Real} \\ \text{Imaginary} \end{array} \quad \begin{cases} \textcircled{1} & a^2 - b^2 = -77 \\ \textcircled{2} & 2ab = -36 \end{cases}$$

$$\textcircled{2} \quad b = -\frac{18}{a}$$

$$\textcircled{1} \quad a^2 - \left(-\frac{18}{a}\right)^2 = -77 \leftarrow \text{Subbing } \textcircled{2} \text{ into } \textcircled{1}$$

$$\text{Multiplying } a^2 \times \textcircled{1} \text{ we get}$$

$$a^4 + 77a^2 - 324 = 0 \Rightarrow a^2 = \frac{-77 \pm \sqrt{7225}}{2}$$

$$a \text{ is real} \Rightarrow a^2 \text{ is positive real}$$

$$\Rightarrow a^2 = \frac{-77 + 85}{2} = \frac{8}{2} = 4$$

$$\begin{array}{l} a = \pm 2 \\ b = \mp 9 \end{array} \Rightarrow \text{roots are}$$

$$\begin{array}{l} 2 - 9i \\ -2 + 9i \end{array}$$

- 36 Use an algebraic method to find the square roots of $11 + (12\sqrt{5})i$. Give your answers in the form $x + iy$, where x and y are exact real numbers. [6]

$$(x + iy)^2 = 11 + (12\sqrt{5})i$$

$$x^2 - y^2 + 2xyi = 11 + 12\sqrt{5}i$$

Real $\textcircled{1} \begin{cases} x^2 - y^2 = 11 \\ 2xy = 12\sqrt{5} \end{cases}$

Imaginary $\textcircled{2} \begin{cases} y = \frac{6\sqrt{5}}{x} \\ x^2 - \frac{180}{x^2} = 11 \end{cases}$ Substitute $\textcircled{1}$ into $\textcircled{2}$

$$x^2 \cdot \textcircled{1} \quad x^4 - 11x^2 - 180 = 0$$

$$x^2 = \frac{11 \pm \sqrt{11^2 + 4 \times 180}}{2},$$

Since x is real, x^2 must be positive

$$x^2 = \frac{11 + 29}{2} = 20 \Rightarrow x = \pm 2\sqrt{5}$$

$$y = \pm 3$$

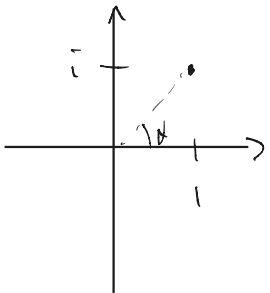
$$\begin{aligned} &2\sqrt{5} + 3i \\ &-2\sqrt{5} - 3i \end{aligned}$$

37 (a) (i) Find the modulus and argument of z_1 , where $z_1 = 1 + i$. [2]

(ii) Given that $|z_2| = 2$ and $\arg(z_2) = \frac{1}{6}\pi$, express z_2 in $a + bi$ form, where a and b are exact real numbers. [2]

(b) Using these results, find the exact value of $\sin \frac{5}{12}\pi$, giving the answer in the form $\frac{\sqrt{m} + \sqrt{n}}{p}$, where m, n and p are integers. [5]

a) i) $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$



$\alpha = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\arg(z_1) = \alpha = \frac{\pi}{4}$

ii) $z_2 = 2 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right)$

$= \sqrt{3} + i$

b) $|z_1 \cdot z_2| = |z_1| \cdot |z_2| = 2\sqrt{2}$

$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

$= \frac{\pi}{4} + \frac{\pi}{6}$

$= \frac{5\pi}{12}$

$z_1 \cdot z_2 = |z_1 \cdot z_2| \left(\cos(\arg(z_1 \cdot z_2)) + i \sin(\arg(z_1 \cdot z_2)) \right)$

thus $\arg(z_1 z_2) = \arg\left(\frac{z_1 z_2}{|z_1 z_2|}\right)$

$$\begin{aligned} \Rightarrow \arg\left(\frac{5\pi}{12}\right) &= \arg\left(\frac{z_1 z_2}{|z_1||z_2|}\right) \\ &= \arg\left(\frac{(1+i)(\sqrt{3}+i)}{2\sqrt{2}}\right) \\ &= \arg\left(\frac{\sqrt{3}-1 + (\sqrt{3}+1)i}{2\sqrt{2}}\right) \end{aligned}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \boxed{\frac{\sqrt{6} + \sqrt{2}}{4}}$$

- 38 (i) Use an algebraic method to find the square roots of the complex number $5 + 12i$. [5]
- (ii) Find $(3 - 2i)^2$. [2]
- (iii) Hence solve the quartic equation $x^4 - 10x^2 + 169 = 0$. [4]

i) $(a + bi)^2 = 5 + 12i$, where a, b are real.

$$a^2 + 2abi - b^2 = 5 + 12i$$

Equating real and imaginary parts we get

$$\begin{cases} a^2 - b^2 = 5 \\ 2ab = 12 \end{cases}$$

$$\begin{cases} b = \frac{6}{a} \\ a^2 - \frac{36}{a^2} = 5 \end{cases}$$

$$\begin{cases} a^4 - 5a^2 - 36 = 0 \\ a^2 = 9 \quad a^2 = -4 \end{cases}$$

$$a^2 = \frac{5 \pm \sqrt{25 + 4 \times 36}}{2} = 9, -4$$

but a is real
so $a^2 \geq 0$

$$\Rightarrow \begin{cases} a = \pm 3 \\ b = \pm 2 \end{cases}$$

the square roots are

$$\begin{array}{l} 3 + 2i \\ -3 - 2i \end{array}$$

$$ii) \quad (3-2i)^2 = 9 - 12i - 4 = 5 - 12i$$

$$iii) \quad x^4 - 10x^2 + 169 = 0$$

$$x^2 = \frac{10 \pm \sqrt{100 - 4 \times 169}}{2}$$

$$= \frac{10 \pm 24i}{2}$$

$$= 5 \pm 12i$$

$$x_{1,2} = \pm (3 + 2i)$$

$$x_{3,4} = \pm (3 - 2i)$$

39 One root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real, is $16 - 30i$.

(i) Write down the other root of the quadratic equation.

[1]

(ii) Find the values of a and b .

[4]

(iii) Use an algebraic method to solve the quartic equation $y^4 + ay^2 + b = 0$.

[7]

i) let $\alpha = x + iy$ denote the second root. Since a, b are real, the conjugate $\bar{\alpha}$ is also a root

$$\alpha = 16 + 30i$$

ii)

$$\begin{aligned} b &= 16x + 30y & a &= x + 16 \\ &= 16^2 + 30^2 & &= 32 \\ &= 1156 \end{aligned}$$

$$\begin{aligned} a &= 32 \\ b &= 1156 \end{aligned}$$

$$iii) \quad y^4 + ay^2 + b = 0 \Rightarrow y^2 = \begin{cases} 16 - 30i \\ 16 + 30i \end{cases}$$

$$y^2 = 16 + 30i \Rightarrow v^2 - w^2 + 2vw i = 16 + 30i$$

Equating real and imaginary parts we get

Real ① $\begin{cases} v^2 - w^2 = 16 \\ 2vw = 30 \end{cases}$

Imaginary ② $\begin{cases} w = \frac{15}{v} \\ v^2 - \frac{225}{v^2} = 16 \end{cases}$ ← Substitute ② into ①

$v^2 \times$ ① $v^4 - 16v^2 - 225 = 0$

$$v^2 = \frac{16 \pm \sqrt{16^2 + 4 \times 225}}{2}$$

v is real, so v^2 must be positive

$$v^2 = \frac{16 + \sqrt{1156}}{2} = 25$$

$v = \pm 5, w = \pm 3,$

$y^2 = 16 - 30i \Rightarrow v^2 - w^2 + 2vw i = 16 - 30i$

$$\begin{cases} w = -\frac{15}{v} \\ v^4 - 16v^2 - 225 = 0 \end{cases} \Rightarrow \begin{cases} v = \pm 5 \\ w = \mp 3 \end{cases}$$

$$z_{1,2,3,4} = \begin{cases} 5 + 3i \\ -5 - 3i \\ 5 - 3i \\ -5 + 3i \end{cases}$$

- 40 (i) Use an algebraic method to find the square roots of the complex number $9 + 40i$. [6]
- (ii) Show that $9 + 40i$ is a root of the quadratic equation $z^2 - 18z + 1681 = 0$. [1]
- (iii) By using the substitution $z = \frac{1}{u^2}$, find the roots of the equation $1681u^4 - 18u^2 + 1 = 0$. Give your answers in the form $x + iy$, where x and y are real. [4]

$$i) \quad (x + iy)^2 = 9 + 40i, \quad x, y \text{ are real.}$$

$$x^2 + 2xyi - y^2 = 9 + 40i$$

Equating real and imaginary parts we get

$$\textcircled{1} \quad \begin{cases} x^2 - y^2 = 9 \end{cases}$$

$$\textcircled{2} \quad \begin{cases} 2xy = 40 \end{cases}$$

$$\textcircled{2} \quad \begin{cases} y = \frac{20}{x} \end{cases}$$

$$\textcircled{1} \quad \begin{cases} x^2 - \frac{400}{x^2} = 9 \end{cases} \leftarrow \text{Substn } \textcircled{2} \text{ into } \textcircled{1}$$

$$x^2 \times \textcircled{1} \Rightarrow x^4 - 9x^2 - 400 = 0 \quad x^2 = \frac{9 \pm \sqrt{9^2 + 1600}}{2}$$

Since x is real, x^2 must be a positive real

$$x^2 = \frac{9 + 41}{2} = 25 \Rightarrow \text{the roots are}$$

$$x = \pm 5$$

$$y = \pm 4$$

$$5 + 4i$$

$$-5 - 4i$$

$$ii) \quad z^2 - 18z + 1681 = 0$$

$$z = \frac{18 \pm \sqrt{18^2 - 4 \times 1681}}{2} = \frac{18 \pm 80i}{2}$$

$$= 9 \pm 40i.$$

$$iii) \quad 1681u^4 - 18u^2 + 1 = 0$$

$$z = \frac{1}{u^2} \Rightarrow \frac{1681}{z^2} - \frac{18}{z} + 1 = 0$$

$$\Rightarrow z^2 - 18z + 1681 = 0$$

$\times z^2$

$$\text{by (ii)} \quad z_{1,2} = \begin{cases} 9 + 40i \\ 9 - 40i \end{cases}$$

$$\frac{1}{u^2} = 9 + 40i \Rightarrow \frac{1}{u} = 5 + 4i, -5 - 4i$$

$$\frac{1}{u^2} = 9 - 40i \Rightarrow \frac{1}{u} = 5 - 4i, -5 + 4i$$

We deduce the case for $9 - 40i$
using the fact that $(z^*)^2 = (z^2)^*$
i.e. $((5 + 4i)^*)^2 = ((5 + 4i)^2)^* = 9 - 40i$.

$$\begin{aligned}
 u_1 &= \frac{1}{5+4i} = \frac{5-4i}{(5+4i)(5-4i)} \\
 &= \frac{5-4i}{25+16} \\
 &= \frac{5}{41} - \frac{4}{41}i
 \end{aligned}$$

$$\begin{aligned}
 u_2 &= \frac{1}{-(5+4i)} \\
 &= -\left(\frac{5}{41} - \frac{4}{41}i\right) \\
 &= -\frac{5}{41} + \frac{4}{41}i
 \end{aligned}$$

$$\begin{aligned}
 u_3 &= \frac{1}{5-4i} \\
 &= \frac{1}{(5+4i)^*} \\
 &= \frac{5}{41} + \frac{4}{41}i
 \end{aligned}$$

$$\begin{aligned}
 u_4 &= \frac{1}{-5+4i} \\
 &= -\frac{1}{(5-4i)} \\
 &= -\frac{5}{41} - \frac{4}{41}i
 \end{aligned}$$

$$u_{1,2,3,4} = \begin{bmatrix} \frac{5}{41} - \frac{4}{41}i \\ -\frac{5}{41} + \frac{4}{41}i \\ -\frac{5}{41} - \frac{4}{41}i \\ \frac{5}{41} + \frac{4}{41}i \end{bmatrix}$$

- 41 (i) Use an algebraic method to find the square roots of the complex number $5 + 12i$. You must show sufficient working to justify your answers. [5]
- (ii) Hence solve the quadratic equation $x^2 - 4x - 1 - 12i = 0$

$$i) \quad (x + iy)^2 = 5 + 12i$$

$$x^2 - y^2 + 2xyi = 5 + 12i$$

Equating real and imaginary parts we get

Real ① $\begin{cases} x^2 - y^2 = 5 \end{cases}$

Imaginary ② $\begin{cases} 2xy = 12 \end{cases}$

$$\Rightarrow \begin{cases} y = \frac{6}{x} \\ x^2 - \frac{36}{x^2} = 5 \end{cases}$$

Substitute ① into ②

$$x^4 - 5x^2 - 36 = 0 \Rightarrow x^2 = \frac{5 \pm \sqrt{25 + 4 \times 36}}{2}$$

Since x is real x^2 must be a positive real

$$\Rightarrow x^2 = \frac{5 + \sqrt{169}}{2} = \frac{5 + 13}{2} = \frac{18}{2} = 9$$

$\Rightarrow x = \pm 3, y = \pm 2$, the square roots are

$$\begin{aligned} &3 + 2i \\ &-3 - 2i \end{aligned}$$

$$(i) \quad x^2 - 4x - 1 - 12i = 0$$

$$x^2 - 4x + 4 - 5 - 12i = 0$$

$$(x-2)^2 = 5 + 12i$$

completing the square.

$$\begin{aligned} \text{by } (i) \quad x_1 - 2 &= 3 + 2i \\ x_2 - 2 &= -3 - 2i \end{aligned} \Rightarrow$$

$$x_1 = 5 + 2i$$

$$x_2 = -1 - 2i$$

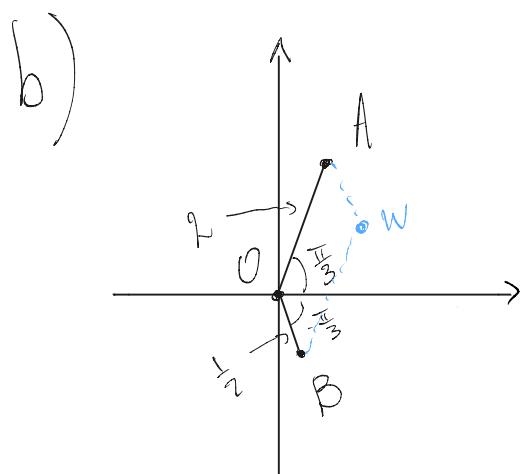
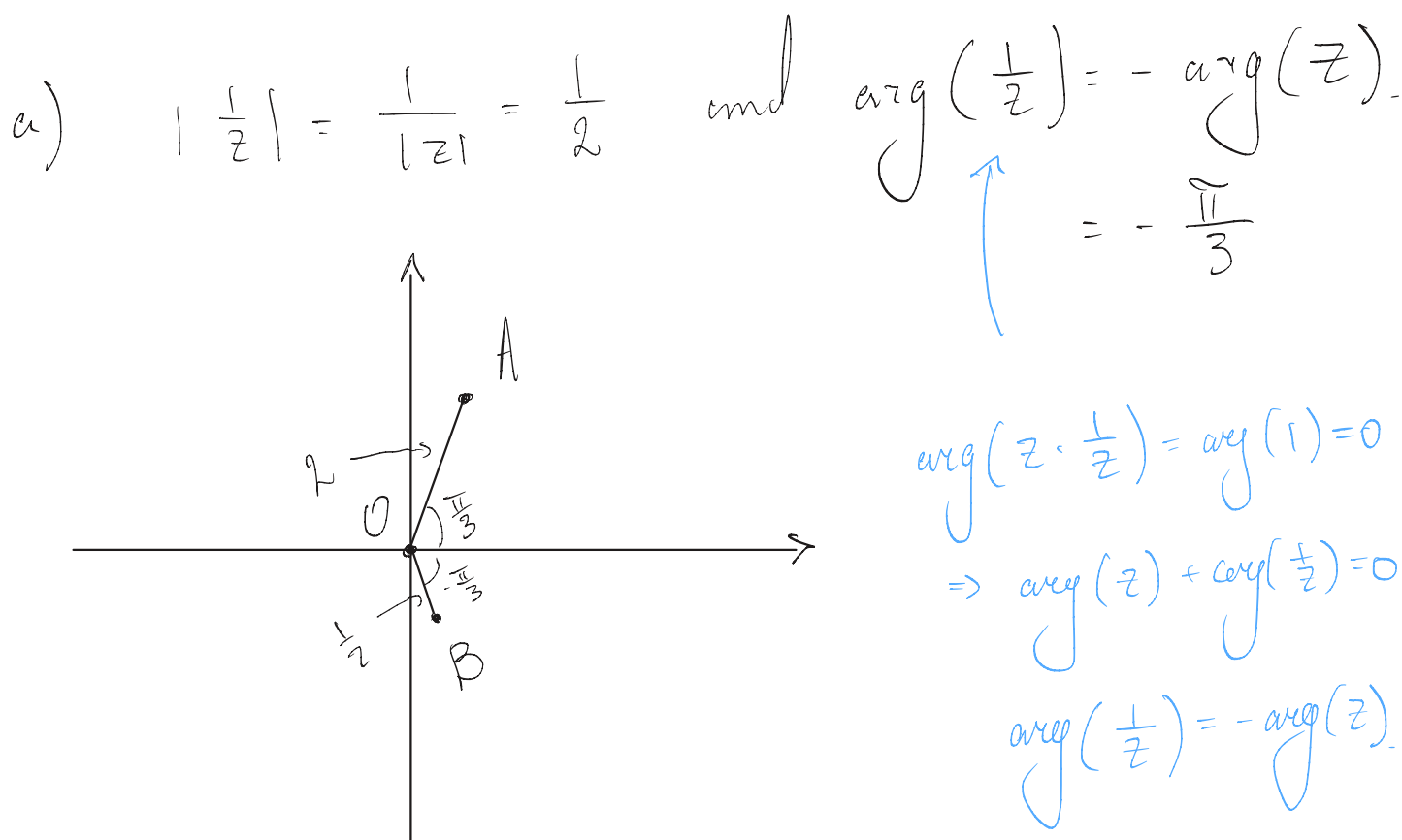
42 On an Argand diagram, the point A represents the complex number z with modulus 2 and argument $\frac{1}{3}\pi$. The point B represents $\frac{1}{z}$.

(a) Sketch an Argand diagram showing the origin O and the points A and B. [2]

(b) The point C is such that OACB is a parallelogram. C represents the complex number w .

Determine each of the following.

- The modulus of w , giving your answer in exact form.
- The argument of w , giving your answer correct to 3 significant figures. [7]



Viewing A and B as vectors we get w as their sum, which corresponds to $z + \frac{1}{z}$

$$\begin{aligned}
 z &= 2 \cdot \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right) \\
 &= 2 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\
 &= 1 + \sqrt{3}i
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{z} &= \frac{1}{2} \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right) \\
 &= \frac{1}{4} - \frac{\sqrt{3}}{4}i
 \end{aligned}$$

$$w = \frac{5}{4} + \frac{3\sqrt{3}}{4}i$$

$$|w| = \sqrt{\left(\frac{5}{4}\right)^2 + \left(\frac{3\sqrt{3}}{4}\right)^2}$$

$$= \sqrt{\frac{25}{16} + \frac{27}{16}}$$

$$= \sqrt{\frac{52}{16}}$$

$$= \boxed{\frac{\sqrt{13}}{2}}$$

$$\arg(w) = \arctan\left(\frac{3\sqrt{3}}{5}\right)$$

$$= 0.804633\dots$$

$$= \boxed{0.805}$$

43 The complex number z , where $0 < \arg z < \frac{1}{2}\pi$, is such that $z^2 = 3 + 4i$.

(i) Use an algebraic method to find z .

[5]

(ii) Show that $z^3 = 2 + 11i$.

[1]

The complex number w is the root of the equation

$$w^6 - 4w^3 + 125 = 0$$

for which $-\frac{1}{2}\pi < \arg w < 0$.

(iii) Find w .

[5]

i) The condition $0 < \arg z < \frac{1}{2}\pi$ indicates that z belongs to first quadrant $\Rightarrow z = a + bi$, where a, b are positive real numbers.

$$(a + bi)^2 = 3 + 4i \Rightarrow a^2 + 2abi - b^2 = 3 + 4i$$

Equating real and imaginary parts we get

Real ① $a^2 - b^2 = 3$

Imaginary ② $2ab = 4$

$$\Rightarrow \begin{cases} b = \frac{2}{a} \\ a^2 - \frac{4}{a^2} = 3 \end{cases}$$

Substg ② into ①

$$a^4 - 3a^2 - 4 = 0 \Rightarrow a^2 = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2}$$

a is a real number, thus a^2 is positive

$a^2 = \frac{3+5}{2} = 4$, $a = 2$ since a must be positive
 due to $0 < \arg(z) < \frac{\pi}{2}$

$$\Rightarrow \boxed{z = 2 + i}$$

$$\begin{aligned}
 \text{ii)} \quad z^3 &= (2+i)^3 = 2^3 + 3 \times 2^2 i - 3 \times 2 - i \\
 &= 8 - 6 + 12i - i \\
 &= 2 + 11i
 \end{aligned}$$

iii) The condition $-\frac{1}{2}\pi < \arg w < 0$, says that w belongs to IV-quadrant, hence for $w = a + bi$, a is a positive real number and b is negative.

$$\begin{aligned}
 w^6 - 4w^3 + 125 &= 0 & w^3 &= \frac{4 \pm \sqrt{4^2 - 4 \times 125}}{2} \\
 & & &= \frac{4 \pm \sqrt{4(4 - 125)}}{2} \\
 & & &= 2 \pm 11i
 \end{aligned}$$

Polynomial with real coefficients with z -root $\Rightarrow \bar{z}$ is a root.

$$(2-i)^3 = 8 - 3 \times 2^2 i - 3 \times 2 + i = 2 - 11i$$

which is one of the roots, so

$$w = 2 - i$$