

Complex Trigonometry: De Moivre's Theorem and Identities

Question Paper

1 In this question you must show detailed reasoning.

- (a)** Use de Moivre's theorem to determine constants A , B and C such that
 $\sin^4 \theta \equiv A \cos 4\theta + B \cos 2\theta + C$.

[5]

- 2** By expanding $\left(z^2 + \frac{1}{z^2}\right)^3$, where $z = e^{i\theta}$, show that $4 \cos^3 2\theta = \cos 6\theta + 3 \cos 2\theta$. **[5]**

- 3 (i) Express $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ and show that

$$\sin^4 \theta \equiv \frac{1}{8}(\cos 4\theta - 4 \cos 2\theta + 3). \quad [4]$$

- (ii) Hence find the exact value of $\int_0^{\frac{1}{6}\pi} \sin^4 \theta \, d\theta$. [4]

- 4 Use de Moivre's theorem to find the constants A , B and C in the identity $\sin^5 \theta \equiv A \sin \theta + B \sin 3\theta + C \sin 5\theta$.

[4]

5 (i) Starting with the result

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

show that

$$(A) \quad (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad [2]$$

$$(B) \quad \cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \quad [2]$$

(ii) Using the result in part (i) (A), obtain the values of the constants a , b , c and d in the identity

$$\cos 6\theta \equiv a \cos^6 \theta + b \cos^4 \theta + c \cos^2 \theta + d. \quad [6]$$

(iii) Using the result in part (i) (B), obtain the values of the constants P , Q , R and S in the identity

$$\cos^6 \theta \equiv P \cos 6\theta + Q \cos 4\theta + R \cos 2\theta + S. \quad [5]$$

$$(iv) \quad \text{Show that } \cos \frac{\pi}{12} = \left(\frac{26 + 15\sqrt{3}}{64} \right)^{\frac{1}{6}}. \quad [3]$$

- 6 (i) By expressing $\cos \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\cos^5 \theta \equiv \frac{1}{16}(\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta). \quad [5]$$

- (ii) Hence solve the equation $\cos 5\theta + 5 \cos 3\theta + 9 \cos \theta = 0$ for $0 \leq \theta \leq \pi$. [4]

7 In this question you must show detailed reasoning.

- (a) By writing $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ show that

$$\sin^6 \theta = \frac{1}{32}(10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta). \quad [5]$$

- (b) Hence show that $\sin \frac{1}{8}\pi = \frac{1}{2}\sqrt[6]{20 - 14\sqrt{2}}.$ [3]

- 8 (i) By expressing $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^5 \theta \equiv \frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta). \quad [4]$$

- (ii) Hence solve the equation

$$\sin 5\theta + 4 \sin \theta = 5 \sin 3\theta$$

$$\text{for } -\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi. \quad [4]$$

9 In this question you must show detailed reasoning.

(a) Show that $\operatorname{Re}(e^{4i\theta}(e^{i\theta} + e^{-i\theta})^4) = a \cos 4\theta \cos^4 \theta$, where a is an integer to be determined. **[3]**

(b) Hence show that $\cos \frac{1}{12}\pi = \frac{1}{2} \sqrt[4]{b + c\sqrt{3}}$, where b and c are integers to be determined. **[6]**

10 (a) Given that $z = \cos \theta + i \sin \theta$, express $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$ in simplified trigonometric form. [2]

(b) By considering $\left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^3$, find constants A and B such that

$$\sin^3 \theta \cos^3 \theta = A \sin 6\theta + B \sin 2\theta. \quad [6]$$

- 11** **(i) (a)** Verify, without using a calculator, that $\theta = \frac{1}{8}\pi$ is a solution of the equation $\sin 6\theta = \sin 2\theta$.
[1]
- (b)** By sketching the graphs of $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \leq \theta \leq \frac{1}{2}\pi$, or otherwise, find the other solution of the equation $\sin 6\theta = \sin 2\theta$ in the interval $0 < \theta < \frac{1}{2}\pi$.
[2]
- (ii)** Use de Moivre's theorem to prove that
- $$\sin 6\theta \equiv \sin 2\theta(16 \cos^4 \theta - 16 \cos^2 \theta + 3). \quad [5]$$
- (iii)** Hence show that one of the solutions obtained in part **(i)** satisfies $\cos^2 \theta = \frac{1}{4}(2 - \sqrt{2})$, and justify which solution it is.
[3]

- 12** (i) By expressing $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^6 \theta \equiv -\frac{1}{32}(\cos 6\theta - 6 \cos 4\theta + 15 \cos 2\theta - 10). \quad [5]$$

- (ii) Replace θ by $(\frac{1}{2}\pi - \theta)$ in the identity in part (i) to obtain a similar identity for $\cos^6 \theta$. [3]

- (iii) Hence find the exact value of $\int_0^{\frac{1}{4}\pi} (\sin^6 \theta - \cos^6 \theta) d\theta$. [4]

- 13** (i) By expressing $\cos \theta$ in terms of $e^{i\theta}$ show that

$$\cos^6 \theta = \frac{1}{32}(\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10). \quad [4]$$

- (ii) Hence solve, for $0 \leq \theta \leq \pi$,

$$\cos 6\theta + 6 \cos 4\theta + 2 \cos 2\theta = 3. \quad [5]$$

- 14** **(i)** Use de Moivre's theorem to express $\cos 4\theta$ as a polynomial in $\cos \theta$. **[4]**
- (ii)** Hence prove that $\cos 4\theta \cos 2\theta \equiv 16 \cos^6 \theta - 24 \cos^4 \theta + 10 \cos^2 \theta - 1$. **[1]**
- (iii)** Use part **(ii)** to show that the only roots of the equation $\cos 4\theta \cos 2\theta = 1$ are $\theta = n\pi$, where n is an integer. **[3]**
- (iv)** Show that $\cos 4\theta \cos 2\theta = -1$ only when $\cos \theta = 0$. **[3]**

15 (i) Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. **[3]**

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2 \cos^2 \theta - 1)(16 \cos^4 \theta - 16 \cos^2 \theta + 1).$$
 [5]

(iii) Hence find the exact value of

$$\cos\left(\frac{1}{12}\pi\right) \cos\left(\frac{5}{12}\pi\right) \cos\left(\frac{7}{12}\pi\right) \cos\left(\frac{11}{12}\pi\right),$$

justifying your answer. **[5]**

- 16** **(i)** Use de Moivre's theorem to prove that

$$\tan 5\theta \equiv \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}. \quad [4]$$

- (ii)** Solve the equation $\tan 5\theta = 1$, for $0 \leq \theta < \pi$. [3]

- (iii)** Show that the roots of the equation

$$t^4 - 4t^3 - 14t^2 - 4t + 1 = 0$$

may be expressed in the form $\tan \alpha$, stating the exact values of α , where $0 \leq \alpha < \pi$. [5]

- 17** **(i)** Use de Moivre's theorem to show that $\cos 5\theta \equiv 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$. **[5]**
- (ii)** Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos \alpha$ where $0 \leq \alpha \leq \pi$. **[4]**
- (iii)** Hence find the exact value of $\cos \frac{1}{10}\pi$. **[3]**

- 18 (i) By expressing $\sin \theta$ and $\cos \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, prove that

$$\sin^3 \theta \cos^2 \theta \equiv -\frac{1}{16}(\sin 5\theta - \sin 3\theta - 2 \sin \theta). \quad [6]$$

- (ii) Hence show that all the roots of the equation

$$\sin 5\theta = \sin 3\theta + 2 \sin \theta$$

are of the form $\theta = \frac{n\pi}{k}$, where n is any integer and k is to be determined. [3]

- 19** **(i)** Use de Moivre's theorem to show that $\tan 4\theta \equiv \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$. **[4]**
- (ii)** Hence find the exact roots of $t^4 + 4\sqrt{3}t^3 - 6t^2 - 4\sqrt{3}t + 1 = 0$. **[5]**

20 **(i)** Use de Moivre's theorem to show that

$$\sin 6\theta \equiv \cos \theta (6 \sin \theta - 32 \sin^3 \theta + 32 \sin^5 \theta). \quad [5]$$

(ii) Hence show that, for $\sin 2\theta \neq 0$,

$$-1 \leq \frac{\sin 6\theta}{\sin 2\theta} < 3. \quad [7]$$

- 21** (i) Use de Moivre's theorem to find an expression for $\cot 7\theta$ in terms of $\cot \theta$ and hence find the exact roots of the equation $u^6 - 21u^4 + 35u^2 - 7 = 0$. [7]
- (ii) State the exact roots of the equation $v^3 - 21v^2 + 35v - 7 = 0$, justifying your answer. Hence find the exact value of

$$\frac{\cot^2\left(\frac{1}{14}\pi\right)\cot^2\left(\frac{3}{14}\pi\right) + \cot^2\left(\frac{3}{14}\pi\right)\cot^2\left(\frac{5}{14}\pi\right) + \cot^2\left(\frac{5}{14}\pi\right)\cot^2\left(\frac{1}{14}\pi\right)}{\cot\left(\frac{1}{14}\pi\right)\cot\left(\frac{3}{14}\pi\right)\cot\left(\frac{5}{14}\pi\right)}.$$
 [4]