

Complex Trigonometry: De Moivre's Theorem and Identities

Worked Solutions

1 In this question you must show detailed reasoning.

- (a) Use de Moivre's theorem to determine constants A , B and C such that
 $\sin^4 \theta \equiv A \cos 4\theta + B \cos 2\theta + C$.

[5]

Let $z = \cos \theta + i \sin \theta$

$$\begin{aligned} (*) \quad z^n - \frac{1}{z^n} &= 2i \sin(n\theta) \\ (*)_2 \quad z^n + \frac{1}{z^n} &= 2 \cos(n\theta) \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} \text{(can be proved easily, but are results you} \\ \text{should have seen already!)} \end{array}$$

$$\left(z - \frac{1}{z}\right)^4 \stackrel{\text{by } (*)}{=} \left(2i \sin \theta\right)^4 = 16 \sin^4 \theta$$

$$\left(z - \frac{1}{z}\right)^4 \stackrel{\text{Binomial Expansion}}{=} z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$$

$$= z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$\stackrel{\text{by } (*)_2}{=} 2 \cos 4\theta - 4(2 \cos 2\theta) + 6$$

$$= 2 \cos 4\theta - 8 \cos 2\theta + 6$$

$$\text{So } 16 \sin^4 \theta = 2 \cos 4\theta - 8 \cos 2\theta + 6$$

$$\Rightarrow \sin^4 \theta = \frac{1}{8} \cos 4\theta - \frac{1}{2} \cos 2\theta + \frac{3}{8}$$

$$A = \frac{1}{8}, \quad B = -\frac{1}{2}, \quad C = \frac{3}{8}$$

- 2 By expanding $\left(z^2 + \frac{1}{z^2}\right)^3$, where $z = e^{i\theta}$, show that $4\cos^3 2\theta = \cos 6\theta + 3\cos 2\theta$. [5]

$$\left(z^2 + \frac{1}{z^2}\right)^3 = z^6 + 3z^2 + 3z^{-2} + z^{-6} \quad \text{By Binomial expansion}$$

$$= z^6 + \frac{1}{z^6} + 3\left(z^2 + \frac{1}{z^2}\right)$$

$$= 2\cos 6\theta + 3(2\cos 2\theta) \quad z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$= 2\cos 6\theta + 6\cos 2\theta$$

$$\left(z^2 + \frac{1}{z^2}\right)^3 = (2\cos 2\theta)^3 = 8\cos^3(2\theta)$$

$$\text{So } 8\cos^3(2\theta) = 2\cos 6\theta + 6\cos 2\theta$$

$$\Rightarrow 4\cos^3(2\theta) = \cos 6\theta + 3\cos 2\theta$$

- 3 (i) Express $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ and show that

$$\sin^4 \theta \equiv \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3). \quad [4]$$

- (ii) Hence find the exact value of $\int_0^{\frac{1}{6}\pi} \sin^4 \theta \, d\theta$. [4]

i) $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ see other questions if you're not sure.

so, $\sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})$

$$\Rightarrow \sin^4 \theta = \left(\frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \right)^4$$

$$= \frac{1}{16} (e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta})$$

$$= \frac{1}{16} (e^{4i\theta} + e^{-4i\theta} - 4(e^{2i\theta} + e^{-2i\theta}) + 6)$$

$$= \frac{1}{16} (2\cos 4\theta - 4(2\cos 2\theta) + 6)$$

$$= \frac{1}{16} (2\cos 4\theta - 8\cos 2\theta + 6)$$

$$= \frac{1}{8} (\cos 4\theta - 4\cos 2\theta + 3)$$

$e^{in\theta} + e^{-in\theta}$
 $= 2\cos(n\theta)$

ii) $\int_0^{\frac{1}{6}\pi} \sin^6 \theta \, d\theta = \int_0^{\frac{1}{6}\pi} \frac{1}{8} (\cos 4\theta - 4\cos 2\theta + 3) \, d\theta$

$$= \frac{1}{8} \int_0^{\frac{1}{6}\pi} (\cos 4\theta - 4\cos 2\theta + 3) \, d\theta$$

$$= \frac{1}{8} \left[\frac{1}{4} \sin 4\theta - 2 \sin 2\theta + 3\theta \right]_0^{\pi/6}$$

$$= \frac{1}{8} \left(\frac{1}{4} \sin \frac{2\pi}{3} - 2 \sin \frac{\pi}{3} + \frac{\pi}{2} \right)$$

$$\sin(0) = 0$$

$$= \frac{1}{8} \left(-\frac{7\sqrt{3}}{8} + \frac{\pi}{2} \right)$$

$$= \frac{\pi}{16} - \frac{7\sqrt{3}}{64} = \frac{4\pi - 7\sqrt{3}}{64}$$

- 4 Use de Moivre's theorem to find the constants A , B and C in the identity
 $\sin^5 \theta \equiv A \sin \theta + B \sin 3\theta + C \sin 5\theta$.

[4]

$$(*) \quad z^n - \frac{1}{z^n} = 2i \sin(n\theta)$$

$$\left(z - \frac{1}{z}\right)^5 = \left(2i \sin \theta\right)^5 = 32i \sin^5 \theta$$

by (*)

$$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$$

Binomial Expansion

$$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$\stackrel{\text{by } (*)}{=} 2i \sin 5\theta - 5(2i \sin 3\theta) + 10(2i \sin \theta)$$

$$= 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$$

So $32i \sin^5 \theta = 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$

$$\sin^5 \theta = \frac{1}{16} \sin 5\theta - \frac{5}{16} \sin 3\theta + \frac{5}{8} \sin \theta$$

$$= \frac{5}{8} \sin \theta - \frac{5}{16} \sin 3\theta + \frac{1}{16} \sin 5\theta$$

$$A = \frac{5}{8}, \quad B = -\frac{5}{16}, \quad C = \frac{1}{16}$$

5 (i) Starting with the result

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

show that

$$(A) \quad (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad [2]$$

$$(B) \quad \cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \quad [2]$$

(ii) Using the result in part (i) (A), obtain the values of the constants a , b , c and d in the identity

$$\cos 6\theta \equiv a \cos^6 \theta + b \cos^4 \theta + c \cos^2 \theta + d. \quad [6]$$

(iii) Using the result in part (i) (B), obtain the values of the constants P , Q , R and S in the identity

$$\cos^6 \theta \equiv P \cos 6\theta + Q \cos 4\theta + R \cos 2\theta + S. \quad [5]$$

$$(iv) \quad \text{Show that } \cos \frac{\pi}{12} = \left(\frac{26 + 15\sqrt{3}}{64} \right)^{\frac{1}{6}}. \quad [3]$$

$$\begin{aligned} i)(A) \quad LHS &= (\cos \theta + i \sin \theta)^n \\ &= (e^{i\theta})^n \\ &= e^{in\theta} \\ &= \cos n\theta + i \sin n\theta = RHS \end{aligned}$$

Note that this is just De Moivre's Theorem

$$\begin{aligned} (B) \quad RHS &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ &= \frac{1}{2}(\cos \theta + i \sin \theta + \cos(-\theta) + i \sin(-\theta)) \\ &= \frac{1}{2}(\cos \theta + i \sin \theta + \cos \theta - i \sin \theta) \\ &= \frac{1}{2}(2 \cos \theta) = \cos \theta = LHS \end{aligned}$$

Since cosine
is even
and sine is odd.

□

$$\text{ii) } (*) (\cos \theta + i \sin \theta)^6 = \cos 6\theta + i \sin 6\theta \quad \text{by (A)}$$

$$(*) (\cos \theta + i \sin \theta)^6 = \cos^6 \theta + 6i \cos^5 \theta \sin \theta - 15 \cos^4 \theta \sin^2 \theta - 20i \cos^3 \theta \sin^3 \theta + 15 \cos^2 \theta \sin^4 \theta + 6i \cos \theta \sin^5 \theta - \sin^6 \theta \quad \text{by Binomial expansion}$$

Equating real parts of $(*)_1$ and $(*)_2$ gives:

$$\cos 6\theta = \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta$$

$$= \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta)^2 - (1 - \cos^2 \theta)^3$$

$$= \cos^6 \theta - 15 \cos^4 \theta + 15 \cos^6 \theta + 15 \cos^2 \theta - 30 \cos^4 \theta + 15 \cos^6 \theta - 1 + 3 \cos^2 \theta - 3 \cos^4 \theta + \cos^6 \theta$$

$$= \boxed{32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1}$$

iii)

$$\cos^6 \theta = (\cos \theta)^6$$

$$= \left(\frac{1}{2} \left(z + \frac{1}{z} \right) \right)^6$$

by (B)

$$= \frac{1}{64} (z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6})$$

Binomial Expansion

$$= \frac{1}{64} \left((z^6 + \frac{1}{z^6}) + 6(z^4 + \frac{1}{z^4}) + 15(z^2 + \frac{1}{z^2}) + 20 \right)$$

$$= \frac{1}{64} (2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20)$$

$$= \boxed{\frac{1}{32} \cos 6\theta + \frac{3}{16} \cos 4\theta + \frac{15}{32} \cos 2\theta + \frac{5}{16}}$$

iv) we know,

$$\cos^6 \theta = \frac{1}{32} \cos 6\theta + \frac{3}{16} \cos 4\theta + \frac{15}{32} \cos 2\theta + \frac{5}{16}$$

Subbing in $\theta = \frac{\pi}{12}$ gives,

$$\cos^6\left(\frac{\pi}{12}\right) = \frac{1}{32} \cos\left(\frac{\pi}{2}\right) + \frac{3}{16} \cos\left(\frac{\pi}{3}\right) + \frac{15}{32} \cos\left(\frac{\pi}{6}\right) + \frac{5}{16}$$

$$= \frac{1}{32} (0) + \frac{3}{16} \left(\frac{1}{2}\right) + \frac{15}{32} \left(\frac{\sqrt{3}}{2}\right) + \frac{5}{16}$$

$$= \frac{13}{32} + \frac{15\sqrt{3}}{64} = \frac{26 + 15\sqrt{3}}{64}$$

By taking the 6th root on both sides we get,

$$\cos\left(\frac{\pi}{12}\right) = \left(\frac{26 + 15\sqrt{3}}{64}\right)^{1/6}$$

- 6 (i) By expressing $\cos \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\cos^5 \theta \equiv \frac{1}{16}(\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta). \quad [5]$$

- (ii) Hence solve the equation $\cos 5\theta + 5 \cos 3\theta + 9 \cos \theta = 0$ for $0 \leq \theta \leq \pi$. [4]

$$\begin{aligned} \text{i)} \quad 2 \cos \theta &= e^{i\theta} + e^{-i\theta} \\ \Rightarrow \cos \theta &= \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \\ \cos^5 \theta &= \frac{1}{32} (e^{i\theta} + e^{-i\theta})^5 \\ &= \frac{1}{32} (e^{5i\theta} + 5e^{3i\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-3i\theta} + e^{-5i\theta}) \\ &= \frac{1}{32} (e^{5i\theta} + e^{-5i\theta} + 5(e^{3i\theta} + e^{-3i\theta}) + 10(e^{i\theta} + e^{-i\theta})) \\ &= \frac{1}{32} (2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos \theta)) \\ &= \frac{1}{32} (2\cos 5\theta + 10\cos 3\theta + 20\cos \theta) \\ &= \frac{1}{16} (\cos 5\theta + 5\cos 3\theta + 10\cos \theta) \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \cos 5\theta + 5\cos 3\theta + 9\cos \theta &= 0 \quad \text{)} + \cos \theta \\ \text{+cos}\theta \left(\cos 5\theta + 5\cos 3\theta + 10\cos \theta \right) &= \cos \theta \end{aligned}$$

$$\frac{1}{16} (\cos 5\theta + 5\cos 3\theta + 10\cos \theta) = \frac{1}{16} \cos \theta$$

$$\cos^5 \theta = \frac{1}{16} \cos \theta$$

P.T.O

$$\cos^4 \theta = \frac{1}{16} \quad \text{or} \quad \cos \theta = 0$$

$$\Rightarrow \cos \theta = \pm \frac{1}{2} \quad \theta = \frac{1}{2} \pi$$

$$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\text{so } \theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$$

7 In this question you must show detailed reasoning.

(a) By writing $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ show that

$$\sin^6 \theta = \frac{1}{32}(10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta). \quad [5]$$

(b) Hence show that $\sin \frac{1}{8}\pi = \frac{1}{2}\sqrt{20 - 14\sqrt{2}}.$ [3]

$$\begin{aligned} a) \quad e^{i\theta} - e^{-i\theta} &= 2i \sin \theta \\ \Rightarrow \sin \theta &= \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \end{aligned}$$

$$\begin{aligned} \text{since } z^n - \frac{1}{z^n} &= 2i \sin n\theta \\ \text{(and } z^n + \frac{1}{z^n} &= 2 \cos n\theta) \end{aligned}$$

$$\begin{aligned} \sin^6 \theta &= \frac{-1}{64} (e^{i\theta} - e^{-i\theta})^6 \\ &= \frac{-1}{64} (e^{6i\theta} - 6e^{4i\theta} + 15e^{2i\theta} - 20 + 15e^{-2i\theta} - 6e^{-4i\theta} + e^{-6i\theta}) \\ &= \frac{-1}{32} (e^{6i\theta} + e^{-6i\theta} - 6(e^{4i\theta} + e^{-4i\theta}) + 15(e^{2i\theta} + e^{-2i\theta}) - 20) \\ &= \frac{-1}{64} (2 \cos 6\theta - 6(2 \cos 4\theta) + 15(2 \cos 2\theta) - 20) \\ &= \frac{1}{32} (10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta) \quad (*) \end{aligned}$$

$$b) \quad \text{Sub } \theta = \frac{1}{8}\pi \text{ into } (*)$$

$$\begin{aligned} \sin^6 \left(\frac{1}{8}\pi \right) &= \frac{1}{32} \left(10 - 15 \cos \left(\frac{\pi}{4} \right) + 6 \cos \left(\frac{\pi}{2} \right) - \cos \left(\frac{3\pi}{4} \right) \right) \\ &= \frac{1}{32} \left(10 - \frac{15\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \frac{1}{32} (10 - 7\sqrt{2}) \end{aligned}$$

$$\begin{aligned}\text{so } \sin\left(\frac{1}{2}\pi\right) &= \left(\frac{1}{32} (10 - 7\sqrt{2})\right)^{1/6} \\ &= \left(\frac{1}{64} (20 - 14\sqrt{2})\right)^{1/6} \\ &= \frac{1}{2} \sqrt[6]{20 - 14\sqrt{2}}\end{aligned}$$

- 8 (i) By expressing $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^5 \theta \equiv \frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta). \quad [4]$$

- (ii) Hence solve the equation

$$\sin 5\theta + 4 \sin \theta = 5 \sin 3\theta$$

$$\text{for } -\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi. \quad [4]$$

i) $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ since $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$

$$\therefore \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})$$

$$\sin^5 \theta = \left(\frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \right)^5$$

$$= \frac{1}{32i} \left(e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta} \right)$$

$$= \frac{1}{32i} \left(e^{5i\theta} - e^{-5i\theta} - 5(e^{3i\theta} - e^{-3i\theta}) + 10(e^{i\theta} - e^{-i\theta}) \right)$$

$$= \frac{1}{32i} \left(2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta \right)$$

$$= \frac{1}{16} (\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$$

ii) $\sin 5\theta + 4 \sin \theta = 5 \sin 3\theta$

$$\sin 5\theta - 5 \sin 3\theta + 4 \sin \theta = 0$$

+6sinθ

+6sinθ

$$\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta = 6 \sin \theta$$

$$\frac{1}{16} (\sin 5\theta - 5\sin 3\theta + 10\sin \theta) = \frac{1}{16} (6\sin \theta)$$

$$\sin^5 \theta = \frac{3}{8} \sin \theta$$

$$\Rightarrow \sin^4 \theta = \frac{3}{8} \quad \text{or} \quad \sin \theta = 0$$

$$\Rightarrow \sin \theta = \pm \sqrt[4]{\frac{3}{8}} \quad \text{or} \quad \theta = 0 \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\theta = \pm 0.849, 0 \quad (\text{to 3sf where appropriate})$$

9 In this question you must show detailed reasoning.

(a) Show that $\operatorname{Re}(e^{4i\theta}(e^{i\theta} + e^{-i\theta})^4) = a \cos 4\theta \cos^4 \theta$, where a is an integer to be determined. [3]

(b) Hence show that $\cos \frac{1}{12}\pi = \frac{1}{2} \sqrt[4]{b+c\sqrt{3}}$, where b and c are integers to be determined. [6]

$$\begin{aligned} \text{a) } e^{4i\theta}(e^{i\theta} + e^{-i\theta})^4 &= e^{4i\theta} (2\cos\theta)^4 \\ &= e^{4i\theta} \times 16\cos^4\theta \\ &= (\cos 4\theta + i\sin 4\theta) 16\cos^4\theta \\ &= 16\cos 4\theta \cos^4\theta + 16i\sin 4\theta \cos^4\theta \end{aligned}$$

$$\text{So, } \operatorname{Re}(e^{4i\theta}(e^{i\theta} + e^{-i\theta})^4) = \boxed{16\cos 4\theta \cos^4\theta}$$

$a=16$

b) The difficult part here is we can't just sub in $\theta = \frac{\pi}{12}$ straight away as we don't have an expression to equate it to. The idea is to write $\operatorname{Re}(e^{4i\theta}(e^{i\theta} + e^{-i\theta})^4)$ in a different form and then we can equate this to $16\cos 4\theta \cos^4\theta$

$$\begin{aligned} (e^{i\theta} + e^{-i\theta})^4 &= e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta} \\ &= e^{4i\theta} + e^{-4i\theta} + 4(e^{2i\theta} + e^{-2i\theta}) + 6 \\ &= 2\cos 4\theta + 8\cos 2\theta + 6 \end{aligned}$$

$$\operatorname{Re} \left\{ e^{4i\theta} (e^{i\theta} + e^{-i\theta})^4 \right\}$$

$$= \operatorname{Re} \left\{ (\cos 4\theta + i \sin 4\theta) (2 \cos 4\theta + 8 \cos 2\theta + 6) \right\}$$

$$= 2 \cos^2 4\theta + 8 \cos 4\theta \cos 2\theta + 6 \cos 4\theta$$

$$\Rightarrow 2 \cos^2 4\theta + 8 \cos 4\theta \cos 2\theta + 6 \cos 4\theta = 16 \cos^4 \theta$$

subbing in $\theta = \pi/12$ (and dividing by $\cos 4\theta$ since $\cos 4\theta \neq 0$ for $\theta = \pi/12$) gives

$$2 \cos \left(\frac{\pi}{3} \right) + 8 \cos \left(\frac{\pi}{6} \right) + 6 = 16 \cos^4 \left(\frac{\pi}{12} \right)$$

$$2 \left(\frac{1}{2} \right) + 8 \left(\frac{\sqrt{3}}{2} \right) + 6 = 16 \cos^4 \left(\frac{\pi}{12} \right)$$

$$7 + 4\sqrt{3} = 16 \cos^4 \left(\frac{\pi}{12} \right)$$

$$\frac{1}{16} (7 + 4\sqrt{3}) = \cos^4 \left(\frac{\pi}{12} \right)$$

$$\cos \left(\frac{\pi}{12} \right) = \left(\frac{1}{16} (7 + 4\sqrt{3}) \right)^{1/4}$$

$$= \frac{1}{2} \sqrt[4]{7 + 4\sqrt{3}} \quad b=7, c=4$$

10 (a) Given that $z = \cos \theta + i \sin \theta$, express $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$ in simplified trigonometric form. [2]

(b) By considering $\left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^3$, find constants A and B such that

$$\sin^3 \theta \cos^3 \theta = A \sin 6\theta + B \sin 2\theta.$$

[6]

a)

$$z^n + \frac{1}{z^n} = 2 \cos(n\theta)$$

$$z^n - \frac{1}{z^n} = 2i \sin(n\theta)$$

Note: if asked to, you should be able to derive these using the definitions. See below:

$$\begin{aligned} z^n + \frac{1}{z^n} &= z^n + z^{-n} \\ &= (\cos \theta + i \sin \theta)^n + (\cos \theta + i \sin \theta)^{-n} \\ &\stackrel{\text{De Moivre's Theorem}}{=} \cos(n\theta) + i \sin(n\theta) + \cos(-n\theta) + i \sin(-n\theta) \\ &= \cos(n\theta) + i \sin(n\theta) + \cos(n\theta) - i \sin(n\theta) \\ &= 2 \cos(n\theta) \end{aligned}$$

Since $\cos \theta$ is even, $\sin \theta$ is odd.

Same method for $z^n - \frac{1}{z^n}$.

$$\begin{aligned}
 b) \quad \left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^3 &= (2\cos\theta)^3 (2i\sin\theta)^3 \quad \text{by (a)} \\
 &= 8\cos^3\theta \times -8i\sin^3\theta \\
 &= -64i\cos^3\theta\sin^3\theta \quad (*)
 \end{aligned}$$

Here note, you could try to expand each of the brackets separately and then multiply them together but this is a lot of work! Much better to notice the difference of two squares...

$$\begin{aligned}
 \left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^3 &= \left(\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right)\right)^3 \\
 &= \left(z^2 - \frac{1}{z^2}\right)^3 \quad \checkmark \text{ Here notice that substituting in } z^2 - \frac{1}{z^2} = 2i\sin(2\theta) \text{ doesn't help me because I need multiple terms.} \\
 &= z^6 - 3z^2 + 3z^{-2} - z^{-6} \\
 &= z^6 - \frac{1}{z^6} - 3\left(z^2 - \frac{1}{z^2}\right) \\
 &= 2i\sin(6\theta) - 6i\sin(2\theta) \quad (*)_2
 \end{aligned}$$

by equating $(*)$ and $(*)_2$, we have

$$-64i\sin^3\theta\cos^3\theta = 2i\sin(6\theta) - 6i\sin(2\theta)$$

$$\sin^3\theta\cos^3\theta = \frac{-1}{32}\sin(6\theta) + \frac{3}{32}\sin(2\theta)$$

$$A = \frac{-1}{32}, B = \frac{3}{32}$$

11 (i) (a) Verify, without using a calculator, that $\theta = \frac{1}{8}\pi$ is a solution of the equation $\sin 6\theta = \sin 2\theta$. [1]

(b) By sketching the graphs of $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \leq \theta \leq \frac{1}{2}\pi$, or otherwise, find the other solution of the equation $\sin 6\theta = \sin 2\theta$ in the interval $0 < \theta < \frac{1}{2}\pi$. [2]

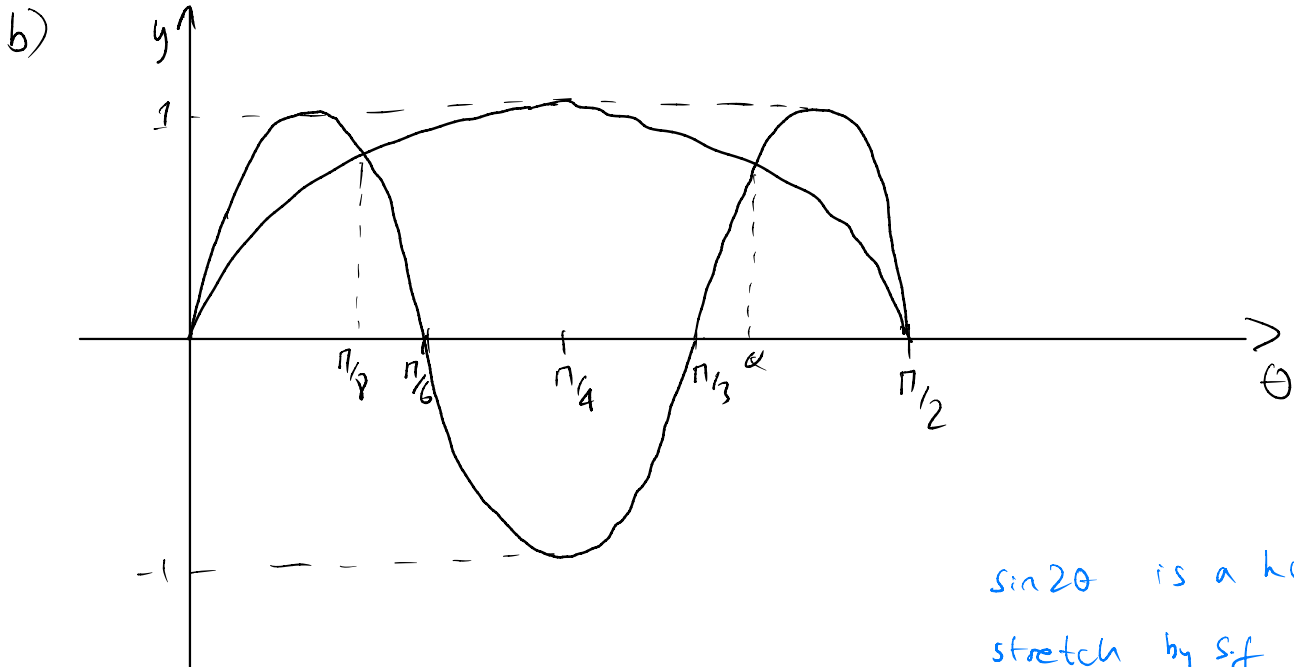
(ii) Use de Moivre's theorem to prove that

$$\sin 6\theta \equiv \sin 2\theta(16 \cos^4 \theta - 16 \cos^2 \theta + 3). \quad [5]$$

(iii) Hence show that one of the solutions obtained in part (i) satisfies $\cos^2 \theta = \frac{1}{4}(2 - \sqrt{2})$, and justify which solution it is. [3]

i)a) $LHS = \sin\left(\frac{6\pi}{8}\right) = \sin\left(\frac{3\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$ since $\sin(\pi - \theta) = \sin \theta$
 $= \frac{\sqrt{2}}{2}$

$RHS = \sin\left(\frac{2\pi}{8}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ so $LHS = RHS \Rightarrow$
 $\theta = \frac{1}{8}\pi$ is a solution.



by symmetry, $\alpha = \frac{\pi}{2} - \frac{\pi}{8}$
 $= \boxed{\frac{3\pi}{8}}$

$\sin 2\theta$ is a horizontal stretch by s.f $\frac{1}{2}$ of $\sin \theta$
 and $\sin 6\theta$ is a horizontal stretch by s.f $\frac{1}{6}$ of $\sin \theta$.

$$ii) (*) (\cos \theta + i \sin \theta)^6 = \cos 6\theta + i \sin 6\theta \quad \text{by De Moivre's Theorem} \quad ****$$

$$(*) (\cos \theta + i \sin \theta)^6 = \cos^6 \theta + 6i \cos^5 \theta \sin \theta - 15 \cos^4 \theta \sin^2 \theta - 20i \cos^3 \theta \sin^3 \theta + 15 \cos^2 \theta \sin^4 \theta + 6i \cos \theta \sin^5 \theta - \sin^6 \theta \quad \text{by Binomial Expansion}$$

Equating imaginary parts of $(*)_1$ and $(*)_2$ gives,

$$\sin 6\theta = 6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta$$

$$= 2 \sin \theta \cos \theta (3 \cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + 3 \sin^4 \theta)$$

$$= \sin 2\theta (3 \cos^4 \theta - 10 \cos^2 \theta (1 - \cos^2 \theta) + 3(1 - \cos^2 \theta)^2)$$

$$= \sin 2\theta (3 \cos^4 \theta - 10 \cos^2 \theta + 10 \cos^4 \theta + 3 - 6 \cos^2 \theta + 3 \cos^4 \theta)$$

$$= \sin 2\theta (16 \cos^4 \theta - 16 \cos^2 \theta + 3)$$

$$iii) \quad \text{To satisfy } \sin 6\theta = \sin 2\theta (16 \cos^4 \theta - 16 \cos^2 \theta + 3)$$

$$\text{and } \sin 6\theta = \sin 2\theta \quad \text{we need,}$$

$$16 \cos^4 \theta - 16 \cos^2 \theta + 3 = 1$$

$$\Rightarrow 8 \cos^4 \theta - 8 \cos^2 \theta + 1 = 0$$

$$\cos^2 \theta = \frac{8 \pm \sqrt{32}}{16}$$

$$= \frac{2 \pm \sqrt{2}}{4}$$

Now $\cos^2 \theta = \frac{2-\sqrt{2}}{4}$ when θ is larger since

$\cos \theta$ is a decreasing function for $0 \leq \theta \leq \pi/2$

$$\Rightarrow \cos\left(\frac{3\pi}{8}\right) < \cos\left(\frac{\pi}{8}\right)$$

$$\Rightarrow \cos^2\left(\frac{3\pi}{8}\right) < \cos^2\left(\frac{\pi}{8}\right) \quad \text{since both are positive.}$$

$\text{So } \cos^2\left(\frac{3\pi}{8}\right) = \frac{2-\sqrt{2}}{4}$	$\text{i.e. for } \theta = \frac{3\pi}{8}, \cos^2 \theta = \frac{2-\sqrt{2}}{4}$
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$$\left(\text{and } \cos^2\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{4} \right)$$

- 12 (i) By expressing $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^6 \theta \equiv -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10). \quad [5]$$

- (ii) Replace θ by $(\frac{1}{2}\pi - \theta)$ in the identity in part (i) to obtain a similar identity for $\cos^6 \theta$. [3]

- (iii) Hence find the exact value of $\int_0^{\frac{1}{4}\pi} (\sin^6 \theta - \cos^6 \theta) d\theta$. [4]

$$i) \quad e^{i\theta} - e^{-i\theta} = 2i \sin \theta$$

see other questions if you haven't seen this result before.

$$\begin{aligned} (2i \sin \theta)^6 &= (e^{i\theta} - e^{-i\theta})^6 \\ &= e^{6i\theta} - 6e^{4i\theta} + 15e^{2i\theta} - 20 + 15e^{-2i\theta} - 6e^{-4i\theta} + e^{-6i\theta} \end{aligned}$$

by the Binomial Expansion

$$= e^{6i\theta} + e^{-6i\theta} - 6(e^{4i\theta} + e^{-4i\theta}) + 15(e^{2i\theta} + e^{-2i\theta}) - 20$$

$$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

$$\text{so} \quad -64 \sin^6 \theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

$$\Rightarrow \sin^6 \theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$

$$ii) \quad \sin^6\left(\frac{\pi}{2} - \theta\right) = -\frac{1}{32}(\cos(6(\frac{\pi}{2} - \theta)) - 6\cos(4(\frac{\pi}{2} - \theta)) + 15\cos(2(\frac{\pi}{2} - \theta)) - 10)$$

$$\cos^6(\theta) = -\frac{1}{32}(\cos(3\pi - 6\theta) - 6\cos(2\pi - 4\theta) + 15\cos(\pi - 2\theta) - 10)$$

$$\cos(3\pi - 6\theta) = \cos(3\pi)\cos(6\theta) + \sin(3\pi)\sin(6\theta)$$

$$= -\cos(6\theta) \quad \text{since } \cos(3\pi) = -1 \quad \text{and } \sin(3\pi)$$

by same argument $\cos(\pi - 2\theta) = -\cos(2\theta)$

$$\text{and } \cos(2\pi - 4\theta) = \cos(4\theta) \quad \text{by } \cos(2\pi - \theta) \equiv \cos(\theta)$$

$$\text{so } \cos^6 \theta = -\frac{1}{32} (-\cos(6\theta) - 6\cos(4\theta) - 15\cos(2\theta) - 10)$$

$$= \boxed{\frac{1}{32} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)}$$

$$\text{ii)} \int_0^{\pi/4} (\sin^6 \theta - \cos^6 \theta) d\theta$$

$$= \int_0^{\pi/4} -\frac{1}{32} (2\cos 6\theta - 30\cos 2\theta) d\theta$$

$$= -\frac{1}{16} \int_0^{\pi/4} (\cos 6\theta + 15\cos 2\theta) d\theta$$

$$= -\frac{1}{16} \left[\frac{1}{6} \sin 6\theta + \frac{15}{2} \sin 2\theta \right]_0^{\pi/4}$$

$$= -\frac{1}{16} \left(\frac{1}{6} \sin\left(\frac{3\pi}{2}\right) + \frac{15}{2} \sin\left(\frac{\pi}{2}\right) \right)$$

$$\sin(0) = 0$$

$$= -\frac{1}{16} \left(-\frac{1}{6} + \frac{15}{2} \right) = -\frac{1}{16} \left(\frac{44}{6} \right) = \boxed{-\frac{11}{24}}$$

13 (i) By expressing $\cos \theta$ in terms of $e^{i\theta}$ show that

$$\cos^6 \theta = \frac{1}{32}(\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10). \quad [4]$$

(ii) Hence solve, for $0 \leq \theta \leq \pi$,

$$\cos 6\theta + 6 \cos 4\theta + 2 \cos 2\theta = 3. \quad [5]$$

i) $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ since $e^{in\theta} + e^{-in\theta} = 2\cos n\theta$

$$(\cos \theta)^6 = \left(\frac{1}{2}(e^{i\theta} + e^{-i\theta}) \right)^6$$

$$\cos^6 \theta = \frac{1}{64} (e^{6i\theta} + 6e^{4i\theta} + 15e^{2i\theta} + 20 + 15e^{-2i\theta} + 6e^{-4i\theta} + e^{-6i\theta})$$

$$= \frac{1}{64} (e^{6i\theta} + e^{-6i\theta} + 6(e^{4i\theta} + e^{-4i\theta}) + 15(e^{2i\theta} + e^{-2i\theta}) + 20)$$

$$= \frac{1}{64} (2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20)$$

$$= \frac{1}{32} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

ii) $\cos 6\theta + 6\cos 4\theta + 2\cos 2\theta = 3$ $+ 13\cos 2\theta + 10$

$$\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 = 13\cos 2\theta + 13$$

$\times \frac{1}{32}$ $\downarrow$ $\times \frac{1}{32}$

$$\frac{1}{32} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10) = \frac{13}{32} (\cos 2\theta + 1)$$

$$\cos^6 \theta = \frac{13}{32} (\cos 2\theta + 1)$$

On the surface, this is quite hard to solve as I have ****
different functions, namely $\cos \theta$ and $\cos 2\theta$. Thus,
I want to rewrite $\cos 2\theta$ in terms of $\cos \theta$:

$$\cos^6 \theta = \frac{13}{32} (\cos 2\theta + 1)$$

$$\cos^6 \theta = \frac{13}{32} (2\cos^2 \theta)$$

$$\text{Since } \cos 2\theta = 2\cos^2 \theta - 1$$

$$\cos^6 \theta = \frac{13}{16} \cos^2 \theta$$

$$\Rightarrow \cos^4 \theta = \frac{13}{16}$$

or

$$\cos^2 \theta = 0$$

$$\Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

$$\Rightarrow \cos \theta = \pm \frac{\sqrt[4]{13}}{2}$$

$$\therefore \theta = 0.319, 2.82, \pi/2 \quad \text{to 3sf (where appropriate)}$$

- 14 (i) Use de Moivre's theorem to express $\cos 4\theta$ as a polynomial in $\cos \theta$. [4]
- (ii) Hence prove that $\cos 4\theta \cos 2\theta \equiv 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1$. [1]
- (iii) Use part (ii) to show that the only roots of the equation $\cos 4\theta \cos 2\theta = 1$ are $\theta = n\pi$, where n is an integer. [3]
- (iv) Show that $\cos 4\theta \cos 2\theta = -1$ only when $\cos \theta = 0$. [3]

1) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ by De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

Equating real parts gives,

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

$$\begin{aligned}\text{ii) } \cos 4\theta \cos 2\theta &= \cos 4\theta (2\cos^2\theta - 1) \\ &= (8\cos^4\theta - 8\cos^2\theta + 1)(2\cos^2\theta - 1) \\ &= 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1\end{aligned}$$

$$\text{iii) } \cos 4\theta \cos 2\theta = 1$$

$$\Rightarrow 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 2 = 0$$

$$8\cos^6\theta - 12\cos^4\theta + 5\cos^2\theta - 1 = 0$$

$$\text{let } \cos^2\theta = u$$

$$f(u) = 8u^3 - 12u^2 + 5u - 1 = 0$$

$$f(1) = 0 \quad \text{so } (u-1) \text{ is a factor.}$$

$$(u-1)(8u^2 - 4u + 1) = 0$$

$$\downarrow \quad \text{No roots since } b^2 - 4ac = 16 - 4(8) = -16 < 0$$

$$u = 1$$

$$\Rightarrow \cos\theta = \pm 1$$

$$\text{so } \theta = 2\pi k \text{ or } \pi + 2\pi k \quad \text{for all } k \in \mathbb{Z}$$

$$= n\pi \quad \text{for all } n \in \mathbb{Z}$$

$$\text{iv) } \cos 4\theta \cos 2\theta = -1$$

$$\Rightarrow 8\cos^6\theta - 12\cos^4\theta + 5\cos^2\theta = 0$$

$$\cos^2\theta (8\cos^4\theta - 12\cos^2\theta + 5) = 0$$

$$\text{let } u = \cos^2\theta$$

$$u (8u^2 - 12u + 5) = 0$$

$$u = 0 \quad \text{or} \quad \downarrow \quad b^2 - 4ac = 144 - 4(8)(5) = -16 < 0$$

so no real roots

$$\text{so } \cos^2\theta = 0 \quad \text{is only solution}$$

$$\Rightarrow \cos\theta = 0$$

15 (i) Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. [3]

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2 \cos^2 \theta - 1)(16 \cos^4 \theta - 16 \cos^2 \theta + 1). \quad [5]$$

(iii) Hence find the exact value of

$$\cos\left(\frac{1}{12}\pi\right) \cos\left(\frac{5}{12}\pi\right) \cos\left(\frac{7}{12}\pi\right) \cos\left(\frac{11}{12}\pi\right),$$

justifying your answer. [5]

$$\begin{aligned} \text{i) } \cos 6\theta &= 0 & 0 < \theta < \pi \\ & & 0 < 6\theta < 6\pi \end{aligned}$$

$$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2} \quad \leftarrow \text{always adjust interval (where appropriate)}$$

$$\theta = \frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}$$

ii)

$$(*)_1 (\cos \theta + i \sin \theta)^6 = \cos 6\theta + i \sin 6\theta \quad \text{by De Moivre's Theorem}$$

$$\begin{aligned} (*)_2 (\cos \theta + i \sin \theta)^6 &= \cos^6 \theta + 6i \cos^5 \theta \sin \theta - 15 \cos^4 \theta \sin^2 \theta - 20i \cos^3 \theta \sin^3 \theta \\ &\quad + 15 \cos^2 \theta \sin^4 \theta + 6i \cos \theta \sin^5 \theta - \sin^6 \theta \quad \text{by Binomial Expansion} \end{aligned}$$

by equating real parts of $(*)_1$ and $(*)_2$ gives,

$$\begin{aligned} \cos 6\theta &= \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta \\ &= \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta) - (1 - \cos^2 \theta)^3 \end{aligned}$$

$$= \dots \quad \text{Expanding.}$$

$$= 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$$

$$\begin{aligned}
&= 32 \cos^6 \theta - 16 \cos^4 \theta - 32 \cos^4 \theta + 18 \cos^2 \theta - 1 \\
&= 16 \cos^4 \theta (2 \cos^2 \theta - 1) + (2 \cos^2 \theta - 1)(-16 \cos^2 \theta + 1) \\
&= (2 \cos^2 \theta - 1)(16 \cos^4 \theta - 16 \cos^2 \theta + 1)
\end{aligned}$$

iii)

We know $\cos 6\theta = 0$

has roots $\frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}$

so $(2 \cos^2 \theta - 1)(16 \cos^4 \theta - 16 \cos^2 \theta + 1) = 0$ must have the same roots, by (ii)

but $2 \cos^2 \theta - 1 = 0 \Rightarrow \cos \theta = \pm \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

so the roots of the quartic must be $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

by Quadratic Formula,

$$\cos^2 \theta = \frac{2 \pm \sqrt{3}}{4}$$

$$\text{so } \cos \theta = \pm \sqrt{\frac{2 \pm \sqrt{3}}{4}}$$

$$\Rightarrow \cos\left(\frac{\pi}{12}\right) \cos\left(\frac{5\pi}{12}\right) \cos\left(\frac{7\pi}{12}\right) \cos\left(\frac{11\pi}{12}\right)$$

$$= \left(\sqrt{\frac{2+\sqrt{3}}{4}}\right) \left(\sqrt{\frac{2-\sqrt{3}}{4}}\right) \left(-\sqrt{\frac{2-\sqrt{3}}{4}}\right) \left(-\sqrt{\frac{2+\sqrt{3}}{4}}\right)$$

$$= \left(\frac{2+\sqrt{3}}{4}\right) \left(\frac{2-\sqrt{3}}{4}\right) = \boxed{\frac{1}{16}}$$

- 16 (i) Use de Moivre's theorem to prove that

$$\tan 5\theta \equiv \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}. \quad [4]$$

- (ii) Solve the equation $\tan 5\theta = 1$, for $0 \leq \theta < \pi$. [3]

- (iii) Show that the roots of the equation

$$t^4 - 4t^3 - 14t^2 - 4t + 1 = 0$$

may be expressed in the form $\tan \alpha$, stating the exact values of α , where $0 \leq \alpha < \pi$. [5]

i) $(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$ by De Moivre's Theorem

$$\begin{aligned} (\cos \theta + i \sin \theta)^5 &= \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta \\ &\quad - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta \end{aligned}$$

By Binomial Expansion

Equating real parts gives,

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$$

Equating imaginary parts gives

$$\sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$

$$= \frac{5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta}{\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta} \times \frac{\frac{1}{\cos^5 \theta}}{\frac{1}{\cos^5 \theta}}$$

$$= \frac{5 \frac{\sin \theta}{\cos \theta} - 10 \frac{\sin^3 \theta}{\cos^3 \theta} + \frac{\sin^5 \theta}{\cos^5 \theta}}{1 - 10 \frac{\sin^2 \theta}{\cos^2 \theta} + 5 \frac{\sin^4 \theta}{\cos^4 \theta}}$$

$$= \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$$

ii) $\tan 5\theta = 1$

for $0 \leq \theta < \pi$
 $0 \leq 5\theta < 5\pi$

$$5\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}$$

$$\theta = \frac{\pi}{20}, \frac{\pi}{4}, \frac{9\pi}{20}, \frac{13\pi}{20}, \frac{17\pi}{20}$$

iii) Tricky question because we only have $\tan 5\theta = 1$ and the expression for $\tan 5\theta$ from i). To combine these two we'll do the following:

$$\tan 5\theta = 1$$

$$\Rightarrow \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta} = 1$$

$$\Rightarrow t^5 - 5t^4 - 10t^3 + 10t^2 + 5t - 1 = 0 \quad \text{where } t = \tan \theta$$

Here we know the solutions from ii), the easiest solution to try to factorise is $\tan(\pi/4) = 1$, i.e. $(t-1)$

$$(t-1)(t^4 - 4t^3 - 4t^2 + 4t + 1) = 0$$

so the solutions of the quartic are

$$t = \tan \alpha \quad \text{where } \alpha = \frac{\pi}{20}, \frac{9\pi}{20}, \frac{13\pi}{20}, \frac{17\pi}{20}$$

i.e. the ones left from (ii)

- 17 (i) Use de Moivre's theorem to show that $\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. [5]
- (ii) Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos\alpha$ where $0 \leq \alpha \leq \pi$. [4]
- (iii) Hence find the exact value of $\cos\frac{1}{10}\pi$. [3]

$$i) (\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

Equating real parts gives,

$$\begin{aligned} \cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \end{aligned}$$

$$ii) \text{ let } \cos 5\theta = 0 \quad \text{i.e.}$$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$$

$$\Rightarrow x(16x^4 - 20x^2 + 5) = 0 \quad \text{where } x = \cos\theta$$

so the roots of $16x^4 - 20x^2 + 5 = 0$ are the roots of $\cos 5\theta = 0$ excluding the root $x=0$
 p.t.o.

$$\cos 5\theta = 0$$

$$\Rightarrow 5\theta = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2, 9\pi/2$$

$$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \cancel{\frac{\pi}{2}}, \frac{7\pi}{10}, \frac{9\pi}{10}$$

This is the solution to $x=0$ ($\cos \theta = 0$) which is why we reject it.

So $x = \cos \alpha$ where $\alpha = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

iii) $16x^4 - 20x^2 + 5 = 0$

$$\Rightarrow x^2 = \frac{20 \pm \sqrt{80}}{32}$$

(Quadratic formula)

$\cos(\frac{\pi}{10})$ is greatest root since $\cos \theta$ is decreasing in the interval $0 \leq \theta \leq \pi$.

$$\text{So, } \cos \frac{\pi}{10} = \sqrt{\frac{20 + \sqrt{80}}{32}} = \sqrt{\frac{5 + \sqrt{5}}{8}}$$

Note for example, $\cos(\frac{7\pi}{10}) = -\sqrt{\frac{20 - \sqrt{80}}{32}}$

and $\cos(\frac{9\pi}{10}) = -\sqrt{\frac{20 + \sqrt{80}}{32}}$

- 18 (i) By expressing $\sin \theta$ and $\cos \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, prove that

$$\sin^3 \theta \cos^2 \theta \equiv -\frac{1}{16}(\sin 5\theta - \sin 3\theta - 2 \sin \theta). \quad [6]$$

- (ii) Hence show that all the roots of the equation

$$\sin 5\theta = \sin 3\theta + 2 \sin \theta$$

are of the form $\theta = \frac{n\pi}{k}$, where n is any integer and k is to be determined. [3]

$$\begin{aligned} i) \quad e^{in\theta} + e^{-in\theta} &= 2 \cos n\theta \\ e^{in\theta} - e^{-in\theta} &= 2i \sin n\theta \end{aligned}$$

$$\text{so } \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})$$

$$\& \quad \cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta})$$

$$\sin^3 \theta \cos^2 \theta = \left(\frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \right)^3 \left(\frac{1}{2} (e^{i\theta} + e^{-i\theta}) \right)^2$$

$$= -\frac{1}{8i} (e^{i\theta} - e^{-i\theta})^3 \frac{1}{4} (e^{i\theta} + e^{-i\theta})^2$$

$$= -\frac{1}{32i} (e^{i\theta} - e^{-i\theta}) (e^{i\theta} - e^{-i\theta})^2 (e^{i\theta} + e^{-i\theta})^2$$

$$= -\frac{1}{32i} (e^{i\theta} - e^{-i\theta}) \left((e^{i\theta} - e^{-i\theta})(e^{i\theta} + e^{-i\theta}) \right)^2$$

$$= -\frac{1}{32i} (e^{i\theta} - e^{-i\theta}) (e^{2i\theta} - e^{-2i\theta})^2$$

$$= -\frac{1}{32i} (e^{i\theta} - e^{-i\theta}) (e^{4i\theta} - 2 + e^{-4i\theta})$$

$$= \frac{-1}{32i} \left(e^{5i\theta} - 2e^{i\theta} + e^{-3i\theta} - e^{3i\theta} + 2e^{-i\theta} - e^{-5i\theta} \right)$$

$$= \frac{-1}{32i} \left(e^{5i\theta} - e^{-5i\theta} - (e^{3i\theta} - e^{-3i\theta}) - 2(e^{i\theta} - e^{-i\theta}) \right)$$

$$= \frac{-1}{32i} \left(2i \sin 5\theta - 2i \sin 3\theta - 4i \sin \theta \right)$$

$$= \frac{-1}{16} \left(\sin 5\theta - \sin 3\theta - 2 \sin \theta \right)$$

$$(i) \quad \sin 5\theta = \sin 3\theta + 2 \sin \theta$$

$$\Rightarrow \sin 5\theta - \sin 3\theta - 2 \sin \theta = 0$$

$$\Rightarrow -\frac{1}{16} (\sin 5\theta - \sin 3\theta - 2 \sin \theta) = 0$$

$$\Rightarrow \sin^3 \theta \cos^2 \theta = 0$$

$$\Rightarrow \sin^3 \theta = 0 \quad \text{or} \quad \cos^2 \theta = 0$$

$$\Rightarrow \sin \theta = 0 \quad \text{or} \quad \cos \theta = 0$$

$$\Rightarrow \theta = n\pi \quad \text{or} \quad \theta = \frac{\pi}{2} + n\pi, \quad \text{for all } n \in \mathbb{Z}$$

$$\Rightarrow \boxed{\theta = \frac{n\pi}{2} \quad \text{for all } n \in \mathbb{Z}}$$

19 (i) Use de Moivre's theorem to show that $\tan 4\theta \equiv \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$. [4]

(ii) Hence find the exact roots of $t^4 + 4\sqrt{3}t^3 - 6t^2 - 4\sqrt{3}t + 1 = 0$. [5]

i) $(\cos \theta + i \sin \theta)^4 = \cos 4\theta + i \sin 4\theta$ De Moivre's Theorem

$$(\cos \theta + i \sin \theta)^4 = \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 6i \cos \theta \sin^3 \theta + \sin^4 \theta$$
 Binomial Expansion

Equate real & imaginary parts :

$$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$$

$$\sin 4\theta = 4 \cos^3 \theta \sin \theta - 6 \cos \theta \sin^3 \theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$$

$$= \frac{4 \cos^3 \theta \sin \theta - 6 \cos \theta \sin^3 \theta}{\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta} \times \frac{\frac{1}{\cos^4 \theta}}{\frac{1}{\cos^4 \theta}}$$

$$= \frac{4 \frac{\sin \theta}{\cos \theta} - 6 \frac{\sin^3 \theta}{\cos^3 \theta}}{1 - \frac{6 \sin^2 \theta}{\cos^2 \theta} + \frac{\sin^4 \theta}{\cos^4 \theta}}$$

$$= \frac{4 \tan \theta - 6 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

ii) The trick here is to set $\tan 4\theta = k$ for some $k \in \mathbb{R}$. *****

$$\therefore k = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

where $t = \tan \theta$

$$kt^4 - 6kt^2 + k = 4t - 4t^3$$

$$kt^4 + 4t^3 - 6kt^2 - 4t + k = 0$$

$\times \sqrt{3}$ (

$$k\sqrt{3}t^4 + 4\sqrt{3}t^3 - 6k\sqrt{3}t^2 - 4\sqrt{3}t + k\sqrt{3} = 0$$

) $\times \sqrt{3}$

to match up the t^3 and t term to the polynomial in the question

$$\text{so } k = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan 4\theta = \frac{1}{\sqrt{3}}$$

$$4\theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$$

$$\theta = \frac{\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}, \frac{19\pi}{24}$$

20 (i) Use de Moivre's theorem to show that

$$\sin 6\theta \equiv \cos \theta (6 \sin \theta - 32 \sin^3 \theta + 32 \sin^5 \theta). \quad [5]$$

(ii) Hence show that, for $\sin 2\theta \neq 0$,

$$-1 \leq \frac{\sin 6\theta}{\sin 2\theta} < 3. \quad [7]$$

$$(\cos \theta + i \sin \theta)^6 = \cos 6\theta + i \sin 6\theta \quad \text{De Moivre's Theorem}$$

$$\begin{aligned} (\cos \theta + i \sin \theta)^6 &= \cos^6 \theta + 6i \cos^5 \theta \sin \theta - 15 \cos^4 \theta \sin^2 \theta \\ &\quad - 20i \cos^3 \theta \sin^3 \theta + 15 \cos^2 \theta \sin^4 \theta + 6i \cos \theta \sin^5 \theta - \sin^6 \theta \end{aligned}$$

By equating imaginary parts we get,

$$\begin{aligned} \sin 6\theta &= 6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta \\ &= \cos \theta (6 \cos^4 \theta \sin \theta - 20 \cos^2 \theta \sin^3 \theta + 6 \sin^5 \theta) \\ &= \cos \theta (6(1 - \sin^2 \theta)^2 \sin \theta - 20(1 - \sin^2 \theta) \sin^3 \theta + 6 \sin^5 \theta) \\ &= \cos \theta (6 \sin \theta - 12 \sin^3 \theta + 6 \sin^5 \theta - 20 \sin^3 \theta + 20 \sin^5 \theta + 6 \sin^5 \theta) \\ &= \cos \theta (6 \sin \theta - 32 \sin^3 \theta + 32 \sin^5 \theta) \end{aligned}$$

$$\text{ii) } \frac{\sin 6\theta}{\sin 2\theta} = \frac{\cos \theta (6 \sin \theta - 32 \sin^3 \theta + 32 \sin^5 \theta)}{2 \sin \theta \cos \theta}$$

$$\begin{aligned} &= 3 - 16 \sin^2 \theta + 16 \sin^4 \theta \quad \left(\begin{array}{l} \text{where } \sin 2\theta \neq 0 \\ \text{i.e. } \theta \neq \frac{\pi}{2} \text{ or } \text{where } n \in \mathbb{Z} \end{array} \right) \\ &= 16(\sin^4 \theta - \sin^2 \theta) + 3 \end{aligned}$$

$$= 16 \left(\left(\sin^2 \theta - \frac{1}{2} \right)^2 - \frac{1}{4} \right) + 3$$

$$= 16 \left(\sin^2 \theta - \frac{1}{2} \right)^2 - 1$$

Now $0 < \sin^2 \theta < 1$ since $\theta \neq \frac{\pi}{2}k$ where $k \in \mathbb{Z}$
 $\sin^2\left(\frac{\pi}{2}k\right) \neq 0$ or 1 for all $k \in \mathbb{Z}$

$$\therefore \text{lower bound} = 16 \left(\frac{1}{2} - \frac{1}{2} \right)^2 - 1$$

$$= 16 \times 0 - 1 = -1$$

$$\text{upper bound (not attained)} = 16 \left(1 - \frac{1}{2} \right)^2 - 1$$

$$= 16 \times \frac{1}{4} - 1$$

$$= 4 - 1 = 3$$

$$\text{So } -1 \leq \frac{\sin 6\theta}{\sin 2\theta} < 3$$

- 21 (i) Use de Moivre's theorem to find an expression for $\cot 7\theta$ in terms of $\cot \theta$ and hence find the exact roots of the equation $u^6 - 21u^4 + 35u^2 - 7 = 0$. [7]
- (ii) State the exact roots of the equation $v^3 - 21v^2 + 35v - 7 = 0$, justifying your answer. Hence find the exact value of

$$\frac{\cot^2\left(\frac{1}{14}\pi\right)\cot^2\left(\frac{3}{14}\pi\right) + \cot^2\left(\frac{3}{14}\pi\right)\cot^2\left(\frac{5}{14}\pi\right) + \cot^2\left(\frac{5}{14}\pi\right)\cot^2\left(\frac{1}{14}\pi\right)}{\cot\left(\frac{1}{14}\pi\right)\cot\left(\frac{3}{14}\pi\right)\cot\left(\frac{5}{14}\pi\right)}.$$
 [4]

i) $(\cos \theta + i \sin \theta)^7 = \cos 7\theta + i \sin 7\theta$ De Moivre's Theorem

$$(\cos \theta + i \sin \theta)^7 = \cos^7 \theta + 7i \cos^6 \theta \sin \theta - 21 \cos^5 \theta \sin^2 \theta - 35i \cos^4 \theta \sin^3 \theta + 35 \cos^3 \theta \sin^4 \theta + 21i \cos^2 \theta \sin^5 \theta - 7 \cos \theta \sin^6 \theta - i \sin^7 \theta$$

Binomial Expansion

Equating Real & Imaginary parts gives,

$$\cos 7\theta = \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta$$

$$\sin 7\theta = 7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta$$

$$\Rightarrow \cot 7\theta = \frac{\cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta}{7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta} \times \frac{\frac{1}{\sin^7 \theta}}{\frac{1}{\sin^7 \theta}}$$

$$= \frac{\cot^7 \theta - 21 \cot^5 \theta + 35 \cot^3 \theta - 7 \cot \theta}{7 \cot^6 \theta - 35 \cot^4 \theta + 21 \cot^2 \theta - 1}$$

Notice the similarities in the coefficients of the numerator.

$$\cot 7\theta = 0 \Rightarrow u^7 - 21u^5 + 35u^3 - 7u = 0 \quad \text{where } u = \cot \theta$$

$$\Rightarrow u(u^6 - 21u^4 + 35u^2 - 7) = 0$$

Thus the roots of $u^6 - 21u^4 + 35u^2 - 7 = 0$ are the roots of $\cot 7\theta = 0$ excluding the root of $u=0$.

$$\cot 7\theta = 0 \Rightarrow 7\theta = \pi/2, 3\pi/2, 5\pi/2, \cancel{7\pi/2}, 9\pi/2, 11\pi/2, 13\pi/2$$
$$\theta = \pi/14, 3\pi/14, 5\pi/14, \cancel{7\pi/14}, 9\pi/14, 11\pi/14, 13\pi/14$$

So the roots of $u^6 - 21u^4 + 35u^2 - 7 = 0$ are:

$$u = \cot(\alpha) \text{ where } \pi/14, 3\pi/14, 5\pi/14, 9\pi/14, 11\pi/14, 13\pi/14$$

$$\text{ii) } v^3 - 21v^2 + 35v - 7 = 0$$

$$v = u^2$$

Now, $\cot(\theta)$ has period of π (since $\cot \theta = \frac{1}{\tan \theta}$)

$$\text{Thus, } \cot\left(\frac{13\pi}{14}\right) = \cot\left(-\frac{\pi}{14}\right)$$

by periodicity

$$= -\cot\left(\frac{\pi}{14}\right)$$

since $\cot(\theta)$ is an odd function. This is something that you can prove!

$$\text{So } v = \cot^2\left(\frac{\pi}{14}\right), \cot^2\left(\frac{3\pi}{14}\right), \cot^2\left(\frac{5\pi}{14}\right)$$

$$\text{since } \cot^2\left(\frac{13\pi}{14}\right) = \cot^2\left(\frac{\pi}{14}\right) \text{ and so on.}$$

ii) Suppose the roots of the cubic $v^3 - 21v^2 + 35v - 7 = 0$ are α, β and γ .

Without loss of generality,

let $\alpha = \cot^2(\pi/14)$, $\beta = \cot^2(3\pi/14)$, $\gamma = \cot^2(5\pi/14)$

Notice that

$$\frac{\cot^2(\frac{1}{14}\pi)\cot^2(\frac{3}{14}\pi) + \cot^2(\frac{3}{14}\pi)\cot^2(\frac{5}{14}\pi) + \cot^2(\frac{5}{14}\pi)\cot^2(\frac{1}{14}\pi)}{\cot(\frac{1}{14}\pi)\cot(\frac{3}{14}\pi)\cot(\frac{5}{14}\pi)} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\sqrt{\alpha\beta\gamma}}$$

using theory of
roots of polynomial from
Year 1 Pure Core.

$$= \frac{\frac{c}{a}}{\sqrt{-\frac{d}{a}}} = \frac{35}{\sqrt{7}}$$