

Proof by Induction - Divisibility

Worked Solutions

1. Prove by induction that for all positive integers n ,

$$f(n) = 3^{4n-2} + 2^{6n-3} \text{ is divisible by } 17$$

(6)

For $n=1$ we have $f(1) = 3^{4 \times 1 - 2} + 2^{6 \times 1 - 3} = 17$.

Assume true for $n=k$, i.e. $f(k) = 3^{4k-2} + 2^{6k-3} = 17M$
for some integer M . $\Rightarrow 3^{4k-2} = 17M - 2^{6k-3}$

Then for $n=k+1$ we get

$$\begin{aligned} f(k+1) &= 3^{4k-2+4} + 2^{6k-3+6} \\ &= 81 \times 3^{4k-2} + 64 \times 2^{6k-3} \\ &= 81(17M - 2^{6k-3}) + 64 \times 2^{6k-3} \\ &= 17 \times 81M - 81 \times 2^{6k-3} + 64 \times 2^{6k-3} \\ &= 17(81M - 2^{6k-3}) \end{aligned}$$

which is divisible by 17.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

2. Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 7^{n-1} + 8^{2n+1}$$

is divisible by 57

(6)

For $n=1$ we have $f(1) = 7^{1-1} + 8^{2 \times 1 + 1} = 57 \times 9$, which is divisible by 57.

Assume that for $n=k$, $f(k)$ is divisible by 57, i.e. $f(k) = 7^{k-1} + 8^{2k+1} = 57 \times M$, where M is some integer. $7^{k-1} = 57 \times M - 8^{2k+1}$

Then for $n=k+1$ we get

$$\begin{aligned} f(k+1) &= 7^{k-1+1} + 8^{2k+1+2} \\ &= 7 \times 7^{k-1} + 64 \times 8^{2k+1} \\ &= 7 \times (57 \times M - 8^{2k+1}) + 64 \times 8^{2k+1} \\ &= 57 \times 7 \times M + 57 \times 8^{2k+1} \\ &= 57(7M + 8^{2k+1}) \end{aligned}$$

which is divisible by 57.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

3. (ii) Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 12^n + 2 \times 5^{n-1}$$

is divisible by 7

(6)

For $n=1$ we have $f(1) = 12^1 + 2 \times 5^{1-1} = 12 + 2 = 14 = 7 \times 2$
which is divisible by 7.

Assume the assertion holds for $n=k$, i.e. $f(k) = 12^k + 2 \times 5^{k-1} = 7M$
for some integer M . Also note $12^k = 7M - 2 \times 5^{k-1}$

Then for $n=k+1$ we have

$$\begin{aligned} f(k+1) &= 12^{k+1} + 2 \times 5^{k-1+1} \\ &= 12 \times 12^k + 5 \times 2 \times 5^{k-1} \\ &= 12(7M - 2 \times 5^{k-1}) + 2 \times 5 \times 5^{k-1} \\ &= 7 \times 12M - 24 \times 5^{k-1} + 10 \times 5^{k-1} \\ &= 7 \times (12M - 2 \times 5^{k-1}) \end{aligned}$$

which is divisible by 7.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

4. (ii) Prove by induction that, for all positive integers n ,

$$f(n) = 3^{3n-2} + 2^{4n-1}$$

is divisible by 11

(5)

For $n=1$ we get $f(1) = 3^{3 \times 1 - 2} + 2^{4 \times 1 - 1} = 11$.

Assume the assertion holds for $n=k$, i.e.

$$f(k) = 3^{3k-2} + 2^{4k-1} = 11M \text{ for some integer } M.$$

$$\Rightarrow 3^{3k-2} = 11M - 2^{4k-1}$$

Then for $n=k+1$ we get

$$f(k+1) = 3^{3k-2+3} + 2^{4k-1+4}$$

$$= 27 \times 3^{3k-2} + 16 \times 2^{4k-1}$$

$$= 27(11M - 2^{4k-1}) + 16 \times 2^{4k-1}$$

$$= 11 \times 27M - 11 \times 2^{4k-1}$$

$$= 11(27M - 2^{4k-1}) \text{ which is divisible by } 11.$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

5. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 7^n - 2^n \text{ is divisible by 5}$$

(6)

For $n=1$ we have $f(1) = 7^1 - 2^1 = 5$

Assume the assertion holds for $n=k$, i.e.

$$f(k) = 7^k - 2^k = 5M \text{ for some integer } M.$$

$$\Rightarrow 7^k = 5M - 2^k.$$

Then for $n=k+1$ we have $f(k+1) = 7^{k+1} - 2^{k+1} =$

$$\begin{aligned} &= 7 \times 7^k - 2 \times 2^k = 7 \times (5M - 2^k) - 2 \times 2^k \\ &= 5 \times 7M - 5 \times 2^k \\ &= 5(7M - 2^k). \end{aligned}$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

6. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

For $n=1$ we have $f(1) = 4^{1+1} + 5^{2 \times 1 - 1} = 21$

Assume true for $n=k$, i.e. $f(k) = 4^{k+1} + 5^{2k-1} = 21M$

$$\Rightarrow 4^{k+1} = 21M - 5^{2k-1}$$

Then for $n=k+1$ we get

$$f(k+1) = 4^{k+1+1} + 5^{2k-1+2}$$

$$= 4 \times 4^{k+1} + 25 \times 5^{2k-1}$$

$$= 4 \times (21M - 5^{2k-1}) + 25 \times 5^{2k-1}$$

$$= 21 \times 4 \times M - 4 \times 5^{2k-1} + 25 \times 5^{2k-1}$$

$$= 21(4M + 5^{2k-1})$$

which is divisible by 21

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

7. Prove by induction, that for $n \in \mathbb{Z}^+$,

(b) $f(n) = 7^{2n-1} + 5$ is divisible by 12.

For $n=1$ we have $f(1) = 7^{2 \times 1 - 1} + 5 = 12$, which is ⁽⁶⁾ divisible by 12.

Assume the assertion holds for $n=k$, i.e.

$$f(k) = 7^{2k-1} + 5 = 12M \text{ for some integer } M$$

$$\Rightarrow 7^{2k-1} = 12M - 5. \text{ Then for } n=k+1$$

$$\begin{aligned} \text{we have } f(k+1) &= 7^{2(k+1)-1} + 5 \\ &= 7^{2k-1+2} + 5 \\ &= 49 \times (12M - 5) + 5 \\ &= 12 \times 49M - 48 \times 5 \\ &= 12(49M - 20) \end{aligned}$$

thus divisible by 12.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

8 Prove by induction that $4 \times 8^n + 66$ is divisible by 14 for all integers $n \geq 0$.

[6]

For $n=0$ we have $4 \times 8^0 + 66 = 14 \times 5$.

Assume true for $n=k$, i.e. $4 \times 8^k + 66 = 14M$ for some integer M , $4 \times 8^k = 14M - 66$

Then for $n=k+1$ we get

$$\underline{4 \times 8^{k+1}} + 66 = 8 \times (14M - 66) + 66$$

$$= 8 \times 14M - 7 \times 66$$

$$= 14(8M - 33), \text{ which is divisible by } 14.$$

The assertion holds for $n=0$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

9. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 8^n - 2^n$$

is divisible by 6

(6)

For $n=1$ we have $f(1) = 8^1 - 2^1 = 6$.

Assume the assertion holds for $n=k$, i.e.

$$f(k) = 8^k - 2^k = 6M \Rightarrow \underline{8^k = 6M + 2^k}$$

Then for $n=k+1$ we have

$$\begin{aligned} f(k+1) &= 8^{k+1} - 2^{k+1} = 8 \times \underline{8^k} - 2 \times 2^k \\ &= 8 \times (6M + 2^k) - 2 \times 2^k \\ &= 8 \times 6 \times M + 8 \times 2^k - 2 \times 2^k \\ &= 6 \times (8M + 2^k). \end{aligned}$$

thus divisible by 6.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

10. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 2^{2n-1} + 3^{2n-1} \text{ is divisible by 5.}$$

(6)

For $n=1$ we have $f(1) = 2^{2 \times 1 - 1} + 3^{2 \times 1 - 1} = 5$.

Assume true for $n=k$: $2^{2k-1} + 3^{2k-1} = 5M$ for some integer M .

$$2^{2k-1} = 5M - 3^{2k-1}$$

Then for $n=k+1$ we have:

$$\begin{aligned} 2^{2k-1+2} + 3^{2k-1+2} &= 4(5M - 3^{2k-1}) + 9 \times 3^{2k-1} \\ &= 5 \times 4M - 5 \times 3^{2k-1} \\ &= 5(4M - 3^{2k-1}) \end{aligned}$$

which is divisible by 5.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

11 Prove that $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$.

[5]

For $n=1$ we get $2^{3 \times 1} - 3^1 = 5$.

Assume true for $n=k$, i.e. $2^{3k} - 3^k = 5M$ for some integer M

$$\underline{2^{3k} = 5M + 3^k}$$

Then for $n=k+1$ we get

$$\begin{aligned} 2^{3k+3} - 3^{k+1} &= 8 \times (5M + 3^k) - 3 \times 3^k \\ &= 5 \times 8M + 5 \times 3^k = 5(8M + 3^k) \end{aligned}$$

which is divisible by 5.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

12 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n .

[7]

For $n=1$ we have $5^1 + 2 \times 11^1 = 27 = 3^3$.

Assume true for $n=k$: $5^k + 2 \times 11^k = 3M$

$$5^k = 3M - 2 \times 11^k$$

$$\begin{aligned} \text{Then for } n=k+1: 5^{k+1} + 2 \times 11^{k+1} &= 5 \times (3M - 2 \times 11^k) + 2 \times 11 \times 11^k \\ &= 3 \times 5M + 3 \times 4 \times 11^k \\ &= 3(5M + 4 \times 11^k) \end{aligned}$$

which is divisible by 3.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

Exercise:

Prove the assertion using modular arithmetic, avoiding induction.

13 Prove by mathematical induction that $8^n - 3^n$ is divisible by 5 for all positive integers n . [5]

For $n=1$ we have $8^1 - 3^1 = 5$.

Assume true for $n=k$: $8^k - 3^k = 5M$ for some integer M

$$8^k = 5M + 3^k$$

Then for $n=k+1$ we have:

$$8^{k+1} - 3^{k+1} = 8 \times (5M + 3^k) - 3 \times 3^k$$

$$= 5 \times 8M + 5 \times 3^k$$

$$= 5(8M + 3^k)$$

which is divisible by 5.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

14. Prove by induction that for all positive integers n ,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17

(6)

For $n=1$ we have $f(1) = 2^{3 \times 1 + 1} + 3 \times 5^{2 \times 1 + 1}$
 $= 391 = 17 \times 23$

Assume holds true for $n=k$, i.e.

$$f(k) = 17M \text{ for some integer } M.$$

$$2^{3k+1} = 17M - 3 \times 5^{2k+1}$$

Then for $n=k+1$ we get

$$2^{3(k+1)+1} + 3 \times 5^{2(k+1)+1} = 2^{3k+1+3} + 3 \times 5^{2k+1+2}$$

$$= 8 \times 2^{3k+1} + 3 \times 25 \times 5^{2k+1}$$

$$= 8 \times (17M - 3 \times 5^{2k+1}) + 3 \times 25 \times 5^{2k+1}$$

$$= 8 \times 17M - 3 \times 8 \times 5^{2k+1} + 3 \times 25 \times 5^{2k+1}$$

$$= 8 \times 17M + 3 \times 17 \times 5^{2k+1}$$

$$= 17 \times (8M + 3 \times 5^{2k+1}) \text{ so divisible by } 17$$

The assertion holds for $n=1$, and if true for $n=k$ **
it holds for $n=k+1$. Hence, by induction, the
assertion holds for all positive integers.

15. (ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

For $n=1$, we have $f(1) = 4^{1+1} + 5^{2 \times 1 - 1} = 21$
Assume the assertion holds for $n=k$, i.e.

$$f(k) = 4^{k+1} + 5^{2k-1} = 21M$$

for some integer M . $4^{k+1} = 21M - 5^{2k-1}$

Then for $n=k+1$ we get

$$\begin{aligned} 4^{k+1+1} + 5^{2(k+1)-1} &= 4 \times 4^{k+1} + 5^{2k-1+2} \\ &= 4 \times \underline{4^{k+1}} + 25 \times 5^{2k-1} \\ &= 4(21M - 5^{2k-1}) + 25 \times 5^{2k-1} \\ &= 21 \times 4 \times M - 4 \times 5^{2k-1} + 25 \times 5^{2k-1} \\ &= 21(4M + 5^{2k-1}) \end{aligned}$$

showing $f(k+1)$ is divisible by 21.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

16. (ii) Prove by induction that, for all positive integers n ,

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(5)

For $n=1$, we have $f(1) = 3^{2 \times 1 + 4} - 2^{2 \times 1} = 5 \times 145$
 thus divisible by 5.

I assume the assertion holds true for $n=k$, i.e.,

$$f(k) = 3^{2k+4} - 2^{2k} = 5M \text{ for some integer } M.$$

$$\underline{3^{2k+4} = 5M + 2^{2k}}$$

Then for $n=k+1$ we have

$$f(k+1) = 3^{2k+4+2} - 2^{2k+2}$$

$$= 9 \times \underline{3^{2k+4}} - 4 \times 2^{2k}$$

$$= 9 \times (5M + 2^{2k}) - 4 \times 2^{2k}$$

$$= 9 \times 5M + 5 \times 2^{2k}$$

$$= 5(9M + 2^{2k}), \text{ thus divisible by } 5.$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

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For students who are taking further pure II/ad-pure. **

Exercise: Using modular arithmetic, show that $3^{2n+4} - 2^{2n}$ is divisible by 5 for all positive integers n .

Solution:

$2 \equiv -3 \pmod{5}$, hence

$$f(n) \equiv 3^{2n+4} - (-3)^{2n} \equiv 3^4 \times 3^{2n} - 3^{2n} \pmod{5}$$

* Fermat's little theorem implies $3^4 \equiv 1 \pmod{5}$

$$3^{2n} - 3^{2n} \equiv 0 \pmod{5}.$$

17. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

(6)

For $n=1$ we have $f(1) = 2^{1+2} + 3^{2 \times 1 + 1} = 7 \times 5$
thus divisible by 7.

Assume the assertion holds true for $n=k$, i.e.

$$f(k) = 2^{k+2} + 3^{2k+1} = 7M \text{ for some integer } M.$$

$$\Rightarrow \underline{2^{k+2} = 7M - 3^{2k+1}}$$

Then for $n=k+1$ we get

$$\begin{aligned} f(k+1) &= 2^{k+2+1} + 3^{2k+1+2} \\ &= \underline{2 \times 2^{k+2}} + 9 \times 3^{2k+1} \\ &= 2(7M - 3^{2k+1}) + 9 \times 3^{2k+1} \\ &= 7 \times 2 \times M - 2 \times 3^{2k+1} + 9 \times 3^{2k+1} \\ &= 7(2M + 3^{2k+1}) \end{aligned}$$

thus divisible by 7.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers. **

18. Prove by induction that $2^{n+1} + 5 \times 9^n$ is divisible by 7 for all integers $n \geq 1$.

[6]

For $n=1$ we have $2^{1+1} + 5 \times 9^1 = 7^2$.

Assume true for $n=k$, i.e. $2^{k+1} + 5 \times 9^k = 7M$ for some integer M

$$\underline{2^{k+1} = 7M - 5 \times 9^k}$$

Then for $n=k+1$ we have

$$2^{(k+1)+1} + 5 \times 9^{k+1} = 2 \times (7M - 5 \times 9^k) + 5 \times 9 \times 9^k$$

$$= 7 \times 2M + 5 \times 7 \times 9^k$$

$$= 7(2M + 5 \times 9^k), \text{ which is divisible by } 7.$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

19. (ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 8^{2n+1} + 6^{2n-1}$$

is divisible by 7

(5)

For $n=1$ we have $f(1) = 8^{2 \times 1 + 1} + 6^{2 \times 1 - 1} = 7 \times 74$

Assume true for $n=k$, i.e. $f(k) = 8^{2k+1} + 6^{2k-1} = 7M$
for some integer M . Let $8^{2k+1} = 7M - 6^{2k-1}$

Then for $n=k+1$ we get

$$\begin{aligned} f(k+1) &= 8^{2(k+1)+1} + 6^{2(k+1)-1} \\ &= 64(7M - 6^{2k-1}) + 36 \times 6^{2k-1} \\ &= 7 \times 64M - 28 \times 6^{2k-1} \\ &= 7(64M - 4 \times 6^{2k-1}) \end{aligned}$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

20 Prove by induction that $11 \times 7^n - 13^n - 1$ is divisible by 3, for all integers $n \geq 0$.

[5]

For $n=0$ we have $11 \times 7^0 - 13^0 - 1 = -3$

Assume true for $n=k$, i.e. $11 \times 7^k - 13^k - 1 = 3M$

$$11 \times 7^k = 3M + 13^k + 1$$

Then for $n=k+1$ we have

$$\begin{aligned} 11 \times 7^{k+1} - 13^{k+1} - 1 &= 7 \times (3M + 13^k + 1) - 13 \times 13^k - 1 \\ &= 3 \times 7M + 6 \times 13^k + 6 \end{aligned}$$

$$= 3(7M + 2 \times 13^k + 2)$$

The assertion holds for $n=0$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

21. Prove by induction that, for $n \in \mathbb{Z}^+$,

(a) $f(n) = 5^n + 8n + 3$ is divisible by 4,

(7)

For $n=1$ we have $f(1) = 5 + 8 \times 1 + 3 = 16 = 4 \times 4$
thus divisible by 4.

Assume assertion holds true for $n=k$, i.e.

$$f(k) = 5^k + 8k + 3 = 4M.$$

$$5^k = 4M - 8k - 3.$$

Then for $n=k+1$ we have

$$f(k+1) = 5^{k+1} + 8(k+1) + 3$$

$$= 5 \times \underline{5^k} + 8k + 8 + 3$$

$$= 5(4M - 8k - 3) + 8k + 11$$

$$= 4 \times 5 \times M - 4 \times 8k - 4$$

$$= 4(5 \times M - 8k - 1)$$

hence $f(k+1)$ is divisible by 4.

The assertion holds for $n=1$, and if true for $n=k$
it holds for $n=k+1$. Hence, by induction, the
assertion holds for all positive integers.

22.

$$f(n) = 2^n + 6^n$$

(a) Show that $f(k+1) = 6f(k) - 4(2^k)$.

(3)

(b) Hence, or otherwise, prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 8.

(4)

$$\begin{aligned} \text{a)} \quad f(k+1) &= 2^{k+1} + 6^{k+1} \\ &= 6 \times 2^k - 4 \times 2^k + 6 \times 6^k \\ &= 6 \times (2^k + 6^k) - 4 \times 2^k \\ &= 6f(k) - 4 \times 2^k. \end{aligned}$$

b) For $n=1$, $f(1) = 8$, which is divisible by 8. $\checkmark$

Assume the assertion holds true for $n=k$, i.e.

$f(k) = 8M$ for some integer M , then for $n=k+1$

we have $f(k+1) = 6 \times 8M - 4 \times 2^k$ (by (i)).

since $k \geq 1$, 2^{k-1} is an integer $f(k) = 8(6M - 2^{k-1})$

thus showing that assertion holds for $k+1$.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

23. (ii) Prove by induction that, for $n \in \mathbb{N}$

$$f(n) = 5^{n+2} - 4n - 9$$

is divisible by 16

(5)

For $n=1$ we have $f(1) = 5^{1+2} - 4 \times 1 - 9 = 112 = 16 \times 7$.

Suppose the assertion holds true for $n=k$, i.e.

$$f(k) = 5^{k+2} - 4 \times k - 9 = 16M \text{ for some integer } M$$

$$\Rightarrow 5^{k+2} = 16M + 4k + 9$$

Then for $n=k+1$ we have

$$f(k+1) = 5^{k+2+1} - 4(k+1) - 9$$

$$= 5 \times 5^{k+2} - 4(k+1) - 9$$

$$= 5(16M + 4k + 9) - 4k - 13$$

$$= 16 \times 5 \times M + 16k + 32$$

$$= 16(5 \times M + k + 2) \quad \text{which is divisible by 16.}$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

24. Prove, by induction, that for $n \in \mathbb{Z}, n \geq 2$

$$4^n + 6n - 10$$

is divisible by 18

(5)

For $n=2$ we have $4^2 + 6 \times 2 - 10 = 18$

Assume the assertion holds for $n=k$, i.e.

$$4^k + 6 \times k - 10 = 18M \text{ for some integer } M$$

$$\Rightarrow \underline{4^k = 18M - 6k + 10}.$$

Then for $n=k+1$ we get

$$\begin{aligned} 4^{k+1} + 6(k+1) - 10 &= 4 \times \underline{4^k} + 6k - 4 \\ &= 4(18M - 6k + 10) + 6k - 4 \\ &= 18 \times 4 \times M - 6 \times 3 \times k - 18 \times 2 \\ &= 18(4M - 3k - 2) \end{aligned}$$

which is a multiple of 18.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

25. (ii) Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 5^{2n} + 3n - 1$$

is divisible by 9

(6)

For $n=1$ we have $f(1) = 5^{2 \times 1} + 3 \times 1 - 1 = 9 \times 3$
which is divisible by 9.

Assume true for $n=k$, i.e. $f(k) = 5^{2k} + 3 \times k - 1 = 9 \times M$
for some integer M . $\Rightarrow \underline{5^{2k} = 9M - 3k + 1}$

Then for $n=k+1$ we have

$$f(k+1) = 5^{2k+2} + 3 \times (k+1) - 1 = 25 \times \underline{5^{2k}} + 3k + 2$$

$$= 25(9M - 3k + 1) + 3k + 2$$

$$= 9 \times 25M + (-75 + 3)k + 27$$

$$= 9(25M - 8k + 3) \text{ which is divisible by } 9.$$

The assertion holds for $n=1$, and if true for $n=k$
it holds for $n=k+1$. Hence, by induction, the
assertion holds for all positive integers.

26. (ii) Prove by induction that, for all positive integers n ,

$$f(n) = 3^{2n+3} + 40n - 27 \text{ is divisible by } 64$$

(6)

For $n=1$ we have $f(1) = 3^{2 \times 1 + 3} + 40 \times 1 - 27 = 64 \times 4$
which is divisible by 64.

Assume true for $n=k$, i.e. $f(k) = 3^{2k+3} + 40k - 27 = 64M$
for some integer M . $\Rightarrow 3^{2k+3} = 64M - 40k + 27$.

Then for $n=k+1$ we have

$$f(k+1) = 3^{2k+3+2} + 40(k+1) - 27$$

$$= 9 \times 3^{2k+3} + 40k + 13$$

$$= 9 \times (64M - 40k + 27) + 40k + 13$$

$$= 64 \times 9M - 8 \times 8 \times 5k + 64 \times 4$$

$$= 64(9M - 5k + 4) \quad \text{which is divisible by } 64.$$

The assertion holds for $n=1$, and if true for $n=k$
it holds for $n=k+1$. Hence, by induction, the
assertion holds for all positive integers.

27. (ii) Prove by induction that,

$$4^n + 6n + 8 \text{ is divisible by } 18$$

for all positive integers n .

(6)

For $n=1$ we have $4^1 + 6 \times 1 + 8 = 18$.

Assume true for $n=k$, i.e. $4^k + 6k + 8 = 18M$ for some integer M . $\Rightarrow 4^k = 18M - 6k - 8$.

Then for $n=k+1$ we get

$$4^{k+1} + 6(k+1) + 8 = 4 \times 4^k + 6k + 14$$

$$= 4(18M - 6k - 8) + 6k + 14$$

$$= 18 \times 4M - 18k - 18$$

$$= 18(4M - k - 1) \quad \text{which is divisible by } 18.$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

28 It is given that $u_n = 13^n + 6^{n-1}$, where n is a positive integer.

(i) Show that $u_n + u_{n+1} = 14 \times 13^n + 7 \times 6^{n-1}$. [3]

(ii) Prove by induction that u_n is a multiple of 7. [4]

$$\begin{aligned} \text{i)} \quad u_n + u_{n+1} &= 13^n + 6^{n-1} + 13^{n+1} + 6^{n+1-1} \\ &= 14 \times 13^n + 7 \times 6^{n-1} \end{aligned}$$

$$\text{ii)} \quad \text{For } n=1 \text{ we have } u_1 = 13^1 + 6^0 = 14 = 7 \times 2.$$

Assume true for $n=k$, i.e. $u_k = 7M$ for some positive integer M .

Then for $n=k+1$ we have

$$\begin{aligned} u_{k+1} &= 7(2 \times 13^k + 6^{k-1}) - u_k \\ &= 7(2 \times 13^k + 6^{k-1} - M) \quad \text{which is divisible by } 7. \end{aligned}$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

29 The sequence $u_1, u_2, u_3, \dots$ is defined by $u_n = 5^n + 2^{n-1}$.

(i) Find u_1, u_2 and u_3 . [2]

(ii) Hence suggest a positive integer, other than 1, which divides exactly into every term of the sequence. [1]

(iii) By considering $u_{n+1} + u_n$, prove by induction that your suggestion in part (ii) is correct. [5]

$$\text{i)} \quad u_1 = 5^1 + 2^0 = 6, \quad u_2 = 5^2 + 2^{2-1} = 27, \quad u_3 = 5^3 + 2^{3-1} = 129. \\ = 3 \times 43.$$

$$\text{ii)} \quad 3$$

$$\text{iii)} \quad u_{n+1} + u_n = 5^{n+1} + 2^{n+1-1} + 5^n + 2^{n-1} \\ = 6 \times 5^n + 3 \times 2^{n-1}$$

For $n=1$ part i) shows this.

Assume true for $n=k$: $u_k = 3M$ for some integer M .

$$\text{Then for } n=k+1: \quad u_{k+1} = 6 \times 5^k + 3 \times 2^{k-1} - u_k \\ = 3(2 \times 5^k + 2^{k-1} - M)$$

which is divisible by 3.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

30 Prove by induction that the sum of the cubes of three consecutive positive integers is divisible by 9. [5]

For $n=1$ we have $1^3 + 2^3 + 3^3 = 9 \times 4$

Assume true for $n=k$, i.e. $k^3 + (k+1)^3 + (k+2)^3 = 9M$
 for some integer M , $(k+1)^3 = 9M - k^3 - (k+2)^3$

Then for $n=k+1$ we get

$$\begin{aligned} \underline{(k+1)^3 + (k+2)^3 + (k+3)^3} &= 9M - k^3 - \cancel{(k+2)^3} + \cancel{(k+2)^3} + (k+3)^3 \\ &= 9M + \cancel{k^3} + 3 \times 3 \times k^2 + 3 \times 9 \times k + 27 - \cancel{k^3} \\ &= 9(M + k^2 + 3k + 3) \end{aligned}$$

, hence divisible by 9.

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

31. (i)

$$f(n) = 7^n(3n+1) - 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is a multiple of 9

(6)

$$\begin{aligned} \text{For } n=1, \quad f(1) &= 7^1(3 \times 1 + 1) - 1 \\ &= 27 \\ &= 9 \times 3 \end{aligned}$$

proving the case for $n=1$.

Assume true for $n=k$, i.e.

$$f(k) = 7^k(3k+1) - 1 = 9M$$

for some integer M . Rearrange

$$7^k(3k+1) = 9M + 1$$

Then for $n=k+1$ we have

$$7^{k+1}(3(k+1)+1) - 1 =$$

$$= 7 \cdot 7^k (3k+4) - 1$$

$$= 7 \cdot 7^k (3k+1) + 21 \cdot 7^k - 1$$

$$= 7 \times (9M+1) + 21 \cdot 7^k (3k+1) - 3k \times 21 \cdot 7^k - 1$$

$$= 7 \times 9M + 7 + 21 \times (9M+1) - 9k \times 7 \times 7^k - 1$$

$$= 7 \times 9M + 9 \times 21M - 9k \times 7^{k+1} + 9 \times 3$$

The assertion holds for $n=1$, and if true for $n=k$ it holds for $n=k+1$. Hence, by induction, the assertion holds for all positive integers.

Alternative method:

Using $7^k(3k+1) - 1 = 9M$ for
 $n = k+1$ we have:

$$7^{k+1}(3(k+1)+1) - 1 =$$

$$= 7^k \times 7(3k+4) - 1$$

$$= 7^k(21k+28) - 1$$

$$= 7^k(3k+1 + 18k+27) - 1$$

$$= 7^k(3k+1) - 1 + 9 \times 7^k(2k+3)$$

$$= 9M + 9 \times 7^k(2k+3)$$

Also showing divisibility by 9.