London Academy of Maths

Proof by Induction - Matrices

Question Paper

 **

1 Prove by mathematical induction that $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 1-2^n & 1 \end{pmatrix}$ for all positive integers n. **[5]**

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2. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

(6)

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3. Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^n = \begin{pmatrix} 6n+1 & -12n \\ 3n & 1-6n \end{pmatrix}$$

(6)

4. Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ \dfrac{a^n - b^n}{a - b} & b^n \end{pmatrix}$$

where a and b are constants and $a \neq b$.

(5)

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5 Prove by induction that $\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^n = \begin{pmatrix}1 & 2^n - 1\\0 & 2^n\end{pmatrix}$ for all positive integers n. **[6]**

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6. Prove by induction, that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}^n = \begin{pmatrix} 1-4n & -2n \\ 8n & 1+4n \end{pmatrix}$$

(5)

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7. Prove by induction that, for $n \in \mathbb{Z}^+$,

(b) $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$

(7)

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8. Prove by induction, that for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$,

(6)

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9. Prove by induction that, for $m \in \mathbb{Z}^+$,

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^m = \begin{pmatrix} 2m+1 & -4m \\ m & 1-2m \end{pmatrix}$$

(5)

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10. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}\left(5^n - 1\right) & 5^n \end{pmatrix}$$

(6)

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11. Prove by induction that for $n \in \mathbb{N}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$$

(5)

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13 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geqslant 1$,

$$\mathbf{M}^n = \begin{pmatrix} 2^n & 2^{n+1} - 2 \\ 0 & 1 \end{pmatrix}.$$

[6]

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14. Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 3n+1 \end{pmatrix}$$

(5)

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15. Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & (2^n - 1)r \\ 0 & 2^n \end{pmatrix}$$

where r is a constant.

(4)

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16 **In this question you must show detailed reasoning.**

$\mathbf{M}$ is the matrix $\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$.

Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & 3(2^{n+1}-2) \\ 0 & 2^n \end{pmatrix}$, for any positive integer n. **[6]**

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17. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}^n = \begin{pmatrix} 1-6n & 9n \\ -4n & 1+6n \end{pmatrix}$$

(6)

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18. Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^n = 3^{n-1}\begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$$

(5)

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19 **(a)** Specify fully the transformation T of the plane associated with the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and λ is a non-zero constant. **[2]**

(b) **(i)** Find $\det \mathbf{M}$. **[1]**

(ii) Deduce **two** properties of the transformation T from the value of $\det \mathbf{M}$. **[2]**

(c) Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix}$, where n is a positive integer. **[4]**

(d) Hence specify fully a **single** transformation which is equivalent to n applications of the transformation T. **[1]**

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20 You are given that $\mathbf{M}=\begin{pmatrix}4 & -9\\ 1 & -2\end{pmatrix}$.

(a) Prove that $\mathbf{M}^n=\begin{pmatrix}1+3n & -9n\\ n & 1-3n\end{pmatrix}$ for all positive integers n. **[6]**

(b) A student thinks that this formula, when $n=0$ and $n=-1$, gives the identity matrix and the inverse matrix $\mathbf{M}^{-1}$ respectively.

Determine whether the student is correct. **[3]**