

Proof by Induction - Matrices

Worked Solutions

1 Prove by mathematical induction that $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 1-2^n & 1 \end{pmatrix}$ for all positive integers n . [5]

For $n=1$ we have $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2^1 & 0 \\ 1-2^1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$.

Assume true for $n=k$, i.e. $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix}$.

Then for $n=k+1$ we have,

$$\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 0 \\ 2-2^{k+1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 0 \\ 1-2^{k+1} & 1 \end{pmatrix} \text{ thus true for } n=k+1.$$

We showed the assertion for $n=1$ and
that if true for k , then true for $k+1$. Hence,
by mathematical induction we conclude that the
assertion holds for all $n \in \mathbb{Z}^+$.

2. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix} \quad (6)$$

For $n=1$ we have $\begin{pmatrix} 4 \times 1 + 1 & -8 \times 1 \\ 2 \times 1 & 1 - 4 \times 1 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1$

Assume true for $n=k$, i.e. $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$

Then for $n=k+1$ we have

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k \cdot \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix} \cdot \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{bmatrix} 20k+5-16k & -32k-8+24k \\ 10k+2-8k & -16k-3+12k \end{bmatrix}$$

$$= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}$$

We showed the assertion for $n = 1$ and
that if true for k , then true for $k+1$. Hence,
by mathematical induction we conclude that the
assertion holds for all $n \in \mathbb{Z}^+$.

3. Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^n = \begin{pmatrix} 6n+1 & -12n \\ 3n & 1-6n \end{pmatrix} \quad (6)$$

For $n=1$ we have $\begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^1 = \begin{pmatrix} 6 \times 1 + 1 & -12 \times 1 \\ 3 \times 1 & 1 - 6 \times 1 \end{pmatrix}$

Assume true for $n=k$, i.e. $\begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^k = \begin{pmatrix} 6k+1 & -12k \\ 3k & 1-6k \end{pmatrix}$

Then for $n=k+1$ we have

$$\begin{aligned} \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^{k+1} &= \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^k \cdot \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 6k+1 & -12k \\ 3k & 1-6k \end{pmatrix} \cdot \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 42k+6-36k & -72k-12+60k \\ 21k+3-18k & -36k-5+30k \end{pmatrix} \\ &= \begin{pmatrix} 6(k+1) & -12(k+1) \\ 3(k+1) & 1-6(k+1) \end{pmatrix} \end{aligned}$$

We showed the assertion for $n=1$ and
that if true for k , then true for $k+1$. Hence,
by mathematical induction we conclude that the
assertion holds for all $n \in \mathbb{Z}^+$.

4. Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ \frac{a^n - b^n}{a - b} & b^n \end{pmatrix}$$

where a and b are constants and $a \neq b$.

(5)

For $n=1$ we have $\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^1 = \begin{pmatrix} a^1 & 0 \\ \frac{a^1 - b^1}{a - b} & b^1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$.

Assume true for $n=k$, i.e. $\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^k = \begin{pmatrix} a^k & 0 \\ \frac{a^k - b^k}{a - b} & b^k \end{pmatrix}$

Then for $n=k+1$ we get

$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^{k+1} = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^k \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^k & 0 \\ \frac{a^k - b^k}{a - b} & b^k \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \frac{a^k - ab^k}{a - b} + b^k & b^{k+1} \end{pmatrix} = \begin{pmatrix} a^{k+1} & 0 \\ \frac{a^{k+1} + ab^k - ab^k - b^{k+1}}{a - b} & b^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \frac{a^{k+1} - b^{k+1}}{a - b} & b^{k+1} \end{pmatrix}$$

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

5 Prove by induction that $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$ for all positive integers n .

[6]

$$\text{For } n=1, \quad \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 2^1 - 1 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\text{Assume true for } n=k, \text{ i.e. } \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$$

Then for $n=k+1$ we have

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 + 2^k - 2 \\ 0 & 2^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^{k+1} - 1 \\ 0 & 2^{k+1} \end{pmatrix} \end{aligned}$$

thus holds for $n=k+1$.

We showed the assertion for $n=1$ and
that if true for k , then true for $k+1$. Hence,
by mathematical induction we conclude that the
assertion holds for all $n \in \mathbb{Z}^+$.

6. Prove by induction, that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}^n = \begin{pmatrix} 1-4n & -2n \\ 8n & 1+4n \end{pmatrix} \quad (5)$$

For $n=1$, we have $\begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}^1 = \begin{pmatrix} 1-4 \times 1 & -2 \times 1 \\ 8 \times 1 & 1+4 \times 1 \end{pmatrix}$

Assume true for $n=k$, then for $n=k+1$ we have

$$\begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}^k \cdot \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}$$

$$= \begin{bmatrix} 1-4 \times k & -2k \\ 8k & 1+4k \end{bmatrix} \cdot \begin{bmatrix} -3 & -2 \\ 8 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3+12k-16k & -2+8k-10k \\ -24k+8+32k & -16k+5+20k \end{bmatrix}$$

$$= \begin{bmatrix} -3-4k & -2-2k \\ 8k+8 & 5+4k \end{bmatrix}$$

$$= \begin{bmatrix} 1-4(k+1) & -2(k+1) \\ 8(k+1)+8 & 1+4(k+1) \end{bmatrix}$$

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

7. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$(b) \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix} \quad (7)$$

$$\text{For } n=1: \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^1 = \begin{pmatrix} 2 \times 1 + 1 & -2 \times 1 \\ 2 \times 1 & 1 - 2 \times 1 \end{pmatrix} \\ = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}.$$

Assume true for $n=k$, then for $n=k+1$ we have

$$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^k \cdot \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6k+3-4k & -4k-2+2k \\ 6k+2-4k & -4k-1+4k \end{pmatrix}$$

$$= \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix} \quad \text{thus holds for } n=k+1.$$

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

8. Prove by induction, that for $n \in \mathbb{Z}^+$,

$$(a) \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix},$$

(6)

For $n=1$ we have: $\begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}^1 = \begin{bmatrix} 3^1 & 0 \\ 3 \times (3^1 - 1) & 1 \end{bmatrix}$

Assume true for $n=k$, then for $n=k+1$ we have

$$\begin{aligned} \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}^{k+1} &= \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}^k \cdot \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3^k & 0 \\ 3 - (3^k - 1) & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3^{k+1} & 0 \\ 3^{k+2} - 9 + 6 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{bmatrix}. \end{aligned}$$

Thus holds for $n=k+1$.

We showed the assertion for $n=1$ and
that if true for k , then true for $k+1$. Hence,
by mathematical induction we conclude that the
assertion holds for all $n \in \mathbb{Z}^+$.

9. Prove by induction that, for $m \in \mathbb{Z}^+$,

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^m = \begin{pmatrix} 2m+1 & -4m \\ m & 1-2m \end{pmatrix} \quad (5)$$

For $n=1$: $\begin{pmatrix} 2 \times 1 + 1 & -4 \times 1 \\ 1 & 1 - 2 \times 1 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^1$

Assume true for $n=k$, then for $n=k+1$ we have.

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2k+1 & -4k \\ k & 1-2k \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6k+3-4k & -8k-4+4k \\ 3k+1-2k & -4k-1+2k \end{pmatrix}$$

$$= \begin{pmatrix} 2(k+1)+1 & -4(k+1) \\ k+1 & 1-2(k+1) \end{pmatrix}$$

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

10. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}$$

(6)

For $n=1$: $\begin{bmatrix} 1 & 0 \\ -\frac{1}{4}(5^1 - 1) & 5^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}^1$

Suppose true for $n=k$, then for $n=k+1$ we have

$$\begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}^k \cdot \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{4}(5^k - 1) & 5^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{4}5^k + \frac{1}{4} - 5^k & 5^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1} - 1) & 5^{k+1} \end{bmatrix}$$

thus holding for $n=k+1$.

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

11. Prove by induction that for $n \in \mathbb{N}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$$

(5)

For $n=1$ we have $\begin{pmatrix} 1 & -2 \times 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^1$

Assume true for $n=k$, then for $n=k+1$ we have

$$\begin{aligned} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^k \cdot \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2-2k \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2(k+1) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

thus holds for $n=k+1$.

We showed the assertion for $n = 1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{N}$.

12 The matrix A is given by $A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geq 1$,

$$A^n = \begin{pmatrix} 3^n & \frac{1}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}. \quad [6]$$

For $n=1$: $A^1 = \begin{pmatrix} 3^1 & \frac{1}{2}(3^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$

Assume true for $n=k$, then for $n=k+1$ we have

$$A^{k+1} = A^k \cdot A = \begin{pmatrix} 3^k & \frac{1}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & 3^k + \frac{1}{2}3^k - \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & \frac{1}{2}(3^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$$

Thus holds for $n=k+1$.

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

- 13 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geq 1$,

$$\mathbf{M}^n = \begin{pmatrix} 2^n & 2^{n+1} - 2 \\ 0 & 1 \end{pmatrix}.$$

[6]

For $n=1$: $\begin{pmatrix} 2^1 & 2^{1+1}-2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \mathbf{M}^1$

Assume true for $n=k$, then for $n=k+1$ we have

$$\mathbf{M}^{k+1} = \mathbf{M}^k \cdot \mathbf{M} = \begin{pmatrix} 2^k & 2^{k+1}-2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 2^{k+1} + 2^{k+1} - 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 2^{k+2} - 2 \\ 0 & 1 \end{pmatrix}$$

thus holds for $n=k+1$,

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

14. Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 3n+1 \end{pmatrix} \quad (5)$$

For $n=1$: $\begin{pmatrix} 1-3 \times 1 & 9 \times 1 \\ -1 & 3 \times 1 + 1 \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^1$

Assume true for $n=k$, then for $n=k+1$ we have

$$\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^k \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1-3k & 9k \\ -k & 3k+1 \end{pmatrix} \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2+6k-9k & 9-27k+36k \\ +2k-3k-1 & -9k+12k+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1-3(k+1) & 9(k+1) \\ -(k+1) & 3(k+1)+1 \end{pmatrix}$$

thus holds for $n=k+1$.

We showed the assertion for $n = 1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

15. Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & (2^n - 1)r \\ 0 & 2^n \end{pmatrix}$$

where r is a constant.

(4)

For $n=1$: $\begin{pmatrix} 1 & (2^1 - 1)r \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}^1$

Assume true for $n=k$, then for $n=k+1$ we have

$$\begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}^k \begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (2^k - 1)r \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & r + 2^{k+1}r - 2r \\ 0 & 2^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (2^{k+1} - 1)r \\ 0 & 2^{k+1} \end{pmatrix}$$

thus it holds true.

We showed the assertion for $n=1$ and
that if true for k , then true for $k+1$. Hence,
by mathematical induction we conclude that the
assertion holds for all $n \in \mathbb{Z}^+$.

16 In this question you must show detailed reasoning.

$\mathbf{M}$ is the matrix $\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$.

Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & 3(2^{n+1}-2) \\ 0 & 2^n \end{pmatrix}$, for any positive integer n .

[6]

$$\text{For } n=1: \begin{pmatrix} 1 & 3(2^{1+1}-2) \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix} = \mathbf{M}^1.$$

Assume true for $n=k$, then for $n=k+1$ we get

$$\begin{aligned} \mathbf{M}^{k+1} &= \mathbf{M}^k \cdot \mathbf{M} = \begin{pmatrix} 1 & 3(2^{k+1}-2) \\ 0 & 2^k \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 6 + 6 \times 2^{k+1} - 12 \\ 0 & 2^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3(2^{k+1}-2) \\ 0 & 2^{k+1} \end{pmatrix} \end{aligned}$$

thus holds for $n=k+1$.

We showed the assertion for $n=1$ and
that if true for k , then true for $k+1$. Hence,
by mathematical induction we conclude that the
assertion holds for all $n \in \mathbb{Z}^+$.

17. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}^n = \begin{pmatrix} 1-6n & 9n \\ -4n & 1+6n \end{pmatrix}$$

(6)

For $n=1$:

$$\begin{pmatrix} 1-6 \times 1 & 9 \times 1 \\ -4 \times 1 & 1+6 \times 1 \end{pmatrix} = \begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}^1$$

Assume true for $n=k$, then for $n=k+1$ we have

$$\begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}^{k+1} = \begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}^k \cdot \begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6k & 9k \\ -4k & 1+6k \end{pmatrix} \cdot \begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}$$

$$= \begin{bmatrix} -5 + 30k - 36k & 9 - 9 \times 6k + 7 \times 9k \\ + 5 \times 4k - 4 - 6 \times 4k & -9 \times 4k + 7 + 7 \times 6k \end{bmatrix}$$

$$= \begin{pmatrix} 1-6(k+1) & 9(k+1) \\ -4(k+1) & 1+6(k+1) \end{pmatrix}$$

We showed the assertion for $n=1$ and
that if true for k , then true for $k+1$. Hence,
by mathematical induction we conclude that the
assertion holds for all $n \in \mathbb{Z}^+$.

18. Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^n = 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix} \quad (5)$$

For $n=1$: $3^{1-1} \begin{pmatrix} 2 \times 1 + 3 & -1 \\ 4 \times 1 & 3 - 2 \times 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^1$

Assume true for $n=k$, then for $n=k+1$ we have

$$\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^k \cdot \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} = 3^{k-1} \begin{pmatrix} 2k+3 & -k \\ 4k & 3-2k \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$$

$$= 3^{k-1} \begin{pmatrix} 10k+15-4k & -2k+3-k \\ 20k+12-8k & -4k+3-2k \end{pmatrix}$$

$$= 3^{k-1} \begin{pmatrix} 3(2(k+1)+3) & -3(k+1) \\ 3 \times 4 \times (k+1) & 3 \times (3-2(k+1)) \end{pmatrix}$$

$$= 3^{(k+1)-1} \begin{pmatrix} 2(k+1)+3 & -(k+1) \\ 4(k+1) & 3-2(k+1) \end{pmatrix}$$

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

- 19 (a) Specify fully the transformation T of the plane associated with the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ and } \lambda \text{ is a non-zero constant.} \quad [2]$$

- (b) (i) Find $\det \mathbf{M}$. [1]

- (ii) Deduce **two** properties of the transformation T from the value of $\det \mathbf{M}$. [2]

- (c) Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix}$, where n is a positive integer. [4]

- (d) Hence specify fully a **single** transformation which is equivalent to n applications of the transformation T . [1]

a) Shear with x -axis fixed and $(0, 1) \mapsto (\lambda, 1)$.

b) i) $\begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix} = 1$

ii) $|\det(A)| = 1 \Rightarrow$ preserves area.

$\det(A) \geq 0 \Rightarrow$ preserves orientation.

c) We proceed by induction.

For $n=1$ we have $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & \lambda \times 1 \\ 0 & 1 \end{pmatrix}$

Assume true for $n=k$, i.e.

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k\lambda \\ 0 & 1 \end{pmatrix}.$$

Then for $n = k+1$ we have

$$\begin{aligned} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & k\lambda \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & (k+1)\lambda \\ 0 & 1 \end{pmatrix} \end{aligned}$$

hence true for $n = k+1$.

We showed the assertion for $n = 1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

d) A shear with x -axis fixed and

$$(0, 1) \longmapsto (n\lambda, 1),$$

20 You are given that $\mathbf{M} = \begin{pmatrix} 4 & -9 \\ 1 & -2 \end{pmatrix}$.

(a) Prove that $\mathbf{M}^n = \begin{pmatrix} 1+3n & -9n \\ n & 1-3n \end{pmatrix}$ for all positive integers n . [6]

(b) A student thinks that this formula, when $n = 0$ and $n = -1$, gives the identity matrix and the inverse matrix $\mathbf{M}^{-1}$ respectively.

Determine whether the student is correct.

[3]

$$a) \text{ For } n=1 : \mathbf{M}^1 = \begin{pmatrix} 4 & -9 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1+3 & -9 \times 1 \\ 1 & 1-3 \end{pmatrix}$$

$$\text{Assume true for } n=k, \text{ i.e. } \mathbf{M}^k = \begin{pmatrix} 1+3k & -9k \\ k & 1-3k \end{pmatrix}$$

Then for $n=k+1$ we have

$$\mathbf{M}^{k+1} = \mathbf{M}^k \cdot \mathbf{M}$$

$$= \begin{pmatrix} 1+3k & -9k \\ k & 1-3k \end{pmatrix} \cdot \begin{pmatrix} 4 & -9 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+12k-9k & -9-27k+18k \\ 4k+1-3k & -9k-2+6k \end{pmatrix}$$

$$= \begin{pmatrix} 1+3(k+1) & -9(k+1) \\ k+1 & 1-3(k+1) \end{pmatrix}$$

We showed the assertion for $n=1$ and that if true for k , then true for $k+1$. Hence, by mathematical induction we conclude that the assertion holds for all $n \in \mathbb{Z}^+$.

$$b) \begin{bmatrix} 1+3 \times 0 & 0 \\ 0 & 1-3 \times 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\det(M) = -8 + 9 = 1 \Rightarrow M^{-1} = \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$$

$$\text{Formula gives: } \begin{bmatrix} 1-3 \times 1 & -9 \times (-1) \\ 1 \times (-1) & 1+3 \times 1 \end{bmatrix} = \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$$

The student is correct.