London Academy of Maths

Trigonometric Identities and Equations

Question Paper

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1 **In this question you must show detailed reasoning.**

Solve the equation $3\sin^4\phi + \sin^2\phi = 4$, for $0 \leqslant \phi < 2\pi$, where ϕ is measured in radians. **[5]**

2 The acute angles α and β are such that

$$2\cot\alpha = 1 \quad \text{and} \quad 24 + \sec^2\beta = 10\tan\beta.$$

(i) State the value of $\tan\alpha$ and determine the value of $\tan\beta$. **[4]**

(ii) Hence find the exact value of $\tan(\alpha + \beta)$. **[3]**

3 Prove that $\sqrt{2}\cos(2\theta + 45°) \equiv \cos^2\theta - 2\sin\theta\cos\theta - \sin^2\theta$, where θ is measured in degrees. **[3]**

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4 **(a)** Given that $7 \sin 2\alpha = 3 \sin \alpha$, where $0^\circ < \alpha < 90^\circ$, find the exact value of $\cos \alpha$. **[3]**

(b) Given that $3 \cos 2\beta + 19 \cos \beta + 13 = 0$, where $90^\circ < \beta < 180^\circ$, find the exact value of $\sec \beta$. **[5]**

5 The angle θ is such that $0^\circ < \theta < 90^\circ$.

(i) Given that θ satisfies the equation $6\sin 2\theta = 5\cos\theta$, find the exact value of $\sin\theta$. **[3]**

(ii) Given instead that θ satisfies the equation $8\cos\theta\,\mathrm{cosec}^2\,\theta = 3$, find the exact value of $\cos\theta$. **[5]**

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6 **(i)** Express $2\tan^2\theta - \dfrac{1}{\cos\theta}$ in terms of $\sec\theta$. **[3]**

(ii) Hence solve, for $0° < \theta < 360°$, the equation

$$2\tan^2\theta - \frac{1}{\cos\theta} = 4.$$ **[4]**

7 It is given that θ is the acute angle such that $\cot\theta = 4$. Without using a calculator, find the exact value of

(i) $\tan(\theta + 45°)$, **[3]**

(ii) $\operatorname{cosec}\theta$. **[2]**

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8 **(i)** Express the equation $\operatorname{cosec}\theta(3\cos 2\theta + 7) + 11 = 0$ in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are constants. **[3]**

(ii) Hence solve, for $-180° < \theta < 180°$, the equation $\operatorname{cosec}\theta(3\cos 2\theta + 7) + 11 = 0$. **[3]**

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9 The angles α and β are such that

$$\tan\alpha = m + 2 \quad \text{and} \quad \tan\beta = m,$$

where m is a constant.

(i) Given that $\sec^2\alpha - \sec^2\beta = 16$, find the value of m. **[3]**

(ii) Hence find the exact value of $\tan(\alpha + \beta)$. **[3]**

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10 **(a)** Express $\tan 2\alpha$ in terms of $\tan \alpha$ and hence solve, for $0° < \alpha < 180°$, the equation

$$\tan 2\alpha \tan \alpha = 8.$$ [6]

(b) Given that β is the acute angle such that $\sin \beta = \frac{6}{7}$, find the exact value of

(i) $\operatorname{cosec} \beta$, [1]

(ii) $\cot^2 \beta$. [2]

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11 The acute angle A is such that $\tan A = 2$.

(i) Find the exact value of $\operatorname{cosec} A$. **[2]**

(ii) The angle B is such that $\tan(A + B) = 3$. Using an appropriate identity, find the exact value of $\tan B$. **[3]**

12 **(a)** Show that $\dfrac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$. **[3]**

(b) **In this question you must show detailed reasoning.**

Solve $\dfrac{2\tan\theta}{1+\tan^2\theta} = 3\cos 2\theta$ for $0 \le \theta \le \pi$. **[3]**

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13 **In this question you must show detailed reasoning.**

The function f is defined by $f(x) = \cos x + \sqrt{3}\sin x$ with domain $0 \leqslant x \leqslant 2\pi$.

(a) Solve the following equations.

(i) $f'(x) = 0$ **[4]**

(ii) $f''(x) = 0$ **[3]**

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14 **In this question you must show detailed reasoning.**

Given that $5\sin 2x = 3\cos x$, where $0^\circ < x < 90^\circ$, find the exact value of $\sin x$. **[4]**

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15 The angle θ, where $90° < \theta < 180°$, satisfies the equation

$$3\sec^2\theta + 10\tan\theta = 11.$$

(i) Find the value of $\tan\theta$. **[3]**

(ii) Without using a calculator, determine the value of

(a) $\tan 2\theta$, **[2]**

(b) $\cot(2\theta + 135°)$. **[3]**

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16 **(a)** Use the result $\cos(A+B) = \cos A \cos B - \sin A \sin B$ to show that

$\cos(A-B) = \cos A \cos B + \sin A \sin B$. **[2]**

The function $\mathrm{f}(\theta)$ is defined as $\cos(\theta+30°)\cos(\theta-30°)$, where θ is in degrees.

(b) Show that $\mathrm{f}(\theta) = \cos^2\theta - \frac{1}{4}$. **[3]**

(c) **(i)** Determine the following.

- The **maximum** value of $\mathrm{f}(\theta)$
- The smallest **positive** value of θ for which this maximum value occurs **[2]**

(ii) Determine the following.

- The **minimum** value of $\mathrm{f}(\theta)$
- The smallest **positive** value of θ for which this minimum value occurs **[2]**

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17 Prove that $\sin^2(\theta + 45)° - \cos^2(\theta + 45)° \equiv \sin 2\theta°$. **[4]**

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18 It is given that the angle θ satisfies the equation $\sin\left(2\theta+\frac{1}{4}\pi\right)=3\cos\left(2\theta+\frac{1}{4}\pi\right)$.

(i) Show that $\tan 2\theta = \frac{1}{2}$. **[3]**

(ii) Hence find, in surd form, the exact value of $\tan\theta$, given that θ is an obtuse angle. **[5]**

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19 **In this question you must show detailed reasoning.**

(a) Prove that $(\cot\theta+\operatorname{cosec}\theta)^2=\dfrac{1+\cos\theta}{1-\cos\theta}$. **[4]**

(b) Hence solve, for $0<\theta<2\pi$, $3(\cot\theta+\operatorname{cosec}\theta)^2=2\sec\theta$. **[5]**

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20 In this question you must show detailed reasoning.

(a) Show that the equation $m\sec\theta + 3\cos\theta = 4\sin\theta$ can be expressed in the form

$m\tan^2\theta - 4\tan\theta + (m+3) = 0$. **[3]**

(b) It is given that there is only one value of θ, for $0 < \theta < \pi$, satisfying the equation $m\sec\theta + 3\cos\theta = 4\sin\theta$.

Given also that m is a negative integer, find this value of θ, correct to **3** significant figures. **[5]**

21 It is given that θ is the acute angle such that $\sec\theta\sin\theta = 36\cot\theta$.

(i) Show that $\tan\theta = 6$. **[3]**

(ii) Hence, using an appropriate formula in each case, find the exact value of

(a) $\tan(\theta - 45°)$, **[2]**

(b) $\tan 2\theta$. **[2]**

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22 **(i)** Express $\cos 4\theta$ in terms of $\sin 2\theta$ and hence show that $\cos 4\theta$ can be expressed in the form $1 - k\sin^2\theta\cos^2\theta$, where k is a constant to be determined. **[3]**

(ii) Hence find the exact value of $\sin^2(\frac{1}{24}\pi)\cos^2(\frac{1}{24}\pi)$. **[2]**

(iii) By expressing $2\cos^2 2\theta - \frac{8}{3}\sin^2\theta\cos^2\theta$ in terms of $\cos 4\theta$, find the greatest and least possible values of

$$2\cos^2 2\theta - \tfrac{8}{3}\sin^2\theta\cos^2\theta$$

as θ varies. **[5]**

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23 The value of $\tan 10°$ is denoted by p. Find, in terms of p, the value of

(i) $\tan 55°$, **[3]**

(ii) $\tan 5°$, **[4]**

(iii) $\tan \theta$, where θ satisfies the equation $3\sin(\theta + 10°) = 7\cos(\theta - 10°)$. **[5]**

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24 In this question you must show detailed reasoning.

(a) (i) Use the formula for $\cos(A+B)$, and the double angle formulae, to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **[2]**

(ii) Use this result to solve the equation $4\cos^3\theta - 3\cos\theta - \frac{\sqrt{2}}{2} = 0$ for $0° \leqslant \theta \leqslant 180°$. **[3]**

(b) (i) Show that $\left(x + \frac{\sqrt{2}}{2}\right)\left(4x^2 - 2\sqrt{2}x - 1\right) = 4x^3 - 3x - \frac{\sqrt{2}}{2}$. **[1]**

(ii) Hence find the exact roots of the equation $4x^3 - 3x - \frac{\sqrt{2}}{2} = 0$. **[2]**

(c) Use the results from parts **(a)(ii)** and **(b)(ii)** to show that $\cos 15° = \frac{\sqrt{2}+\sqrt{6}}{4}$. **[2]**

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25 **(a)** Show that the equation $2\cot^2 x - 9\operatorname{cosec} x - 3 = 0$ can be expressed in the form

$5\sin^2 x + 9\sin x - 2 = 0$. **[3]**

(b) **(i)** **In this question you must show detailed reasoning.**

Hence solve, for $0 < \theta < \pi$,

$2\cot^2 2\theta - 9\operatorname{cosec} 2\theta - 3 = 0$.

Give your answers correct to **3** decimal places. **[4]**

The small angle approximation for $\sin 2\theta$ is used to find an approximation for the smallest positive solution of the equation $2\cot^2 2\theta - 9\operatorname{cosec} 2\theta - 3 = 0$.

(ii) Show that this approximate solution is accurate to **2** decimal places. **[2]**

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26 **(a)** Given that α satisfies the equation

$$3\sin(\alpha+60°)-3\cos(\alpha+30°)=\operatorname{cosec}^2\alpha,$$

find the exact value of $\sin\alpha$. **[4]**

(b) It is given that β satisfies the equation

$$\sin 4\beta \sec^3\beta = 8\sin\beta+2.$$

By first expressing $\sin 4\beta$ in terms of $\sin 2\beta$ and $\cos 2\beta$, find the exact value of $\sin\beta$. **[7]**

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27 **(i)** Show that $\sin 2\theta(\tan\theta + \cot\theta) \equiv 2$. **[4]**

(ii) Hence

(a) find the exact value of $\tan\frac{1}{12}\pi + \tan\frac{1}{8}\pi + \cot\frac{1}{12}\pi + \cot\frac{1}{8}\pi$, **[3]**

(b) solve the equation $\sin 4\theta(\tan\theta + \cot\theta) = 1$ for $0 < \theta < \frac{1}{2}\pi$, **[3]**

(c) express $(1-\cos 2\theta)^2\left(\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta\right)^3$ in terms of $\sin\theta$. **[2]**

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28 It is given that $f(\theta) = \sin(\theta + 30°) + \cos(\theta + 60°)$.

(i) Show that $f(\theta) = \cos\theta$. Hence show that

$$f(4\theta) + 4f(2\theta) \equiv 8\cos^4\theta - 3.$$ **[6]**

(ii) Hence

(a) determine the greatest and least values of $\dfrac{1}{f(4\theta) + 4f(2\theta) + 7}$ as θ varies, **[3]**

(b) solve the equation

$$\sin(12\alpha + 30°) + \cos(12\alpha + 60°) + 4\sin(6\alpha + 30°) + 4\cos(6\alpha + 60°) = 1$$

for $0° < \alpha < 60°$. **[4]**

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29 **(i)** By first expanding $\cos(2\theta + \theta)$, prove that

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta.$$ **[4]**

(ii) Hence prove that

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$ **[3]**

(iii) Show that the only solutions of the equation

$$1 + \cos 6\theta = 18\cos^2\theta$$

are odd multiples of $90°$. **[5]**

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30 **(a)** **(i)** Sketch the graph of $y = \operatorname{cosec} x$ for $0 < x < 4\pi$. **[3]**

(ii) It is given that $\operatorname{cosec} \alpha = \operatorname{cosec} \beta$, where $\frac{1}{2}\pi < \alpha < \pi$ and $2\pi < \beta < \frac{5}{2}\pi$. By using your sketch, or otherwise, express β in terms of α. **[2]**

(b) **(i)** Write down the identity giving $\tan 2\theta$ in terms of $\tan \theta$. **[1]**

(ii) Given that $\cot \phi = 4$, find the exact value of $\tan \phi \cot 2\phi \tan 4\phi$, showing all your working. **[6]**

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31 **(i)** Prove that

$$\cos^2(\theta + 45°) - \tfrac{1}{2}(\cos 2\theta - \sin 2\theta) \equiv \sin^2\theta. \qquad \textbf{[4]}$$

(ii) Hence solve the equation

$$6\cos^2(\tfrac{1}{2}\theta + 45°) - 3(\cos\theta - \sin\theta) = 2$$

for $-90° < \theta < 90°$. **[3]**

(iii) It is given that there are two values of θ, where $-90° < \theta < 90°$, satisfying the equation

$$6\cos^2(\tfrac{1}{3}\theta + 45°) - 3(\cos\tfrac{2}{3}\theta - \sin\tfrac{2}{3}\theta) = k,$$

where k is a constant. Find the set of possible values of k. **[3]**