

Trigonometric Identities and Equations

Worked Solutions

1 In this question you must show detailed reasoning.

Solve the equation $3\sin^4\phi + \sin^2\phi = 4$, for $0 \leq \phi < 2\pi$, where ϕ is measured in radians. [5]

$$\text{Let } x = \sin^2\phi \Rightarrow 3x^2 + x - 4 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1 + 3 \times 4 \times 4}}{6}$$

$$= \frac{-1 \pm 7}{6}$$

$$= \frac{-8}{6} \text{ or } 1$$

$$|\sin\theta| \leq 1$$
$$\sin\theta \neq -\frac{8}{6}$$

$$\Rightarrow \sin^2\theta = 1 \Rightarrow \sin\theta = \pm 1$$

$$\Rightarrow \theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}$$

2 The acute angles α and β are such that

$$2 \cot \alpha = 1 \quad \text{and} \quad 24 + \sec^2 \beta = 10 \tan \beta.$$

(i) State the value of $\tan \alpha$ and determine the value of $\tan \beta$.

[4]

(ii) Hence find the exact value of $\tan(\alpha + \beta)$.

[3]

$$i) \quad 2 \cot \alpha = 1 \Rightarrow \boxed{\tan \alpha = 2}$$

$$24 + \sec^2 \beta = 10 \tan \beta$$

$$24 + (1 + \tan^2 \beta) = 10 \tan \beta$$

$$25 + \tan^2 \beta - 10 \tan \beta = 0$$

$$\tan \beta = \frac{10 \pm \sqrt{10^2 - 4 \times 25}}{2} = 5$$

$$\Rightarrow \boxed{\tan \beta = 5}$$

$$ii) \quad \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{2 + 5}{1 - 2 \times 5}$$

$$= \boxed{-\frac{7}{9}}$$

3 Prove that $\sqrt{2}\cos(2\theta + 45^\circ) \equiv \cos^2\theta - 2\sin\theta\cos\theta - \sin^2\theta$, where θ is measured in degrees. [3]

$$\begin{aligned}\sqrt{2}\cos(2\theta + 45^\circ) &= \sqrt{2}\left(\cos 2\theta \underbrace{\cos 45^\circ}_{\frac{\sqrt{2}}{2}} - \underbrace{\sin 2\theta}_{\sin 2\theta} \underbrace{\sin 45^\circ}_{\frac{\sqrt{2}}{2}}\right) \\&= \sqrt{2}\left((\cos^2\theta - \sin^2\theta) \frac{\sqrt{2}}{2} - 2\sin\theta\cos\theta \frac{\sqrt{2}}{2}\right) \\&= \cos^2\theta - 2\sin\theta\cos\theta - \sin^2\theta.\end{aligned}$$

- 4 (a) Given that $7 \sin 2\alpha = 3 \sin \alpha$, where $0^\circ < \alpha < 90^\circ$, find the exact value of $\cos \alpha$. [3]
- (b) Given that $3 \cos 2\beta + 19 \cos \beta + 13 = 0$, where $90^\circ < \beta < 180^\circ$, find the exact value of $\sec \beta$. [5]

a)

$$7 \sin 2\alpha = 3 \sin \alpha$$

$$14 \sin \alpha \cos \alpha = 3 \sin \alpha$$

$$\sin \alpha (14 \cos \alpha - 3) = 0$$

$0 < \alpha < 90^\circ$
so $\sin \alpha \neq 0$

$$\Rightarrow \boxed{\cos \alpha = \frac{3}{14}}$$

b)

$$3 \cos 2\beta + 19 \cos \beta + 13 = 0$$

$$3(2 \cos^2 \beta - 1) + 19 \cos \beta + 13 = 0$$

$$6 \cos^2 \beta - 3 + 19 \cos \beta + 13 = 0$$

$$6 \cos^2 \beta + 19 \cos \beta + 10 = 0$$

$$\cos \beta = \frac{-19 \pm \sqrt{19^2 - 4 \times 10 \times 6}}{2 \times 6}$$

$$= \frac{-19 \pm 11}{12} = -\frac{30}{12}, -\frac{2}{3}$$

$\hookrightarrow |\cos \beta| \leq 1$

so $\cos \beta \neq -\frac{30}{12}$

$$\boxed{\sec \beta = \frac{1}{\cos \beta} = -\frac{3}{2}}$$

5 The angle θ is such that $0^\circ < \theta < 90^\circ$.

(i) Given that θ satisfies the equation $6 \sin 2\theta = 5 \cos \theta$, find the exact value of $\sin \theta$. [3]

(ii) Given instead that θ satisfies the equation $8 \cos \theta \operatorname{cosec}^2 \theta = 3$, find the exact value of $\cos \theta$. [5]

$$i) \quad 6 \sin 2\theta = 5 \cos \theta$$

$$12 \sin \theta \cos \theta = 5 \cos \theta$$

$$\sin \theta = \frac{5}{12}$$

$$ii) \quad 8 \cos \theta \operatorname{cosec}^2 \theta = 3$$

$$\frac{8 \cos \theta}{1 - \cos^2 \theta} = 3$$

$$8 \cos \theta = 3 - 3 \cos^2 \theta$$

$$3 \cos^2 \theta + 8 \cos \theta - 3 = 0$$

$$\cos \theta = \frac{-8 \pm \sqrt{64 + 36}}{6}$$

$$= \frac{-8 \pm 10}{6} = \frac{1}{3}, -\frac{3}{1}$$

$$\hookrightarrow |\cos \theta| \leq 1$$

$$\text{so } \cos \theta \neq -3$$

$\Rightarrow$

$$\cos \theta = \frac{1}{3}$$

6 (i) Express $2 \tan^2 \theta - \frac{1}{\cos \theta}$ in terms of $\sec \theta$. [3]

(ii) Hence solve, for $0^\circ < \theta < 360^\circ$, the equation

$$2 \tan^2 \theta - \frac{1}{\cos \theta} = 4. \quad [4]$$

$$\begin{aligned} \text{i)} \quad 2 \tan^2 \theta - \frac{1}{\cos \theta} &= 2(\sec^2 \theta - 1) - \sec \theta \\ &= 2 \sec^2 \theta - \sec \theta - 2 \end{aligned}$$

$$\text{ii)} \quad 2 \tan^2 \theta - \frac{1}{\cos \theta} = 4$$

$$2 \sec^2 \theta - \sec \theta - 2 = 4$$

$$2 \sec^2 \theta - \sec \theta - 6 = 0$$

$$\sec \theta = \frac{1 \pm \sqrt{1 + 4 \cdot 6 \cdot 2}}{4}$$

$$= \frac{1 \pm \sqrt{49}}{4}$$

$$= \frac{1 \pm 7}{4}$$

$$= 2, -\frac{3}{2}$$

$$\sec \theta = 2 \Rightarrow \theta = 60^\circ, 300^\circ$$

$$\sec \theta = -\frac{3}{2} \Rightarrow \theta = 131.8^\circ, 228.2^\circ$$

$$60^\circ$$

$$131.8^\circ$$

$$228.2^\circ$$

$$300^\circ$$

7 It is given that θ is the acute angle such that $\cot \theta = 4$. Without using a calculator, find the exact value of

(i) $\tan(\theta + 45^\circ)$, [3]

(ii) $\operatorname{cosec} \theta$. [2]

$$\begin{aligned} \text{i) } \tan(\theta + 45^\circ) &= \frac{\tan \theta + \tan 45^\circ}{1 - \tan \theta \tan 45^\circ} \\ &= \frac{\tan \theta + 1}{1 - \tan \theta} \\ &= \frac{\frac{1}{4} + 1}{1 - \frac{1}{4}} \\ &= \boxed{\frac{5}{3}} \end{aligned}$$

$\cot \theta = 4$
 $\Rightarrow \tan \theta = \frac{1}{4}$

ii) $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$

$\Rightarrow \operatorname{cosec} \theta = \sqrt{1 + 4^2} = \boxed{\sqrt{17}}$, ~~$-\sqrt{17}$~~

Since θ is acute, making
 $\operatorname{cosec} \theta > 0$.

- 8 (i) Express the equation $\operatorname{cosec} \theta(3 \cos 2\theta + 7) + 11 = 0$ in the form $a \sin^2 \theta + b \sin \theta + c = 0$, where a , b and c are constants. [3]
- (ii) Hence solve, for $-180^\circ < \theta < 180^\circ$, the equation $\operatorname{cosec} \theta(3 \cos 2\theta + 7) + 11 = 0$. [3]

i) $\operatorname{cosec} \theta(3 \cos 2\theta + 7) + 11 = 0$

$$\frac{1}{\sin \theta} (3(1 - 2\sin^2 \theta) + 7) + 11 = 0$$

$\times \sin \theta \downarrow$ $-6\sin^2 \theta + 10 + 11\sin \theta = 0$ $\downarrow \times \sin \theta$

$$6\sin^2 \theta - 11\sin \theta - 10 = 0$$

$$\sin \theta = \frac{11 \pm \sqrt{121 + 4 \times 10 \times 6}}{2 \times 6}$$

$$= \frac{11 \pm 19}{2 \times 6}$$

$$= -\frac{2}{3}, \quad \frac{30}{12}$$

$|\sin \theta| < 1$
so $\sin \theta \neq \frac{30}{12}$

ii) For the range given $\sin \theta = -\frac{1}{2}$

$$\theta = -41.8^\circ, -138^\circ$$

9 The angles α and β are such that

$$\tan \alpha = m + 2 \quad \text{and} \quad \tan \beta = m,$$

where m is a constant.

(i) Given that $\sec^2 \alpha - \sec^2 \beta = 16$, find the value of m . [3]

(ii) Hence find the exact value of $\tan(\alpha + \beta)$. [3]

i) $\sec^2 \alpha - \sec^2 \beta = 16$ $\sec^2 = 1 + \tan^2$

$\cancel{1} + \tan^2 \alpha - \cancel{1} - \tan^2 \beta = 16$

$$(m+2)^2 - m^2 = 16$$

$$m^2 + 4m + 4 - m^2 = 16$$

$$4m = 12$$

$$m = 3$$

ii) $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

$$= \frac{5 + 3}{1 - 5 \times 3}$$

$$= \frac{8}{-14}$$

$$= -\frac{4}{7}$$

- 10 (a) Express $\tan 2\alpha$ in terms of $\tan \alpha$ and hence solve, for $0^\circ < \alpha < 180^\circ$, the equation

$$\tan 2\alpha \tan \alpha = 8.$$

[6]

- (b) Given that β is the acute angle such that $\sin \beta = \frac{6}{7}$, find the exact value of

(i) $\operatorname{cosec} \beta$,

[1]

(ii) $\cot^2 \beta$.

[2]

a) $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

$$\tan 2\alpha \tan \alpha = 8$$

$$\frac{2 \tan^2 \alpha}{1 - \tan^2 \alpha} = 8$$

$$2 \tan^2 \alpha = 8 - 8 \tan^2 \alpha$$

$$\tan^2 \alpha = \frac{8}{10}$$

$$= \frac{4}{5}$$

$$\tan \alpha = \pm \frac{2}{\sqrt{5}}$$

$$\alpha = \arctan \left(\pm \frac{2}{\sqrt{5}} \right)$$

$\Rightarrow$

$$\alpha = 41.8^\circ, 138.2^\circ$$

$$b) \quad i) \quad \omega_{sl} \beta = \frac{1}{\sin \beta} = \frac{7}{6}$$

$$ii) \quad \omega_f^2 \beta = \omega_{sl}^2 \beta - 1$$

$$= \frac{49}{36} - 1$$

$$= \frac{13}{36}$$

11 The acute angle A is such that $\tan A = 2$.

(i) Find the exact value of $\operatorname{cosec} A$. [2]

(ii) The angle B is such that $\tan(A + B) = 3$. Using an appropriate identity, find the exact value of $\tan B$. [3]

$$\begin{aligned} \text{i)} \quad \operatorname{cosec}^2 A &= 1 + \tan^2 A \\ &= 1 + \frac{1}{\tan^2 A} \\ &= 1 + \frac{1}{4} \\ &= \frac{5}{4} \end{aligned}$$

$$\operatorname{cosec} A = \pm \frac{\sqrt{5}}{2}$$

Since A is acute, we have

$$\operatorname{cosec} A = \frac{\sqrt{5}}{2}$$

$$\text{ii) } \tan(A+B) = 3$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 3$$

$$2 + \tan B = 3 - 6 \tan B$$

$$7 \tan B = 1$$

$$\tan B = \boxed{\frac{1}{7}}$$

12 (a) Show that $\frac{2 \tan \theta}{1 + \tan^2 \theta} = \sin 2\theta$. [3]

(b) In this question you must show detailed reasoning.

Solve $\frac{2 \tan \theta}{1 + \tan^2 \theta} = 3 \cos 2\theta$ for $0 \leq \theta \leq \pi$. [3]

a)
$$\frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2 \tan \theta}{\sec^2 \theta}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$= \frac{2 \sin \theta}{\cos \theta \cdot \frac{1}{\cos^2 \theta}}$$

$$= 2 \sin \theta \cos \theta = \sin 2\theta$$

b)
$$\frac{2 \tan \theta}{1 + \tan^2 \theta} = 3 \cos 2\theta$$

$$\sin 2\theta = 3 \cos 2\theta$$

$$\tan 2\theta = 3$$

$$2\theta = \arctan(3) + \pi k, \quad k \in \mathbb{Z}$$

$$\theta = \arctan(3) + \frac{\pi}{2} k$$

$$\theta = 0.625, \underline{2.20} \rightarrow 0.625 + \frac{\pi}{2}$$

adding $\frac{\pi}{2}k$, $k \neq 1$ gives
values outside of $0 \leq \theta \leq \pi$.

13 In this question you must show detailed reasoning.

The function f is defined by $f(x) = \cos x + \sqrt{3} \sin x$ with domain $0 \leq x \leq 2\pi$.

(a) Solve the following equations.

(i) $f'(x) = 0$

[4]

(ii) $f''(x) = 0$

[3]

a) i) $f(x) = \cos x + \sqrt{3} \sin x$

$$f'(x) = -\sin x + \sqrt{3} \cos x$$

$$f'(x) = 0 \Rightarrow -\sin x + \sqrt{3} \cos x = 0$$

$$\sin x = \sqrt{3} \cos x$$

$$\tan x = \sqrt{3}$$

$$x = \arctan(\sqrt{3}) = \frac{\pi}{3}, \frac{4\pi}{3}$$

ii) $f''(x) = -\cos x - \sqrt{3} \sin x$

$$f''(x) = 0$$

$$-\cos x - \sqrt{3} \sin x = 0$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}, \frac{11\pi}{6}$$

14 In this question you must show detailed reasoning.

Given that $5 \sin 2x = 3 \cos x$, where $0^\circ < x < 90^\circ$, find the exact value of $\sin x$.

[4]

$$5 \sin 2x = 3 \cos x \quad 0 < x < 90^\circ$$

$$10 \sin x \cos x = 3 \cos x$$

You lose mark if you do not state it.

Since $0 < x < 90^\circ$, $\cos x \neq 0$ so we can divide both sides by $\cos x$

$$10 \sin x = 3$$

Be careful with dividing by variables

$$\sin x = \frac{3}{10}$$

- 15 The angle θ , where $90^\circ < \theta < 180^\circ$, satisfies the equation

$$3 \sec^2 \theta + 10 \tan \theta = 11.$$

- (i) Find the value of $\tan \theta$. [3]

- (ii) Without using a calculator, determine the value of

- (a) $\tan 2\theta$, [2]

- (b) $\cot(2\theta + 135^\circ)$. [3]

i)

$$3 \sec^2 \theta + 10 \tan \theta = 11 \quad \sec^2 \theta = 1 + \tan^2 \theta$$

$$3(1 + \tan^2 \theta) + 10 \tan \theta = 11$$

$$3 \tan^2 \theta + 10 \tan \theta - 8 = 0$$

$$\begin{aligned} \tan \theta &= \frac{-10 \pm \sqrt{100 + 4 \times 3 \times 8}}{6} \\ &= \frac{-10 \pm 14}{6} \\ &= \frac{2}{3}, -4 \end{aligned}$$

$$\text{Since } 90^\circ < \theta < 180^\circ \Rightarrow \tan \theta = -4.$$

$$\text{ii) a) } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$= \frac{2 \times (-4)}{1 - (-4)^2}$$

$$= \boxed{\frac{8}{15}}$$

b)

$$\cot(2\theta + 135^\circ) = \frac{1}{\tan(2\theta + 135^\circ)}$$

$$= \frac{1 - \tan(2\theta) \tan(135^\circ)}{\tan(2\theta) \tan(135^\circ)}$$

$$= - \frac{1 + \tan(2\theta)}{\tan(2\theta)}$$

$$= \boxed{-\frac{23}{7}}$$

16 (a) Use the result $\cos(A+B) = \cos A \cos B - \sin A \sin B$ to show that

$$\cos(A-B) = \cos A \cos B + \sin A \sin B. \quad [2]$$

The function $f(\theta)$ is defined as $\cos(\theta+30^\circ)\cos(\theta-30^\circ)$, where θ is in degrees.

(b) Show that $f(\theta) = \cos^2 \theta - \frac{1}{4}$. [3]

(c) (i) Determine the following.

- The **maximum** value of $f(\theta)$
- The smallest **positive** value of θ for which this maximum value occurs [2]

(ii) Determine the following.

- The **minimum** value of $f(\theta)$
- The smallest **positive** value of θ for which this minimum value occurs [2]

$$a) \quad \cos(A+(-B)) = \cos A \cos(-B) - \sin A \sin(-B)$$

$\cos(-B) = \cos(B)$
 $\sin(-B) = -\sin(B)$

$$= \cos A \cos B + \sin A \sin B$$

→ must be stated for full marks

$$b) \quad \cos(\theta+30^\circ)\cos(\theta-30^\circ) =$$

$$= (\cos \theta \cos 30^\circ - \sin \theta \sin 30^\circ)(\cos \theta \cos 30^\circ + \sin \theta \sin 30^\circ)$$

$$= (\cos \theta \cos 30^\circ)^2 - (\sin \theta \sin 30^\circ)^2$$

$$= \frac{3}{4} \cos^2 \theta - \frac{1}{4} \sin^2 \theta = \frac{3}{4} \cos^2 \theta - \frac{1}{4} (1 - \cos^2 \theta)$$

$$= \cos^2 \theta - \frac{1}{4}$$

c) $\cos \theta$ ranges from -1 to 1 , so
 $\cos^2 \theta$ ranges from 0 to 1 .

i) Maximum $f(\theta) = 1 - \frac{1}{4} = \frac{3}{4}$

$$\cos^2 \theta = 1 \Rightarrow \cos \theta = \pm 1$$

$$\Rightarrow \theta = 180^\circ k, \quad k \in \mathbb{Z}$$

Smallest positive $\Rightarrow \boxed{\theta = 180^\circ}$

ii) Minimum $f(\theta) = 0 - \frac{1}{4} = -\frac{1}{4}$

$$\cos^2 \theta = 0 \Rightarrow \cos \theta = 0 \Rightarrow$$

$$\theta = 90^\circ + 180^\circ k, \quad k \in \mathbb{Z}$$

Smallest positive value here is

$$\boxed{90^\circ}$$

17 Prove that $\sin^2(\theta + 45^\circ) - \cos^2(\theta + 45^\circ) \equiv \sin 2\theta$.

[4]

$$\begin{aligned} & \sin^2(\theta + 45^\circ) - \cos^2(\theta + 45^\circ) = \\ &= -(\cos^2(\theta + 45^\circ) - \sin^2(\theta + 45^\circ)) \\ &= -\cos(2\theta + 90^\circ) \quad \text{Double angle formula.} \\ &= -\cos 2\theta \cos 90^\circ + \sin 2\theta \sin 90^\circ \\ &= \sin 2\theta \end{aligned}$$

18 It is given that the angle θ satisfies the equation $\sin(2\theta + \frac{1}{4}\pi) = 3 \cos(2\theta + \frac{1}{4}\pi)$.

(i) Show that $\tan 2\theta = \frac{1}{2}$.

[3]

(ii) Hence find, in surd form, the exact value of $\tan \theta$, given that θ is an obtuse angle.

[5]

$$i) \sin(2\theta + \frac{\pi}{4}) = 3 \cos(2\theta + \frac{\pi}{4})$$

$$\sin 2\theta \cos \frac{\pi}{4} + \cos 2\theta \sin \frac{\pi}{4} = 3(\cos 2\theta \cos \frac{\pi}{4} - \sin 2\theta \sin \frac{\pi}{4})$$

Divide both sides by $\cos 2\theta$ gives

$$\tan 2\theta \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} \tan 2\theta$$

$$\frac{4\sqrt{2}}{2} \tan 2\theta = \frac{2\sqrt{2}}{2}$$

$$\tan 2\theta = \frac{1}{2}$$

$$ii) \tan 2\theta = \frac{1}{2}$$

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{1}{2}$$

$$4 \tan \theta = 1 - \tan^2 \theta$$

$$\tan^2 \theta + 4 \tan \theta - 1 = 0$$

$$\tan \theta = \frac{-4 \pm \sqrt{16 + 4}}{2}$$

$$= \frac{-4 \pm \sqrt{20}}{2}$$

Since θ - obtuse

$$\tan \theta < 0 \Rightarrow \tan \theta = \frac{-\sqrt{20} - 4}{2}$$

$$= -2 - \sqrt{5}$$

19 In this question you must show detailed reasoning.

(a) Prove that $(\cot \theta + \operatorname{cosec} \theta)^2 = \frac{1 + \cos \theta}{1 - \cos \theta}$. [4]

(b) Hence solve, for $0 < \theta < 2\pi$, $3(\cot \theta + \operatorname{cosec} \theta)^2 = 2 \sec \theta$. [5]

$$\begin{aligned} \text{a)} \quad (\cot \theta + \operatorname{cosec} \theta)^2 &= \left(\frac{\cos \theta + 1}{\sin \theta} \right)^2 \\ &= \frac{(\cos \theta + 1)^2}{\sin^2 \theta} \\ &= \frac{(\cos \theta + 1)^2}{1 - \cos^2 \theta} \\ &= \frac{(\cos \theta + 1)^2}{(1 - \cos \theta)(1 + \cos \theta)} \\ &= \frac{1 + \cos \theta}{1 - \cos \theta} \end{aligned}$$

$$b) \quad 3 (\cos \theta + \sec \theta)^2 = 2 \sec \theta$$

|| by a)

$$\frac{1 + \cos \theta}{1 - \cos \theta} = \frac{2}{3} \sec \theta$$

$$\cos \theta \times \left(1 + \cos \theta = \frac{2}{3} \sec \theta - \frac{2}{3} \right) \times \cos \theta$$

$$\cos \theta + \cos^2 \theta = \frac{2}{3} - \frac{2}{3} \cos \theta$$

$$\cos^2 \theta + \frac{5}{3} \cos \theta - \frac{2}{3} = 0$$

$$\cos \theta = \frac{-\frac{5}{3} \pm \sqrt{\frac{25}{9} + \frac{8}{3}}}{2}$$

$$= \frac{-\frac{5}{3} \pm \frac{7}{3}}{2} = \frac{1}{3}, -2 \rightarrow \begin{matrix} | \cos \theta | \leq 1 \\ \text{so} \\ \cos \theta \neq -2 \end{matrix}$$

$$\Rightarrow \cos \theta = \frac{1}{3}$$

$$\Rightarrow \theta = 1.23, 5.05$$

20 In this question you must show detailed reasoning.

- (a) Show that the equation $m \sec \theta + 3 \cos \theta = 4 \sin \theta$ can be expressed in the form

$$m \tan^2 \theta - 4 \tan \theta + (m + 3) = 0. \quad [3]$$

- (b) It is given that there is only one value of θ , for $0 < \theta < \pi$, satisfying the equation $m \sec \theta + 3 \cos \theta = 4 \sin \theta$.

Given also that m is a negative integer, find this value of θ , correct to 3 significant figures.

[5]

$$\begin{aligned} \text{a)} \quad m \sec \theta + 3 \cos \theta &= 4 \sin \theta \\ \sec \theta \times (\quad) \times \sec \theta & \\ m \sec^2 \theta + 3 &= 4 \tan \theta \end{aligned}$$

$$m (1 + \tan^2 \theta) + 3 = 4 \tan \theta$$

$$m \tan^2 \theta - 4 \tan \theta + 3 + m = 0$$

b) The discriminant of this quadratic

$$\text{is } 16 - 4m(3+m)$$

As there is only one value, we must have
 $16 - 4m(3+m) = 0$

$$4 - m(3+m) = 0$$

$$m^2 + 3m - 4 = 0$$

$$m = -4$$

$$m = 1$$

$$m \text{ is negative} \Rightarrow m = -4$$

Subbing it back into the equation we get

$$-4 + \tan^2 \theta - 4 + \tan \theta - 1 = 0$$

$$-(2 + \tan \theta + 1)^2 = 0$$

$$\tan \theta = -\frac{1}{2}$$

$$\Rightarrow \theta = \arctan\left(-\frac{1}{2}\right)$$

$$= \boxed{2.678}$$

21 It is given that θ is the acute angle such that $\sec \theta \sin \theta = 36 \cot \theta$.

(i) Show that $\tan \theta = 6$. [3]

(ii) Hence, using an appropriate formula in each case, find the exact value of

(a) $\tan(\theta - 45^\circ)$, [2]

(b) $\tan 2\theta$. [2]

$$i) \quad \sec \theta \sin \theta = 36 \cot \theta$$

$$\tan \theta = \frac{36}{\tan \theta}$$

$$\tan^2 \theta = 36$$

$$\tan \theta = \pm 6, \text{ since } \theta \text{ is acute}$$

$$\tan \theta = 6$$

$$ii) \quad a) \quad \tan(\theta - 45^\circ) = \frac{\tan \theta - \boxed{\tan(45^\circ)}}{1 + \tan \theta \tan(45^\circ)}$$

$$= \frac{\tan \theta - 1}{1 + \tan \theta}$$

$$= \frac{6 - 1}{6 + 1}$$

$$= \boxed{\frac{5}{7}}$$

$$\begin{aligned} \text{b) } \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ &= \frac{2 \times 6}{1 - 6^2} \\ &= \boxed{-\frac{12}{35}} \end{aligned}$$

- 22 (i) Express $\cos 4\theta$ in terms of $\sin 2\theta$ and hence show that $\cos 4\theta$ can be expressed in the form $1 - k \sin^2 \theta \cos^2 \theta$, where k is a constant to be determined. [3]
- (ii) Hence find the exact value of $\sin^2(\frac{1}{24}\pi) \cos^2(\frac{1}{24}\pi)$. [2]
- (iii) By expressing $2 \cos^2 2\theta - \frac{8}{3} \sin^2 \theta \cos^2 \theta$ in terms of $\cos 4\theta$, find the greatest and least possible values of $2 \cos^2 2\theta - \frac{8}{3} \sin^2 \theta \cos^2 \theta$ as θ varies. [5]

$$\begin{aligned} \text{i)} \quad \cos 4\theta &= \cos^2 2\theta - \sin^2 2\theta = \\ &= 1 - \sin^2 2\theta - \sin^2 2\theta \\ &= 1 - 2 \sin^2 2\theta \end{aligned}$$

Hence

$$\begin{aligned} \cos 4\theta &= 1 - 2 (2 \sin \theta \cos \theta)^2 \\ &= 1 - 8 \sin^2 \theta \cos^2 \theta \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad 1 - 8 \sin^2\left(\frac{\pi}{24}\right) \cos^2\left(\frac{\pi}{24}\right) &= \cos\left(\frac{4\pi}{24}\right) \\ 1 - 8 \sin^2\left(\frac{\pi}{24}\right) \cos^2\left(\frac{\pi}{24}\right) &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\Rightarrow \sin^2\left(\frac{\pi}{24}\right) \cos^2\left(\frac{\pi}{24}\right) = \frac{2 - \sqrt{3}}{16}$$

$$\text{iii)} \quad 2 \cos^2 2\theta - \frac{8}{3} \sin^2 \theta \cos^2 \theta =$$

$$= \underbrace{2 \cos^2 2\theta}_{\parallel} - \frac{1 - \cos 4\theta}{3}$$

$$= \underbrace{\cos 4\theta + 1}_{\parallel} - \frac{1 - \cos 4\theta}{3}$$

$$= \frac{4 \cos 4\theta}{3} + \frac{2}{3}, \quad -1 \leq \cos 4\theta \leq 1$$

hence

Greatest value $\frac{4}{3} + \frac{2}{3} = 2$

Minimal value $-\frac{4}{3} + \frac{2}{3} = -\frac{1}{3}$

23 The value of $\tan 10^\circ$ is denoted by p . Find, in terms of p , the value of

(i) $\tan 55^\circ$, [3]

(ii) $\tan 5^\circ$, [4]

(iii) $\tan \theta$, where θ satisfies the equation $3 \sin(\theta + 10^\circ) = 7 \cos(\theta - 10^\circ)$. [5]

$$\begin{aligned} \text{i)} \quad \tan(55^\circ) &= \tan(45^\circ + 10^\circ) \\ &= \frac{\tan(45^\circ) + \tan(10^\circ)}{1 - \tan(45^\circ)\tan(10^\circ)} \\ &= \boxed{\frac{1 + p}{1 - p}} \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \tan 5^\circ &= \tan(60^\circ - 55^\circ) \\ &= \frac{\tan(60^\circ) - \tan(55^\circ)}{1 + \tan(60^\circ)\tan(55^\circ)} \\ &= \frac{\sqrt{3} - \frac{1+p}{1-p}}{1 + \sqrt{3} \frac{1+p}{1-p}} \\ &= \frac{\sqrt{3} - \sqrt{3}p - 1 - p}{1 - p + \sqrt{3} + \sqrt{3}p} \end{aligned}$$

$$= \frac{\sqrt{3}-1 - (\sqrt{3}+1)p}{1+\sqrt{3} + (\sqrt{3}-1)p}$$

$$\text{ii)} \quad 3 \sin(10^\circ + \theta) = 7 \cos(\theta - 10^\circ)$$

$$3 \sin \theta \cos 10^\circ + 3 \sin 10^\circ \cos \theta = 7 \cos \theta \cos 10^\circ + 7 \sin \theta \sin 10^\circ$$

Divide both sides by $\cos \theta \cos 10^\circ$

$$3 \tan \theta + 3 \tan 10^\circ = 7 + 7 \tan 10^\circ \tan \theta$$

$$(3 - 7 \tan 10^\circ) \tan \theta = 7 - 3 \tan 10^\circ$$

$$\tan \theta = \frac{7 - 3 \tan 10^\circ}{3 - 7 \tan 10^\circ}$$

$$= \frac{7 - 3p}{3 - 7p}$$

24 In this question you must show detailed reasoning.

(a) (i) Use the formula for $\cos(A+B)$, and the double angle formulae, to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [2]

(ii) Use this result to solve the equation $4\cos^3\theta - 3\cos\theta - \frac{\sqrt{2}}{2} = 0$ for $0^\circ \leq \theta \leq 180^\circ$. [3]

(b) (i) Show that $\left(x + \frac{\sqrt{2}}{2}\right)(4x^2 - 2\sqrt{2}x - 1) = 4x^3 - 3x - \frac{\sqrt{2}}{2}$. [1]

(ii) Hence find the exact roots of the equation $4x^3 - 3x - \frac{\sqrt{2}}{2} = 0$. [2]

(c) Use the results from parts **(a)(ii)** and **(b)(ii)** to show that $\cos 15^\circ = \frac{\sqrt{2} + \sqrt{6}}{4}$. [2]

$$\begin{aligned}
 a) \quad i) \quad & \cos(3\theta) = \cos(2\theta + \theta) \\
 &= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\
 &= (\cos^2 \theta - \sin^2 \theta) \cos \theta - 2 \sin \theta \cos \theta \sin \theta \\
 &= \cos^3 \theta - \sin^2 \theta \cos \theta - 2 \sin^2 \theta \cos \theta \\
 &= \cos^3 \theta - 3 \sin^2 \theta \cos \theta \\
 &= \cos^3 \theta - 3(1 - \cos^2 \theta) \cos \theta \\
 &= \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta \\
 &= 4 \cos^3 \theta - 3 \cos \theta
 \end{aligned}$$

$$ii) \quad 4 \cos^3 \theta - 3 \cos \theta - \frac{\sqrt{2}}{2} = 0$$

$$\cos 3\theta - \frac{\sqrt{2}}{2} = 0$$

$$\cos 3\theta = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad 3\theta = 45^\circ, 315^\circ, 405^\circ, \dots$$

$$\theta = 15^\circ, 105^\circ, 135^\circ$$

$$b) \quad i) \quad \left(x + \frac{\sqrt{2}}{2}\right)(4x^2 - 2\sqrt{2}x - 1) =$$

$$= 4x^3 - 2\sqrt{2}x^2 - x + 2\sqrt{2}x^2 - 2x - \frac{\sqrt{2}}{2}$$

$$= 4x^3 - 3x - \frac{\sqrt{2}}{2}$$

$$ii) \quad \lim_{x \rightarrow \infty} \left(x + \frac{\sqrt{2}}{2}\right)(4x^2 - 2\sqrt{2}x - 1)$$

we get $x = -\frac{\sqrt{2}}{2}$, $x = \frac{2\sqrt{2} \pm \sqrt{8+16}}{8}$

$$= \frac{2\sqrt{2} \pm 2\sqrt{6}}{8}$$

$$= \frac{\sqrt{2} \pm \sqrt{6}}{8}$$

Roots are

$$-\frac{\sqrt{2}}{2}, \frac{\sqrt{2} - \sqrt{6}}{8}, \frac{\sqrt{2} + \sqrt{6}}{8}$$

c) Using a) i) $\cos \theta$ is a root of the above polynomial for $\theta = 15^\circ, 105^\circ, 135^\circ$.
these yield distinct values for $\cos \theta$.

$270^\circ > 135^\circ > 105^\circ > 90^\circ \Rightarrow \cos \theta$ is negative for these values of θ

$$\Rightarrow \cos 15^\circ = \frac{\sqrt{2} + \sqrt{6}}{4}$$

- 25 (a) Show that the equation $2 \cot^2 x - 9 \operatorname{cosec} x - 3 = 0$ can be expressed in the form

$$5 \sin^2 x + 9 \sin x - 2 = 0. \quad [3]$$

- (b) (i) In this question you must show detailed reasoning.

Hence solve, for $0 < \theta < \pi$,

$$2 \cot^2 2\theta - 9 \operatorname{cosec} 2\theta - 3 = 0.$$

Give your answers correct to 3 decimal places. [4]

The small angle approximation for $\sin 2\theta$ is used to find an approximation for the smallest positive solution of the equation $2 \cot^2 2\theta - 9 \operatorname{cosec} 2\theta - 3 = 0$.

- (ii) Show that this approximate solution is accurate to 2 decimal places. [2]

a)

$$2 \cot^2 x - 9 \operatorname{cosec} x - 3 = 0 \quad \cot^2 x = \operatorname{cosec}^2 x - 1$$

$$-2 + 2 \operatorname{cosec}^2 x - 9 \operatorname{cosec} x - 3 = 0$$

$$\sin^2 x \times \quad \downarrow \times \sin^2 x$$

$$-5 \sin^2 x + 2 - 9 \sin x = 0$$

$$(-1) \times \quad \downarrow \times (-1)$$

$$5 \sin^2 x + 9 \sin x - 2 = 0$$

b) i) We solve for $\sin 2\theta$ to get

$$\sin 2\theta = \frac{-9 \pm \sqrt{9^2 + 4 \times 5 \times 2}}{2 \times 5}$$

$$= \frac{-9 \pm 11}{10} = -2, \frac{1}{5}$$

$$\Rightarrow \sin 2\theta = \frac{1}{5}$$

$$2\theta = \arcsin\left(\frac{1}{5}\right) = 0.2014$$

$$= 2.94023$$

$\Rightarrow$

$$\theta = 0.101, 1.47$$

Using $\sin 2\theta \approx 2\theta \Rightarrow 2\theta = \frac{1}{5}$

$\Rightarrow \theta = 0.1$, which is accurate to 2 decimal places.

This sentence costs 1 mark.

- 26 (a) Given that α satisfies the equation

$$3 \sin(\alpha + 60^\circ) - 3 \cos(\alpha + 30^\circ) = \operatorname{cosec}^2 \alpha,$$

find the exact value of $\sin \alpha$.

[4]

- (b) It is given that β satisfies the equation

$$\sin 4\beta \sec^3 \beta = 8 \sin \beta + 2.$$

By first expressing $\sin 4\beta$ in terms of $\sin 2\beta$ and $\cos 2\beta$, find the exact value of $\sin \beta$.

[7]

a)

$$3 \sin(\alpha + 60^\circ) - 3 \cos(\alpha + 30^\circ) = \operatorname{cosec}^2 \alpha$$

$$3 \left(\sin \alpha \overset{\frac{1}{2}}{\cos 60^\circ} + \overset{\frac{\sqrt{3}}{2}}{\sin 60^\circ} \cos \alpha - \cos \alpha \overset{\frac{\sqrt{3}}{2}}{\cos 30^\circ} + \sin \alpha \overset{\frac{1}{2}}{\sin 30^\circ} \right) = \operatorname{cosec}^2 \alpha$$

$$3 \left(\frac{1}{2} \sin \alpha + \frac{\sqrt{3}}{2} \cancel{\cos \alpha} - \frac{\sqrt{3}}{2} \cancel{\cos \alpha} + \frac{1}{2} \sin \alpha \right) = \operatorname{cosec}^2 \alpha$$

$$3 \sin \alpha = \frac{1}{\sin^2 \alpha}$$

$$\sin^3 \alpha = \frac{1}{3}$$

$$\sin \alpha = \frac{1}{\sqrt[3]{3}}$$

$$b) \quad \sin 4\beta \sec^3 \beta = 8 \sin \beta + 2$$

$$2 \sin 2\beta \cos 2\beta \sec^3 \beta = 8 \sin \beta + 2$$

$$\frac{2 \sin 2\beta (1 - 2 \sin^2 \beta)}{\cos^3 \beta} = 8 \sin \beta + 2$$

$$\frac{4 \sin \beta \cancel{\cos \beta} (1 - 2 \sin^2 \beta)}{\cos^3 \beta} = 8 \sin \beta + 2$$

$$4 \sin \beta - \cancel{8 \sin^3 \beta} = (8 \sin \beta + 2)(1 - \sin^2 \beta)$$

$$= 8 \sin \beta + 2 - \cancel{8 \sin^3 \beta} - 2 \sin^2 \beta$$

$$2 \sin^2 \beta - 4 \sin \beta - 2 = 0$$

$$\sin^2 \beta - 2 \sin \beta - 1 = 0$$

$$\sin \beta = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}$$

$$|\sin \beta| \leq 1$$

$$\text{so } \sin \beta \neq 1 + \sqrt{2} \Rightarrow$$

$$\boxed{\sin \beta = 1 - \sqrt{2}}$$

27 (i) Show that $\sin 2\theta(\tan \theta + \cot \theta) \equiv 2$.

[4]

(ii) Hence

(a) find the exact value of $\tan \frac{1}{12}\pi + \tan \frac{1}{8}\pi + \cot \frac{1}{12}\pi + \cot \frac{1}{8}\pi$, [3](b) solve the equation $\sin 4\theta(\tan \theta + \cot \theta) = 1$ for $0 < \theta < \frac{1}{2}\pi$, [3](c) express $(1 - \cos 2\theta)^2(\tan \frac{1}{2}\theta + \cot \frac{1}{2}\theta)^3$ in terms of $\sin \theta$. [2]

$$i) \quad \sin 2\theta (\tan \theta + \cot \theta)$$

$$= 2 \cancel{\sin \theta} \cancel{\cos \theta} \cdot \frac{\cancel{\sin \theta}}{\cancel{\cos \theta}} + 2 \cancel{\sin \theta} \cancel{\cos \theta} \frac{\cancel{\cos \theta}}{\cancel{\sin \theta}}$$

$$= 2 \sin^2 \theta + 2 \cos^2 \theta = 2$$

$$ii) \quad a) \quad \text{Using } i) \quad \tan \theta + \cot \theta = \frac{2}{\sin 2\theta}$$

$$\Rightarrow \tan\left(\frac{\pi}{12}\right) + \cot\left(\frac{\pi}{12}\right) + \tan\left(\frac{\pi}{8}\right) + \cot\left(\frac{\pi}{8}\right) =$$

$$= \frac{2}{\sin\left(\frac{\pi}{6}\right)} + \frac{2}{\sin\left(\frac{\pi}{4}\right)} = \frac{2}{\frac{1}{2}} + \frac{2}{\frac{\sqrt{2}}{2}}$$

$$= 4 + 2\sqrt{2}$$

$$b) \quad \sin 4\theta (\tan \theta + \cot \theta) = 1$$

$$2 \sin 2\theta \cos 2\theta (\tan \theta + \cot \theta) = 1$$

$$4 \cos 2\theta = 1$$

$$\cos 2\theta = \frac{1}{4}$$

$$2\theta = \arccos\left(\frac{1}{4}\right)$$

$$\theta = 0.659$$

$$2\theta = \arccos\left(\frac{1}{4}\right),$$

$$2\pi - \arccos\left(\frac{1}{4}\right)$$

$$\Rightarrow 2\theta = 2\pi - \arccos\left(\frac{1}{4}\right)$$

$\Rightarrow \theta \geq \frac{\pi}{2}$, which is outside requested range.

$$c) \quad (1 - \cos 2\theta)^2 \left(\tan \frac{1}{2}\theta + \cot \frac{1}{2}\theta \right)^3 =$$

$$= (2 \sin^2 \theta)^2 \left(\tan \frac{1}{2}\theta + \cot \frac{1}{2}\theta \right)^3 =$$

$$= 4 \sin^4 \theta \left(\tan \frac{1}{2}\theta + \cot \frac{1}{2}\theta \right)^3 =$$

$$= 4 \cdot 2^3 \sin \theta = 32 \sin \theta$$

28 It is given that $f(\theta) = \sin(\theta + 30^\circ) + \cos(\theta + 60^\circ)$.

(i) Show that $f(\theta) = \cos \theta$. Hence show that

$$f(4\theta) + 4f(2\theta) \equiv 8\cos^4\theta - 3. \quad [6]$$

(ii) Hence

(a) determine the greatest and least values of $\frac{1}{f(4\theta) + 4f(2\theta) + 7}$ as θ varies, [3]

(b) solve the equation

$$\sin(12\alpha + 30^\circ) + \cos(12\alpha + 60^\circ) + 4\sin(6\alpha + 30^\circ) + 4\cos(6\alpha + 60^\circ) = 1$$

for $0^\circ < \alpha < 60^\circ$. [4]

$$\begin{aligned} \text{i)} \quad & \sin(\theta + 30^\circ) + \cos(\theta + 60^\circ) = \\ & \sin \theta \overset{\frac{\sqrt{3}}{2}}{\cancel{\cos 30^\circ}} + \overset{\frac{1}{2}}{\cancel{\sin 30^\circ}} \cos \theta + \cos \theta \overset{\frac{1}{2}}{\cancel{\cos 60^\circ}} - \sin \theta \overset{\frac{\sqrt{3}}{2}}{\cancel{\sin 60^\circ}} = \\ & = \frac{1}{2} \cos \theta + \frac{1}{2} \cos \theta \\ & = \cos \theta \end{aligned}$$

$$\begin{aligned} f(4\theta) + 4f(2\theta) &= \cos(4\theta) + 4\cos(2\theta) \\ &= 2\cos^2 2\theta - 1 + 4(2\cos^2 \theta - 1) \\ &= 2(2\cos^2 \theta - 1)^2 - 1 + 8\cos^2 \theta - 4 \\ &= 2(4\cos^4 \theta - 4\cos^2 \theta + 1) - 1 + 8\cos^2 \theta - 4 \end{aligned}$$

$$= 8 \cos^4 \theta - 3$$

$$a) \frac{1}{f(4\theta) + 4f(2\theta) + 7} = \frac{1}{8 \cos^4 \theta + 4}$$

$\cos^4 \theta$ ranges from 0 to 1

$\Rightarrow$ the minimal value is $\frac{1}{12}$ and

the maximum is $\frac{1}{4}$.

$$b) \sin(12\alpha + 30^\circ) + \cos(12\alpha + 60^\circ) + 4\sin(6\alpha + 30^\circ) + 4\cos(6\alpha + 60^\circ) =$$

$$= f(4(3\alpha)) + 4f(2(3\alpha)) = 1$$

$$8 \cos^4(3\alpha) - 3 = 1$$

$$\cos^4(3\alpha) = \frac{1}{2}$$

$$\cos(3\alpha) = \pm \frac{1}{\sqrt[4]{2}}$$

$$3\alpha = \arccos\left(\pm \frac{1}{\sqrt{2}}\right).$$

$$= 32.965, 147.235$$

$$\Rightarrow \alpha = 10.922, 49.078$$

→ there are more values of α satisfying $\cos(3\alpha) = \pm \frac{1}{\sqrt{2}}$, but they are outside the range $0 < \alpha < 60^\circ$.

29 (i) By first expanding $\cos(2\theta + \theta)$, prove that

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta. \quad [4]$$

(ii) Hence prove that

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1. \quad [3]$$

(iii) Show that the only solutions of the equation

$$1 + \cos 6\theta = 18\cos^2\theta$$

are odd multiples of 90° .

[5]

$$\begin{aligned} \text{i)} \quad \cos(2\theta + \theta) &= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta \cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\ &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \cos 6\theta &= 4\cos^3(2\theta) - 3\cos(2\theta) \\ &= 4(2\cos^2\theta - 1)^3 - 3(2\cos^2\theta - 1) \\ &= 4(8\cos^6\theta - 12\cos^4\theta + 6\cos^2\theta - 1) - 3(2\cos^2\theta - 1) \\ &= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 \end{aligned}$$

$$i) \quad 1 + \cos 6\theta = 18 \cos^2 \theta$$

$$\cancel{1 + 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1 = 18 \cos^2 \theta}$$

$$8 \cos^4 \theta (4 \cos^2 \theta - 6) = 0$$

$$\cos^2 \theta = \frac{6}{4} \quad \text{or} \quad \cos^4 \theta = 0$$

impossible

$$|\cos x| \leq 1$$

$$\Rightarrow \cos^2 x \leq 1$$

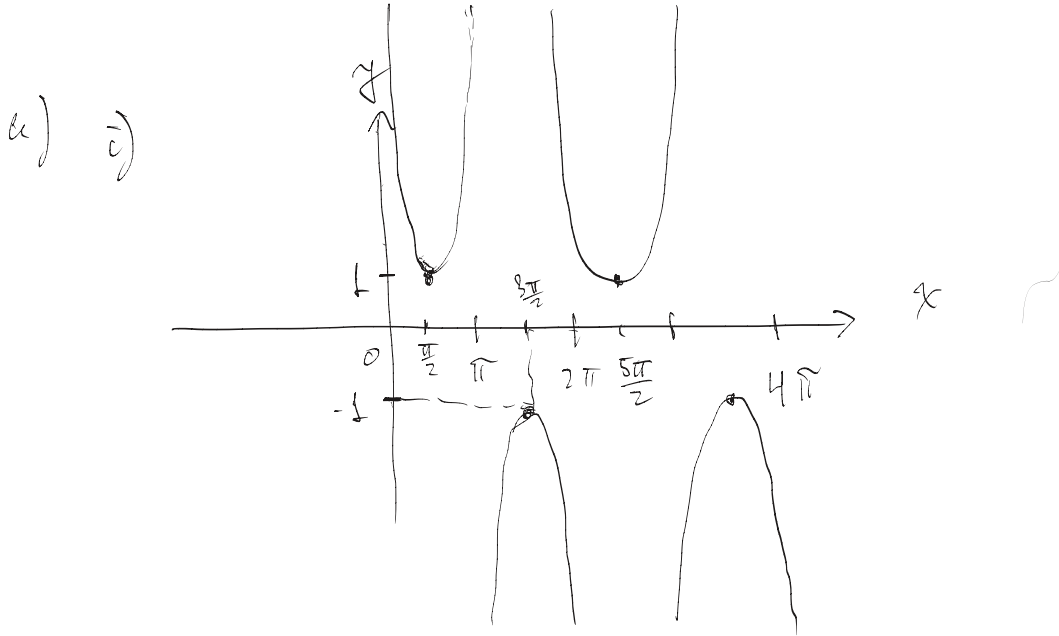
$$\Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = 90^\circ + 180^\circ k$$

$$= (2k+1) \cdot 90^\circ$$

$$\text{for } k \in \mathbb{Z}.$$

- 30 (a) (i) Sketch the graph of $y = \operatorname{cosec} x$ for $0 < x < 4\pi$. [3]
- (ii) It is given that $\operatorname{cosec} \alpha = \operatorname{cosec} \beta$, where $\frac{1}{2}\pi < \alpha < \pi$ and $2\pi < \beta < \frac{5}{2}\pi$. By using your sketch, or otherwise, express β in terms of α . [2]
- (b) (i) Write down the identity giving $\tan 2\theta$ in terms of $\tan \theta$. [1]
- (ii) Given that $\cot \phi = 4$, find the exact value of $\tan \phi \cot 2\phi \tan 4\phi$, showing all your working. [6]



ii) $\operatorname{cosec} \alpha = \operatorname{cosec} \beta$

$$\frac{\pi}{2} < \alpha < \pi$$

$$2\pi < \beta < \frac{5\pi}{2}$$

$$\beta = 2\pi + (\pi - \alpha) = 3\pi - \alpha$$

b) i) $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

ii) $\cot \phi = 4 \Rightarrow \tan \phi = \frac{1}{4}$

$$\cot 2\phi = \frac{1}{\tan 2\theta}$$

$$= \frac{1 - \tan^2 \theta}{2 \tan \theta}$$

$$= \frac{1 - \frac{1}{16}}{2 \times \frac{1}{4}}$$

$$= \frac{15}{16 \cdot \frac{1}{2}}$$

$$= \frac{15}{8} \Rightarrow \tan 2\phi = \frac{8}{15}$$

$$\tan 4\phi = \frac{2 \tan 2\phi}{1 - \tan^2 2\phi}$$

$$= \frac{2 \times \frac{8}{15}}{1 - \left(\frac{8}{15}\right)^2}$$

$$= \frac{240}{161}$$

$$\Rightarrow \tan \theta \cdot \cot 2\theta \cdot \tan 4\theta = \frac{1}{4} \cdot \frac{15}{8} \cdot \frac{240}{161}$$

$$= \frac{225}{322}$$

31 (i) Prove that

$$\cos^2(\theta + 45^\circ) - \frac{1}{2}(\cos 2\theta - \sin 2\theta) \equiv \sin^2 \theta. \quad [4]$$

(ii) Hence solve the equation

$$6 \cos^2\left(\frac{1}{2}\theta + 45^\circ\right) - 3(\cos \theta - \sin \theta) = 2$$

$$\text{for } -90^\circ < \theta < 90^\circ. \quad [3]$$

(iii) It is given that there are two values of θ , where $-90^\circ < \theta < 90^\circ$, satisfying the equation

$$6 \cos^2\left(\frac{1}{3}\theta + 45^\circ\right) - 3(\cos \frac{2}{3}\theta - \sin \frac{2}{3}\theta) = k,$$

$$\text{where } k \text{ is a constant. Find the set of possible values of } k. \quad [3]$$

$$\begin{aligned} \text{i)} \quad & \cos^2(\theta + 45^\circ) - \frac{1}{2}(\cos 2\theta - \sin 2\theta) = \\ & = (\cos \theta \cos 45^\circ - \sin \theta \sin 45^\circ)^2 - \frac{1}{2}(\cos 2\theta - \sin 2\theta) \\ & = \frac{1}{2}(\cos^2 \theta - 2\sin \theta \cos \theta + \sin^2 \theta) - \frac{1}{2}(\cos 2\theta - \sin 2\theta) \\ & = \frac{1}{2}(1 - \sin 2\theta) - \frac{1}{2}(1 - 2\sin^2 \theta - \sin 2\theta) \\ & = \sin^2 \theta \end{aligned}$$

$$\text{ii)} \quad 6 \left(\cos^2\left(\frac{1}{2}\theta + 45^\circ\right) - \frac{1}{2}(\cos \theta - \sin \theta) \right) = 2$$

$$\text{By i)} \quad 6 \sin^2 \frac{1}{2} \theta = 2$$

$$\sin^2 \frac{1}{2} \theta = \frac{1}{3}$$

$$\sin \frac{1}{2} \theta = \pm \frac{1}{\sqrt{3}}$$

$$\theta = 2 \arcsin \left(\pm \frac{1}{\sqrt{3}} \right).$$

$$= 70.5^\circ, -70.5^\circ$$

$$\text{iii) } 6 \sin^2 \frac{1}{3} \theta = k$$

$$\sin^2 \frac{1}{3} \theta = \frac{k}{6}$$

$$-90 < \theta < 90 \Rightarrow -30 < \frac{\theta}{3} < 30^\circ$$

$$\Rightarrow \sin(-30) < \sin\left(\frac{\theta}{3}\right) < \sin(30^\circ)$$

$$-\frac{1}{2} < \sin\left(\frac{\theta}{3}\right) < \frac{1}{2}$$

$$\Rightarrow \sin^2\left(\frac{\theta}{3}\right) < \frac{1}{4}$$

$$\Rightarrow 0 < \frac{k}{6} < \frac{1}{4}$$

$$0 < k < \frac{3}{2}$$

If $k=0$,
there is only
one value
 θ s.t.

$$6 \sin^2 \frac{\theta}{3} = 0$$